MICROPTERON TRAVELING WAVES IN DIATOMIC FERMI-PASTA-ULAM-TSINGOU LATTICES UNDER THE EQUAL MASS LIMIT

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Abstract. The diatomic Fermi-Pasta-Ulam-Tsingou (FPUT) lattice is an infinite chain of alternating particles connected by identical nonlinear springs. We prove the existence of micropteron traveling waves in the diatomic FPUT lattice in the limit as the ratio of the two alternating masses approaches 1, at which point the diatomic lattice reduces to the well-understood monatomic FPUT lattice. These are traveling waves whose profiles asymptote to a small periodic oscillation at infinity, instead of vanishing like the classical solitary wave. We produce these micropteron waves using a functional analytic method, originally due to Beale, that was successfully deployed in the related long wave and small mass diatomic problems. Unlike the long wave and small mass problems, this equal mass problem is not singularly perturbed, and so the amplitude of the micropteron's oscillation is not necessarily small beyond all orders (i.e., the traveling wave that we find is not necessarily a nanopteron). The central challenge of this equal mass problem hinges on a hidden solvability condition in the traveling wave equations, which manifests itself in the existence and fine properties of asymptotically sinusoidal solutions (Jost solutions) to an auxiliary advance-delay differential equation.

1. Introduction

1.1. The diatomic FPUT lattice.

A diatomic Fermi-Pasta-Ulam-Tsingou (FPUT) lattice is an infinite one-dimensional chain of particles of alternating masses connected by identical springs. These lattices, also called mass dimers, are a material generalization of the finite lattice of identical particles studied numerically by Fermi, Pasta, and Ulam [FPU55] and Tsingou [Dau08]; such lattices are valued in applications as models of wave propagation in discrete and granular materials [Bri53, Kev11].

We index the particles and their masses by $j \in \mathbb{Z}$ and let u_j denote the position of the jth particle. After a routine nondimensionalization, we may assume that the jth particle has mass

$$m_j = \begin{cases} 1, & j \text{ is odd} \\ m, & j \text{ is even} \end{cases} \tag{1.1.1}$$

and that each spring exerts the force $F(r) = r + r^2$ when stretched a distance r from its equilibrium length. Newton's law then implies that the position functions u_j satisfy the

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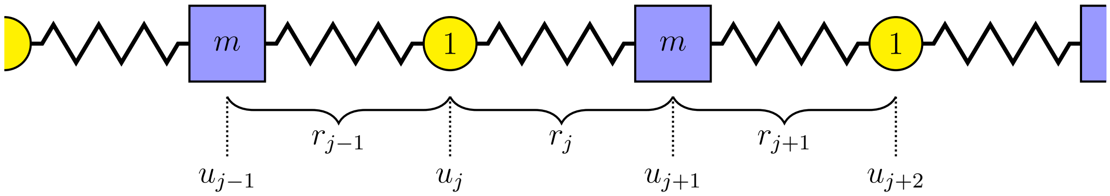

FIGURE 1. The mass dimer ($r_j := u_{j+1} - u_j$)

system

$$\begin{cases} \ddot{u}_j = F(u_{j+1} - u_j) - F(u_j - u_{j-1}), & j \text{ is odd} \\ \\ m\ddot{u}_j = F(u_{j+1} - u_j) - F(u_j - u_{j-1}), & j \text{ is even}. \end{cases} \tag{1.1.2}$$

We sketch a diatomic FPUT lattice in Figure 1.

We are interested in the existence and properties of traveling waves in diatomic lattices in the limit as the ratio of the two alternating masses approaches 1. Specifically, we define the relative displacement between the jth and the $(j+1)$st particle to be

$$r_j := u_{j+1} - u_j,$$

and then we make the traveling wave ansatz

$$r_j(t) = \begin{cases} p_1(j - ct), & j \text{ is even} \\ p_2(j - ct), & j \text{ is odd}. \end{cases} \tag{1.1.3}$$

Here p_1 and p_2 are the traveling wave profiles and $c \in \mathbb{R}$ is the wave speed.

When the masses are identical, the lattice reduces to a *monatomic* lattice, which, due to the work of Friesecke and Wattis [FW94] and Friesecke and Pego [FP99, FP02, FP04a, FP04b] is known to bear solitary traveling waves, i.e., waves whose profiles vanish exponentially fast at spatial infinity; see also Pankov [Pan05] for a comprehensive overview of monatomic traveling waves. Our interest, then, is to determine how the monatomic solitary traveling wave changes when the mass ratio is close to 1.

1.2. **Parameter regimes.** We derive our motivation for this equal mass situation from two recent papers studying traveling waves in diatomic lattices under different limits. Faver and Wright [FW18] fix the mass ratio[1] and consider the long wave limit, i.e., they look for traveling waves where the wave speed is close to a special m-dependent threshold called the "speed of sound" and where the traveling wave profile is close to a certain KdV sech^2-type soliton. Hoffman and Wright [HW17] fix the wave speed and consider the small mass limit, in which the ratio of the alternating masses approaches zero, thereby reducing the lattice from diatomic to monatomic[2]. Figure 3 sketches the bands of long wave (in yellow) and

[1] They work with $w := 1/m > 1$; after rescaling and relabeling the lattice, we may equivalently think of their results for $m \in (0, 1)$.

[2] This is not the same monatomic lattice that results from the equal mass limit; the springs in this small mass limiting lattice are double the length of the original springs in the diatomic lattice, and so they exert twice the original force. This factor of 2 ends up affecting the wave speed of the traveling waves that Hoffman and Wright construct: theirs have speed close to $\sqrt{2}$.

small mass (in red) traveling waves and indicates, roughly, how they depend on wave speeds and mass ratios.

In both problems, the solitary wave that exists in the limiting case perturbs into a traveling wave whose profile asymptotes to a small amplitude periodic oscillation or "ripple." That is, the wave is not "localized" in the "core" of the classical solitary wave, and so, per Boyd [Boy98], it is a *nonlocal* solitary wave. Moreover, the long wave and small mass problems are singularly perturbed, which causes the amplitude of their periodic ripples to be small beyond all orders of the long wave/small mass parameter. So, these nonlocal traveling waves are, in Boyd's parlance, *nanopterons*; see [Boy98] for an overview of the nanopteron's many incarnations in applied mathematics and nature.

The question of the equal mass limit then follows naturally from the success of these two studies. It was raised in the conclusion of [FW18], where the authors wondered if the diatomic long wave profiles would converge to those found by Friesecke and Pego in the monatomic long wave limit [FP99], and appears as far back as Brillouin's book [Bri53], which examines both the small and equal mass limits for lattices with linear spring forces.

The articles [FW18] and [HW17] both derive their nanopteron traveling waves via a method due to Beale [Bea91] for a capillary-gravity water wave problem. Beale's method was later adapted by Amick and Toland [AT92] for a model singularly perturbed KdV-type fourth order equation. More recently, Faver [Fav, Fav18] used Beale's method to study the long wave problem in spring dimer lattices (FPUT lattices with alternating spring forces but constant masses), and Johnson and Wright [JW] adapted it for a singularly perturbed Whitham equation.

Although the equal mass problem is not, ultimately, singularly perturbed, its structure has enough in common with these predecessors that we are able to adapt Beale's method to this situation, too. We discover that the monatomic traveling wave perturbs into a "micropteron" traveling wave as the mass ratio hovers around 1. Boyd uses this term to refer to a nonlocal solitary wave whose ripples are only algebraically small in the relevant small parameter, not small beyond *all* algebraic orders. We do not find such small beyond all orders estimates in our equal mass problem, and so we consciously avoid using the term "nanopteron" for our profile.

We sketch this micropteron wave in Figure 2 and provide in Figure 3 an informal, but evocative, cartoon comparing the three families of nonlocal solitary waves (long wave, small mass, equal mass) that now exist for the diatomic FPUT lattice. We state our main result below in Theorem 1.2.

Beyond these nanopteron problems, wave propagation in diatomic and more generally heterogeneous lattices has received considerable recent attention; we mention, among others, the papers [CBCPS12, BP13, Qin15, VSWP16, SV17, Wat19, Lus] for theoretical and numerical examples of waves in FPUT lattices under different material regimes. See [GMWZ14] for a discussion of long wave KdV approximations in polyatomic FPUT lattices and [GSWW19] for a further discussion of the metastability of these waves in diatomic lattices. The paper [HMSZ13] studies a regular perturbation problem in monatomic FPUT lattices in which the spring force is perturbed from a known piecewise quadratic potential; the resulting solutions are asymptotic, like ours, to a sinusoid whose amplitude is algebraically small. Lattice differential equations abound in contexts beyond the FPUT model that we study here; see, for example, [HMSSVV] for a survey of traveling wave results for the Nagumo lattice equation and related models.

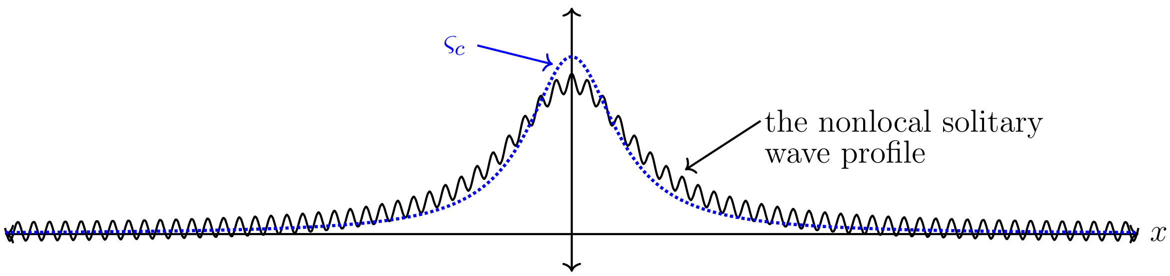

FIGURE 2. One of the micropteron profiles (p_1 or p_2) sketched close to the monatomic solitary wave ς_c. In a nanopteron, the periodic ripple would be so small as to be invisible relative to the monatomic profile; this is not the case in the equal mass limit.

1.3. **The traveling wave problem.** After making the traveling wave ansatz (1.1.3) for the original equations of motion (1.1.2), we write $1/m = 1 + \mu$ for $\mu \in (-1, 1)$ and make the linear change of variables

$$\rho_1 = \frac{p_1 + p_2}{2} \qquad \text{and} \qquad \rho_2 = \frac{p_1 - p_2}{2}$$

to obtain the equivalent system

$$\underbrace{c^2 \boldsymbol{\rho}'' + \mathcal{D}_\mu \boldsymbol{\rho} + \mathcal{D}_\mu \mathcal{Q}(\boldsymbol{\rho}, \boldsymbol{\rho})}_{\mathcal{G}_c(\boldsymbol{\rho}, \mu)} = 0. \tag{1.3.1}$$

The details of this change of variables are discussed in Appendix B. For now, we focus on the definitions and properties of the operators $\mathcal{D}_\mu$ and $\mathcal{Q}$.

First, let S^d be the "shift-by-d" operator defined by $(S^d f)(x) := f(x + d)$ and set

$$A := S^1 + S^{-1}, \qquad \delta := S^1 - S^{-1}.$$

Then we have

$$\mathcal{D}_\mu := \frac{1}{2} \begin{bmatrix} (2+\mu)(2-A) & \mu\delta \\ -\mu\delta & (2+\mu)(2+A) \end{bmatrix}. \tag{1.3.2}$$

Next, we define

$$\mathcal{Q}(\boldsymbol{\rho}, \grave{\boldsymbol{\rho}}) = \begin{pmatrix} \mathcal{Q}_1(\boldsymbol{\rho}, \grave{\boldsymbol{\rho}}) \\ \mathcal{Q}_2(\boldsymbol{\rho}, \grave{\boldsymbol{\rho}}) \end{pmatrix} := \begin{pmatrix} \rho_1 \grave{\rho}_1 + \rho_2 \grave{\rho}_2 \\ \rho_1 \grave{\rho}_2 + \grave{\rho}_1 \rho_2 \end{pmatrix}. \tag{1.3.3}$$

The version (1.3.1) is particularly useful because it preserves a number of symmetries. Namely, $\mathcal{G}_c$ maps

$$\{\text{even functions}\} \times \{\text{odd functions}\} \times \mathbb{R} \to \{\text{even mean-zero functions}\} \times \{\text{odd functions}\}. \tag{1.3.4}$$

We prove these symmetries in Appendix B. We say that a function f is "mean-zero" if $\widehat{f}(0) = 0$; here $\widehat{f}$ is the Fourier transform of f, and our conventions for the Fourier transform are outlined in Appendix A.1.

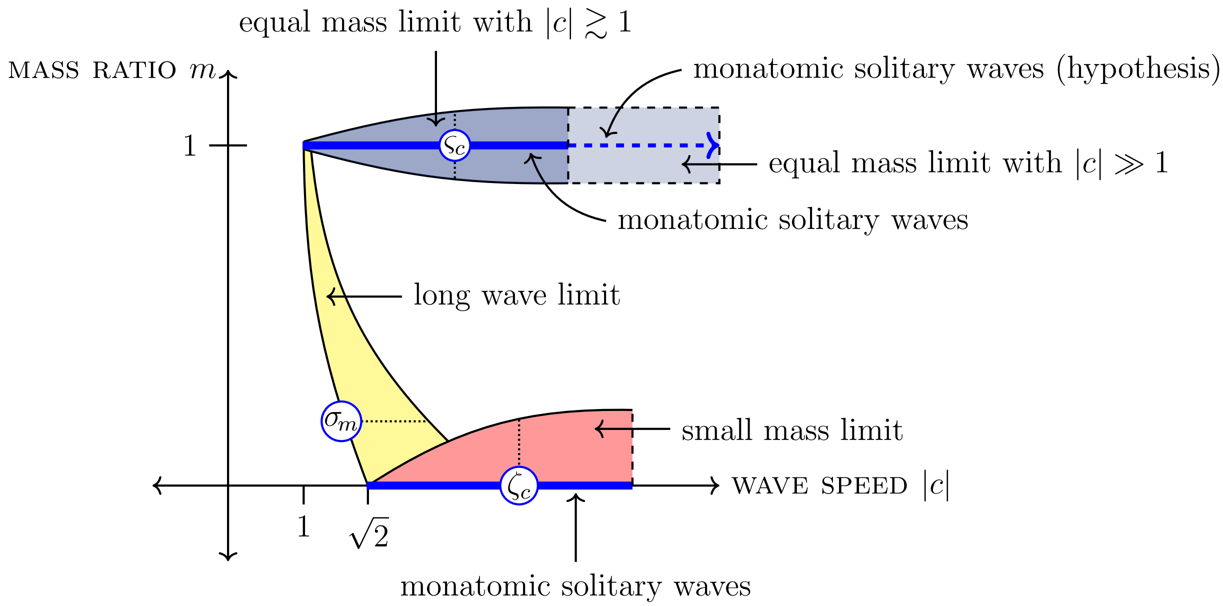

FIGURE 3. A comparison of nonlocal solitary waves across the different diatomic FPUT problems. In this paper, for $|c| \gtrsim 1$ fixed and $m \approx 1$, we find micropterons close to a Friesecke-Pego solitary wave profile ς_c. Under suitable hypotheses on the existence of a monatomic solitary wave for $|c| \gg 1$, we may still find micropterons close to that profile. For $|c| \gtrsim \sqrt{2}$ fixed and $m \approx 0$, Hoffman and Wright found nanopterons close to a (different) Friesecke-Pego solitary wave ζ_c. For $m \in (0, 1)$ fixed and $|c|$ close to an m-dependent threshold called the "speed of sound," Faver and Wright found nanopterons close to a KdV sech^2-type profile σ_m. That the bands collapse as $|c| \to 1^+$ or $|c| \to \sqrt{2}^+$ is intentional; see (the proofs of) Lemma C.7 in [FW18], Lemma 7.1 in [HW17], and Lemma C.1 in this paper. This graphic elides the interesting, but difficult, question of how the different families of waves interact; for example, for $|c| \gtrsim 1$ and $m \approx 1$, how do the long wave nanopterons and equal mass micropterons compare? See Section 7 in [FW18] for a further discussion of the challenges involved in addressing these questions.

When $\mu = 0$, the diatomic lattice reverts to a monatomic lattice, and the traveling wave problem (1.3.1) reduces to

$$\begin{cases} c^2\rho_1'' + (2 - A)(\rho_1 + \rho_1^2 + \rho_2^2) = 0 \\ c^2\rho_2'' + (2 + A)(2\rho_1\rho_2) = 0. \end{cases}$$

If we take $\rho_2 = 0$, then the second equation is satisfied, and the first becomes

$$c^2\rho_1'' + (2 - A)(\rho_1 + \rho_1^2) = 0.$$

This is the equation for the traveling wave profile of a monatomic FPUT lattice. For $|c| \gtrsim 1$, it has an even exponentially decaying (or localized) solution due to Friesecke and Pego [FP99], which we call ς_c. We discuss the properties of ς_c in greater detail in Theorem 6.1.

1.4. **Linearizing at the Friesecke-Pego solution.** If we set $\boldsymbol{\varsigma}_c := (\varsigma_c, 0)$, we see that $\mathcal{G}_c(\boldsymbol{\varsigma}_c, 0) = 0$. Then for μ small, we are interested in solutions $\boldsymbol{\rho}$ to $\mathcal{G}_c(\boldsymbol{\rho}, \mu)$ that are close to the Friesecke-Pego solution $\boldsymbol{\varsigma}_c$. In order to perturb from $\boldsymbol{\varsigma}_c$, we first define Sobolev spaces of exponentially localized functions.

Definition 1.1. *Let*

$$H_q^r := \{f \in H^r(\mathbb{R}) \mid \cosh^q(\cdot) f \in H^r(\mathbb{R})\}, \qquad \|f\|_{H_q^r} = \|f\|_{r,q} := \|\cosh^q(\cdot) f\|_{H^r(\mathbb{R})}. \tag{1.4.1}$$

and

$$E_q^r := H_q^r \cap \{\textit{even functions}\} \qquad \textit{and} \qquad O_q^r := H_q^r \cap \{\textit{odd functions}\}.$$

For a function $\mathbf{f} = (f_1, f_2) \in H_q^r \times H_q^r$, *we set*

$$\|\mathbf{f}\|_{r,q} := \|f_1\|_{r,q} + \|f_2\|_{r,q}.$$

Under this notation, we have $\varsigma_c \in \cap_{r=0}^{\infty} E_q^r$ for q sufficiently small. We set $\boldsymbol{\rho} = \boldsymbol{\varsigma}_c + \boldsymbol{\varrho}$, where $\boldsymbol{\varrho} = (\varrho_1, \varrho_2) \in E_q^2 \times O_q^2$, and compute that $\mathcal{G}_c(\boldsymbol{\varsigma}_c + \boldsymbol{\varrho}, \mu) = 0$ if and only if ϱ_1 and ϱ_2 satisfy the system

$$c^2 \boldsymbol{\varrho}'' + \mathcal{D}_0 \boldsymbol{\varrho} + 2\mathcal{D}_0 \mathcal{Q}(\boldsymbol{\varsigma}_c, \boldsymbol{\varrho}) = \boldsymbol{\mathcal{R}}_c(\boldsymbol{\varrho}, \mu) = \begin{pmatrix} \mathcal{R}_{c,1}(\boldsymbol{\varrho}, \mu) \\ \mathcal{R}_{c,2}(\boldsymbol{\varrho}, \mu) \end{pmatrix}. \tag{1.4.2}$$

The right side $\boldsymbol{\mathcal{R}}_c(\boldsymbol{\varrho}, \mu)$ is "small" in the sense that it consists, roughly, of linear combinations of terms of the form μ, $\mu\boldsymbol{\varrho}$, and $\boldsymbol{\varrho}^{.2}$.

The first component of this system has the form

$$\underbrace{c^2 \varrho_1'' + (2 - A)(1 + 2\varsigma_c)\varrho_1}_{\mathcal{H}_c \varrho_1} = \mathcal{R}_{c,1}(\boldsymbol{\varrho}, \mu). \tag{1.4.3}$$

The operator $\mathcal{H}_c$ is the linearization of the monatomic traveling wave problem at ς_c. Proposition 3.1 from [HW17] tells us that, for q sufficiently small, $\mathcal{H}_c$ is invertible from E_q^{r+2} to $E_{q,0}^r$ for any $r \geq 0$, where

$$E_{q,0}^r := \left\{ f \in E_q^r \;\middle|\; \widehat{f}(0) = 0 \right\}.$$

In some sense, this can be seen as a spectral stability result for the monatomic wave; see also Lemma 4.2 in [FP04a] and Lemma 6 in [HM17] for Fredholm properties of $\mathcal{H}_c$, under different guises, in exponentially weighted Sobolev spaces. It follows that (1.4.3) is equivalent to

$$\varrho_1 = \mathcal{H}_c^{-1} \mathcal{R}_{c,1}(\boldsymbol{\varrho}, \mu), \tag{1.4.4}$$

which is a fixed point equation for ϱ_1 on the function space E_q^r. We now attempt to construct a similar fixed point equation for ϱ_2; our failure in this attempt will be quite instructive.

1.5. **The operator $\mathcal{L}_c$.** The second component of (1.4.2) is

$$\underbrace{c^2 \varrho_2'' + (2 + A)(1 + 2\varsigma_c)\varrho_2}_{\mathcal{L}_c \varrho_2} = \mathcal{R}_{c,2}(\boldsymbol{\varrho}, \mu). \tag{1.5.1}$$

The operator $\mathcal{L}_c$ is the sum of a constant-coefficient second-order advance-delay differential operator and a nonlocal term, which we write more explicitly as

$$\mathcal{L}_c f = \underbrace{c^2 f'' + (2 + A) f}_{\mathcal{B}_c f} + \underbrace{2(2 + A)(\varsigma_c f)}_{-\Sigma_c f}. \tag{1.5.2}$$

The minus sign on Σ_c is purely for convenience. If $\mathcal{L}_c$ were invertible from O_q^{r+2} to O_q^r, then (1.5.1) would rearrange to a fixed point equation for ϱ_2, which we could combine with (1.4.4) to get a fixed point problem for $\boldsymbol{\varrho}$. In this attempt we fail: $\mathcal{L}_c$ is injective but not surjective.

1.5.1. *Injectivity.* We first sketch how, at least in the case $|c| \gtrsim 1$, the operator $\mathcal{L}_c$ is injective from O_q^{r+2} to O_q^r. We begin with the constant-coefficient part $\mathcal{B}_c$. This operator is a Fourier multiplier with symbol

$$\widetilde{\mathcal{B}}_c(k) := -c^2k^2 + 2 + 2\cos(k). \tag{1.5.3}$$

That is, if $f \in L^2$ or $f \in L^2_{\text{per}}$, then

$$\widehat{\mathcal{B}_c f}(k) = \widetilde{\mathcal{B}}_c(k)\widehat{f}(k).$$

See Appendix A.1.1 for further definitions and properties of Fourier multipliers. A straightforward application of the intermediate value theorem yields a unique $\omega_c > 0$ such that $\widetilde{\mathcal{B}}_c(\omega_c) = 0$. This Fourier condition in fact characterizes the range of $\mathcal{B}_c$: using results of Beale (phrased as Lemma A.3 in this paper) and some further properties of $\widetilde{\mathcal{B}}_c$ from Proposition 4.2, one can show that $\mathcal{B}_c$ is invertible from O_q^{r+2} to the subspace

$$\mathfrak{D}^r_{c,q} := \left\{ f \in O_q^r \;\middle|\; \widehat{f}(\omega_c) = 0 \right\}.$$

Next, when $|c| \gtrsim 1$, the Friesecke-Pego solution ς_c is small in the sense that $\|\varsigma_c\|_{L^\infty} = \mathcal{O}((c-1)^2)$; see part (v) of Proposition D.2. Consequently, the operator Σ_c is also small. However, Σ_c does not map O_q^{r+2} to $\mathfrak{D}^r_{q,c}$; otherwise, we could use the Neumann series to invert $\mathcal{B}_c - \Sigma_c$. Nonetheless, one can parley the smallness of Σ_c and the invertibility of $\mathcal{B}_c$ into a coercive estimate of the form

$$\mathcal{L}_c f = g,\ f \in O_q^2,\ g \in O_q^0 \Longrightarrow \|f\|_{2,q} \le C(c,q)\|g\|_{0,q}, \tag{1.5.4}$$

which implies that $\mathcal{L}_c$ is injective on O_q^2 and, by the containment $O_q^{r+2} \subseteq O_q^2$ for $r \ge 0$, on O_q^r for all $r \ge 0$.

1.5.2. *A characterization of the range of $\mathcal{L}_c$.* We show, abstractly, that $\mathcal{L}_c$ is not surjective from O_q^{r+2} to O_q^r. Since the operator Σ_c localizes functions, it is compact from O_q^{r+2} to O_q^r, and so the Fredholm index of $\mathcal{L}_c = \mathcal{B}_c + \Sigma_c$ equals the index of $\mathcal{B}_c$ The Fourier analysis in Section 1.5.1 shows that the index of $\mathcal{B}_c$ is -1, so $\mathcal{L}_c$ also has index -1. Since $\mathcal{L}_c$ is injective, we conclude that $\mathcal{L}_c$ has a one-dimensional cokernel in O_q^r and thus is not surjective.

However, we can characterize the range of $\mathcal{L}_c$ in O_q^r more precisely. Classical functional analysis tells us there is a nontrivial bounded linear functional $\mathfrak{z}_c$ on O_q^0 such that

$$\mathcal{L}_c f = g,\ f \in O_q^{r+2},\ g \in O_q^r \iff \mathfrak{z}_c[g] = 0. \tag{1.5.5}$$

Let

$$\mathcal{Z}_q^\star := \left\{ f \in L^1_{\text{loc}} \;\middle|\; \operatorname{sech}^q(\cdot) f \in L^2 \right\} \cap \{\text{odd functions}\}.$$

The Riesz representation theorem then furnishes an nonzero function $\widetilde{\mathfrak{z}}_c \in \mathcal{Z}_q^\star$ such that

$$\mathfrak{z}_c[g] = \int_{-\infty}^{\infty} g(x)\widetilde{\mathfrak{z}}_c(x)\,dx,\ g \in O_q^0.$$

It follows that $\mathcal{L}_c^* \widetilde{\mathfrak{z}}_c = 0$, where

$$\mathcal{L}_c^* g := \mathcal{B}_c g + \underbrace{2\varsigma_c(x)(2+A)g}_{-\Sigma_c^* g} \tag{1.5.6}$$

is the L^2-adjoint of $\mathcal{L}_c$. We conclude from (1.5.5) the "solvability condition"

$$\mathcal{L}_c f = g, \ f \in O_q^{r+2}, \ g \in O_q^r \iff \int_{-\infty}^{\infty} g(x)\widetilde{\mathfrak{z}}_c(x)\ dx = 0. \tag{1.5.7}$$

It appears, then, that our attempt to solve the traveling wave problem $\mathcal{G}_c(\boldsymbol{\rho}, \mu) = 0$ by perturbing from the Friesecke-Pego solution $\boldsymbol{\varsigma}_c$ will fail, since $\mathcal{L}_c$ is not surjective. Moreover, although we can characterize the range of $\mathcal{L}_c$ precisely via (1.5.7), all we know about $\widetilde{\mathfrak{z}}_c$ is that $\mathcal{L}_c^* \widetilde{\mathfrak{z}}_c = 0$ and $\widetilde{\mathfrak{z}}_c \in \mathcal{Z}_q^\star$. The kernel of $\mathcal{L}_c^*$ in $\mathcal{Z}_q^\star$ must be one-dimensional, as otherwise, $\mathcal{L}_c^*$ would have Fredholm index -2 or lower, and so $\widetilde{\mathfrak{z}}_c$ is unique up to scalar multiplication. But this function space $\mathcal{Z}_q^\star$ is quite large — it contains, for example, all odd functions in L^2 and L^∞ — and so further features of $\widetilde{\mathfrak{z}}_c$ are not immediately apparent.

To determine our next steps, we look back to the work of our predecessors in the long wave [FW18] and small mass [HW17] problems. In each of these problems, an operator similar to $\mathcal{L}_c$ appears; each of these operators on O_q^{r+2} has a one-dimensional cokernel in O_q^r because of a solvability condition like (1.5.7). Moreover, the authors were able to construct odd solutions in $W^{2,\infty}$ to their versions of $\mathcal{L}_c^* g = 0$ that asymptote to a sinusoid. We refer to such solutions as "Jost solutions," due to their similarity to the classical Jost solutions for the Schrodinger equation [RS79]. Seeing how the Jost solutions in the long wave and small mass problems are determined both guides us to the features of the Jost solution that we seek for $\mathcal{L}_c^*$ and help us appreciate what is intrinsically different about $\mathcal{L}_c^*$ when compared to its analogues in the prior nanopteron problems.

1.5.3. *Jost solutions in the long wave limit* [FW18]. The analogue of $\mathcal{L}_c$ in this problem is, roughly, the operator

$$\mathcal{W}_\varepsilon f := (1+\varepsilon^2)\varepsilon^2 f'' + \mathcal{M}_\varepsilon f,$$

where $\varepsilon \approx 0$ and $\mathcal{M}_\varepsilon$ is a Fourier multiplier with the real-valued symbol $\widetilde{\mathcal{M}}(\varepsilon \cdot)$, i.e., $\widehat{\mathcal{M}_\varepsilon f}(k) = \widetilde{\mathcal{M}}(\varepsilon k)\widehat{f}(k)$. The wave speed is intrinsically linked to the small long wave parameter ε, and so c does not appear here. Since $\mathcal{W}_\varepsilon$ is a Fourier multiplier with real-valued symbol $\widetilde{\mathcal{W}}_\varepsilon(k) := -(1+\varepsilon^2)\varepsilon^2 k^2 + \widetilde{\mathcal{M}}(\varepsilon k)$, it is self-adjoint in L^2. An intermediate value theorem argument similar to the one referenced in Section 1.5.1 gives the existence of a unique $\Omega_\varepsilon > 0$ such that $\widetilde{\mathcal{W}}_\varepsilon(K) = 0$ if and only if $K = \pm\Omega_\varepsilon$, and so $\mathcal{W}_\varepsilon \sin(\Omega_\varepsilon \cdot) = \mathcal{W}_\varepsilon \cos(\Omega_\varepsilon \cdot) = 0$. That is, the Jost solutions are exactly sinusoidal.

1.5.4. *Jost solutions in the small mass limit* [HW17]. Here the analogue of $\mathcal{L}_c$ is, roughly,

$$\mathcal{T}_c(m) f := \underbrace{c^2 m f'' + (1 + 2\zeta_c(x))f}_{\mathcal{S}_c(m)f} + m\mathcal{M}_m f + m\mathcal{J}_m(\zeta_c(x) f).$$

We take $m \approx 0$, where m is the mass ratio of the diatomic lattice, per (1.1.1). The operators $\mathcal{M}_m$ and $\mathcal{J}_m$ are Fourier multipliers that are $\mathcal{O}(1)$ in m, so the perturbation terms $m\mathcal{M}_m$ and $m\mathcal{J}_m$ are indeed small, but Hoffman and Wright found it essential to retain these terms rather than absorb them into their analogues of our right side $\mathcal{R}_{c,2}$ from (1.5.1). Here the exponentially localized function ζ_c is the Friesecke-Pego solitary wave profile corresponding to the monatomic lattice formed by taking $m = 0$ in (1.1.1).

The L^2-adjoint of $\mathcal{T}_c(m)$ is

$$\mathcal{T}_c(m)^* g = \mathcal{S}_c(m)g + m\mathcal{M}_m g + m\varsigma_c(x)\mathcal{J}_m g.$$

Hoffman and Wright construct a nontrivial solution to $\mathcal{T}_c(m)^* g = 0$ as follows. First, they show that $\mathcal{S}_c(m)$ vanishes on certain asymptotically sinusoidal functions, i.e., there exists $j_c^m \in W^{2,\infty}$ such that

$$\mathcal{S}_c(m) j_c^m = 0 \qquad \text{and} \qquad \lim_{x\to\infty} |j_c^m(x) - \sin(\Omega_c^m(x + \theta_c^m))| = 0 \tag{1.5.8}$$

for some critical frequency Ω_c^m and phase shift θ_c^m. This solution j_c^m is indeed a classical Jost solution for the Schrodinger operator $\mathcal{S}_c(m)$. The proof of the asymptotics in (1.5.8) uses a polar coordinate decomposition that closely relies on the structure of $\mathcal{S}_c(m)$ as a second-order linear differential operator.

Second, it is clear that $\mathcal{T}_c(m)^*$ is a small nonlocal perturbation of $\mathcal{S}_c(m)$. Using these facts, Hoffman and Wright perform an intricate variation of parameters argument (which again relies on the differential operator structure of $\mathcal{S}_c(m)$) in an asymptotically sinusoidal subspace of $W^{2,\infty}$ to construct a function $\gamma_c^m \in W^{2,\infty}$ with the properties that

$$\mathcal{T}_c(m)^*\gamma_c^m = 0 \qquad \text{and} \qquad \lim_{x\to\infty} |\gamma_c^m(x) - \sin(\Omega_c^m(x + \vartheta_c^m))| = 0$$

for some (new) phase shift ϑ_c^m, which is a small perturbation of θ_c^m. A corollary to this analysis is the frequency-phase shift "resonance" relation $\sin(\Omega_c^m\vartheta_c^m) \neq 0$ for almost all values of m close to zero, which turns out, in a subtle and surprising way, to be critical for their subsequent analysis.

1.5.5. *Toward Jost solutions for the equal mass operator $\mathcal{L}_c^*$.* Unlike the long wave operator $\mathcal{W}_\varepsilon$, the operator $\mathcal{L}_c^*$ is not a "pure" Fourier multiplier as it has the variable-coefficient piece Σ_c^*. And unlike the small mass operator $\mathcal{T}_c(m)^*$, we cannot decompose $\mathcal{L}_c^*$ as the sum of a classical differential operator and a perturbation term that is small in μ. So, we cannot directly import prior results to produce the Jost solutions of $\mathcal{L}_c^*$.

Our approach is to take advantage of two particular aspects of the structure of $\mathcal{L}_c^*$. First, the constant-coefficient part $\mathcal{B}_c$ is an advance-delay operator formed by a simple linear combination of shift operators. The Fredholm properties of such operators have received significant attention from Mallet-Paret [MP99]. Next, the variable-coefficient piece Σ_c^* is both exponentially localized and small for $|c| \gtrsim 1$. These facts are sufficient to solve the equation $(\mathcal{B}_c - \Sigma_c^*)f = 0$ in a class of "one-sided" exponentially weighted Sobolev spaces, whose features we specify below in Definition 2.2.

In broad strokes, then, we first use an adaptation of Mallet-Paret's theory due to Hupkes and Verduyn-Lunel [HL07] to invert $\mathcal{B}_c$ on these one-sided spaces and, moreover, obtain a precise formula for its inverse. Next, it turns out that Σ_c^* does map between these one-sided spaces[3] and so, since Σ_c^* is still "small," we are able to invert $\mathcal{B}_c - \Sigma_c^*$ with the Neumann series. This procedure yields a function $\gamma_c \in W^{2,\infty}$ that satisfies $\mathcal{L}_c^*\gamma_c = 0$ and is asymptotic to a phase-shifted sinusoid of frequency ω_c at ∞, where ω_c is the "critical frequency" of $\mathcal{B}_c$ that appeared in Section 1.5.1. The exact formulas that we enjoy for the inverses of $\mathcal{B}_c$ and then $\mathcal{B}_c - \Sigma_c^*$ permit us to calculate an asymptotic expansion for the phase shift.

[3] Unlike its failure to map between O_q^{r+2} and $\mathfrak{D}_{c,q}^r$, as we saw in Section 1.5.1).

1.6. **Main result for the case $|c| \gtrsim 1$.** Now that we understand precisely the range of $\mathcal{L}_c$, as well as its lack of surjectivity, we can confront again the traveling wave problem $\mathcal{G}_c(\boldsymbol{\rho}, \mu) = 0$ from (1.3.1), which we seek to solve for $\boldsymbol{\rho} \approx \boldsymbol{\varsigma}_c$ and $\mu \approx 0$. The general structure of this problem, most especially the solvability condition (1.5.7), puts us in enough concert with our predecessors in [FW18] and [HW17] that we may follow their modifications of a method due to Beale [Bea91] for solving problems with such a solvability condition. Specifically, we replace the perturbation ansatz $\boldsymbol{\rho} = \boldsymbol{\varsigma}_c + \boldsymbol{\varrho}$ with the *nonlocal solitary wave* anzatz $\boldsymbol{\rho} = \boldsymbol{\varsigma}_c + a\boldsymbol{\phi}_c^{\mu}[a] + \boldsymbol{\eta}$, where $\boldsymbol{\eta} = (\eta_1, \eta_2)$ is exponentially localized and $a\boldsymbol{\phi}_c^{\mu}[a]$ is a periodic solution to the traveling wave problem with amplitude roughly a and frequency (very) roughly ω_c. Of course, proving the existence of periodic traveling wave solutions is a fundamental part of our analysis.

Under Beale's ansatz, one easily finds a fixed point equation for η_1, similar to how we converted (1.4.3) into (1.4.4). Then we can extract from the solvability condition a fixed point equation for a, and, in turn, an equation for η_2. We carry out this construction in Section 5, where we prove (as Theorem 5.4) a more technical version of our main theorem below.

Theorem 1.2. *Suppose $|c| \gtrsim 1$. For $|\mu|$ sufficiently small, there are functions $\Upsilon_{c,1}^{\mu}$, $\Upsilon_{c,2}^{\mu}$, $\varphi_{c,1}^{\mu}$, $\varphi_{c,2}^{\mu} \in \mathcal{C}^{\infty}(\mathbb{R})$ such that the traveling wave profiles*

$$\rho_{c,1}^{\mu} := \varsigma_c + \Upsilon_{c,1}^{\mu} + \varphi_{c,1}^{\mu} \qquad \text{and} \qquad \rho_{c,2}^{\mu} := \Upsilon_{c,2}^{\mu} + \varphi_{c,2}^{\mu}$$

satisfy $\mathcal{G}_c((\rho_{c,1}^{\mu}, \rho_{c,2}^{\mu}), \mu) = 0$. The functions $\Upsilon_{c,1}^{\mu}$ and $\Upsilon_{c,2}^{\mu}$ are exponentially localized, while $\varphi_{c,1}^{\mu}$ and $\varphi_{c,2}^{\mu}$ are periodic. The amplitude of all four functions is $\mathcal{O}(\mu)$.

When $\mu = 0$ and our lattice is monatomic, the micropteron in Theorem 1.2 reduces to the Friesecke-Pego solitary wave. We remark that the stability of these micropteron traveling wave solutions for the equal mass limit, as well as the stability of the long wave and small mass nanopterons, is an intriguing open problem; see [JW] for some initial forays into this arena.

1.7. **Toward the case $|c| \gg 1$.** Throughout this introduction, we have assumed that $|c|$ is close to 1. This first allows us to summon a Friesecke-Pego solitary wave solution ς_c for the monatomic problem and next shows, through the coercive estimate (1.5.4), that $\mathcal{L}_c$ is injective from O_q^{r+2} to O_q^r. Third, taking $|c|$ close to 1 is fundamental for the Neumann series argument that allows us to invert $\mathcal{B}_c - \Sigma_c^*$ in the one-sided spaces. Ostensibly, then, it appears that we are working with two small parameters, μ and $|c| - 1$, in our equal mass problem.

This turns out, from a certain point of view, to be superfluous. In Section 2, we present four simple and natural hypotheses that, if satisfied for an arbitrary wave speed $|c| > 1$ guarantee a micropteron solution in the equal mass limit. All of these hypotheses are satisfied for $|c| \gtrsim 1$. Among these hypotheses is the existence of an exponentially localized and spectrally stable traveling wave solution for the monatomic problem with wave speed possibly much greater 1; this is the origin of the dashed blue line in Figure 3. We emphasize that the existence of the Jost solutions to $\mathcal{L}_c^* g = 0$ is not one of these hypotheses; their construction, for an arbitrary $|c| > 1$, follows from a weaker hypothesis.

This approach has several advantages and justifications over working only in the near-sonic regime $|c| \gtrsim 1$. First, it decouples the starring small parameter μ from the deuteragonist[4] $|c|-1$; after all, we are interested in the equal mass limit, not the near-sonic limit. Next, it frees us from overreliance on the Friesecke-Pego traveling wave and leaves our results open to interpretation and invocation in the case of high-speed monatomic traveling waves. For example, Herrmann and Matthies [HM15, HM17, HM19] have developed a number of results on the asymptotics, uniqueness, and stability of solitary traveling wave solutions to the monatomic FPUT problem (albeit with different spring forces from ours) in the "high-energy limit," which inherently assumes a large wave speed. Additionally, the original monatomic solitary traveling wave of Friesecke and Wattis [FW94] does not come with a near-sonic restriction on its speed, and so, in principle, that wave could have speed much greater than 1. Further study of the existence and properties of solutions to the monatomic traveling wave problem in the high wave speed regime remains an interesting open problem, and we are eager to see how our hypotheses may exist in concert with future solutions to that problem.

1.8. **Remarks on notation.** We define some basic notation that will be used throughout the paper.

• The letter C will always denote a positive, finite constant; if C depends on some other quantities, say, q and r, we will write this dependence in function notation, i.e., $C = C(q,r)$. Frequently C will depend on the wave speed c, in which case we write $C(c)$.

• We employ the usual big-$\mathcal{O}$ notation: if $x_\epsilon \in \mathbb{C}$ for $\epsilon \in \mathbb{R}$, then we write $x_\epsilon = \mathcal{O}(\epsilon^p)$ if there are constants ϵ_0, $C(\epsilon_0,p) > 0$ such that

$$0 < \epsilon < \epsilon_0 \Longrightarrow |x_\epsilon| \le C(\epsilon_0,p)\epsilon^p.$$

• Next, we need a c-dependent notion of big-$\mathcal{O}$ notation. Suppose that $x_c^\mu \in \mathbb{C}$ for c, $\mu \in \mathbb{R}$. Then we write $x_c^\mu = \mathcal{O}_c(\mu^p)$ if there are constants μ_c, $C(c,p) > 0$ such that

$$|\mu| \le \mu_c \Longrightarrow |x_c^\mu| \le C(c,p)|\mu|^p.$$

• If $\mathcal{X}$ and $\mathcal{Y}$ are normed spaces, then $\mathbf{B}(\mathcal{X},\mathcal{Y})$ is the space of bounded linear operators from $\mathcal{X}$ to $\mathcal{Y}$ and $\mathbf{B}(\mathcal{X}) := \mathbf{B}(\mathcal{X},\mathcal{X})$.

1.9. **Outline of the remainder of the paper.** Here we briefly discuss the structure of the rest of the paper.

• Section 2 contains the precise statements of the four hypotheses under which we will work.

• Section 3 constructs a family of small-amplitude periodic traveling wave solutions to the traveling wave problem (1.3.1) that possess a number of uniform estimates in the small parameter μ. These periodic traveling waves exist for arbitrary $|c| > 1$.

• Section 4 characterizes the range of $\mathcal{L}_c$ as an operator from O_q^{r+2} to O_q^r by constructing a Jost solution for $\mathcal{L}_c^* g = 0$.

[4]In fact, a third small parameter also lurks in nonlocal solitary wave problems: the precise decay rate q of the exponentially localized spaces E_q^r and O_q^r. Our hypotheses make explicit the decay rates that we employ and highlight the relationships between the rates at different stages of the problem.

• Section 5 converts the nonlocal solitary wave equations of Section 5.1 into a fixed point problem, which we then solve with a modified contraction mapping argument.

• Section 6 shows that the four main hypotheses of Section 2 are valid in the case $|c| \gtrsim 1$.

• The appendices contain various technical proofs and ancillary background material.

2. The Traveling Wave Problem for Arbitrary Wave Speeds

In this section we discuss the four hypotheses that are sufficient to guarantee micropteron traveling wave solutions in the equal mass limit for arbitrary wave speeds. Throughout, we fix $c \in \mathbb{R}$ with $|c| > 1$. Since $|c|$ may not be close to 1, we are not guaranteed a monatomic traveling wave solution from Friesecke and Pego, and so we make its existence our first hypothesis.

Hypothesis 1. *There exist $q_\varsigma(c) > 0$ and a real-valued function $\varsigma_c \in E^2_{q_\varsigma(c)}$ such that*

$$c^2\varsigma_c'' + (2 - A)(\varsigma_c + \varsigma_c^2) = 0. \tag{2.0.1}$$

That is, ς_c is an exponentially localized traveling wave profile with wave speed c for the monatomic FPUT equations of motion. With ς_c satisfying (2.0.1), it follows that if $\boldsymbol{\varsigma}_c := (\varsigma_c, 0)$, then $\mathcal{G}_c(\boldsymbol{\varsigma}_c, 0) = 0$. A straightforward bootstrapping argument, discussed in Appendix F.1, endows arbitrary regularity to ς_c.

Proposition 2.1. *The monatomic traveling wave solution ς_c from Hypothesis 1 also satisfies $\varsigma_c \in \cap_{r=1}^\infty E^r_{q_\varsigma(c)}$.*

Next, we add the invertibility of the linearization of the monatomic traveling wave problem at ς_c as a hypothesis for the situation when $|c|$ is not necessarily close to 1.

Hypothesis 2. *There exists $q_{\mathcal{H}}(c) \in (0, \min\{1, q_\varsigma(c)\})$ such that the operator $\mathcal{H}_c$ from* (1.4.3) *is invertible from $E^2_{q_{\mathcal{H}}(c)}$ to $E^0_{q_{\mathcal{H}}(c),0}$.*

In Section 1.5.5, we claimed that the key to characterizing the range of $\mathcal{L}_c$, defined in (1.5.1), was the invertibility of its L^2-adjoint, $\mathcal{L}_c^*$, defined in (1.5.6), on a class of one-sided exponentially weighted spaces. Now we make precise the definitions of these spaces and the weights for which $\mathcal{L}_c^*$ needs to be invertible.

Definition 2.2. *For $q \in \mathbb{R}$ and m, $r \in \mathbb{N}$, we define*

$$W^{r,\infty}_q(\mathbb{R}, \mathbb{C}^m) := \left\{ f \in L^1_{\text{loc}}(\mathbb{R}, \mathbb{C}^m) \;\middle|\; e^{-q\cdot}\mathbf{f} \in W^{r,\infty}(\mathbb{R}, \mathbb{C}^m) \right\}$$

and

$$L^\infty_q(\mathbb{R}, \mathbb{C}^m) := W^{0,\infty}_q(\mathbb{R}, \mathbb{C}^m).$$

When $m = 1$, we write just $W^{r,\infty}_q$ and L^∞_q. We set

$$\|\mathbf{f}\|_{W^{r,\infty}_q(\mathbb{R},\mathbb{C}^m)} := \|e^{-q\cdot}\mathbf{f}\|_{W^{r,\infty}(\mathbb{R},\mathbb{C}^m)}.$$

Hypothesis 3. *There exists $q_{\mathcal{L}}(c) \in (0, \min\{q_{\mathcal{H}}(c), q_\varsigma(c)/2, 1\})$ such that the operator $\mathcal{L}_c^*$ is invertible from $W^{2,\infty}_{-q_{\mathcal{L}}(c)}$ to $L^\infty_{-q_{\mathcal{L}}(c)}$.*

Assuming this hypothesis we obtain a precise characterization of the range of $\mathcal{L}_c$. We prove this theorem in Section 4.

Proposition 2.3. *There is a nonzero odd function $\gamma_c \in W^{2,\infty}$ (a "Jost solution" to $\mathcal{L}_c^* g = 0$) such that for $f \in O_q^{r+2}$ and $g \in O_q^r$, with $r \geq 0$ and $q \in [q_{\mathcal{L}}(c), 1)$, we have*

$$\mathcal{L}_c f = g \iff \int_{-\infty}^{\infty} g(x)\gamma_c(x)\ dx = 0.$$

Moreover, γ_c is asymptotically sinusoidal in the sense that, for some $\theta_c \in \mathbb{R}$,

$$\lim_{x\to\infty} |\gamma_c(x) - \sin(\omega_c(x+\vartheta_c))| = \lim_{x\to\infty} |(\gamma_c)'(x) - \omega_c \cos(\omega_c(x+\vartheta_c))| = 0.$$

Last, like Hoffman and Wright, we need one more condition on the interaction between the critical frequency ω_c from Section 1.5.1 and the phase shift ϑ_c of the Jost solution to $\mathcal{L}_c^* f = 0$ from Theorem 2.3. This condition arises in practice much later for quite a technical reason; see Section 5.3.

Hypothesis 4. $\sin(\omega_c\vartheta_c) \neq 0$.

All four hypothesis hold when $|c| \gtrsim 1$; we give the proof of the next theorem in Section 6. The verification of Hypotheses 1 and 2 is merely a matter of quoting results from [FP99] and [HW17], On the other hand, verifying Hypotheses 3 and 4 relies on results from [HL07], and the proof of their validity is among the central technical results of this paper.

Theorem 2.4. *There exists $c_\star > 1$ such that Hypotheses 1, 2, 3, and 4 hold if $|c| \in (1, c_\star)$.*

Under these hypotheses we retain the existence of micropterons in the equal mass limit. We prove the next theorem, our central result, as Theorem 5.4.

Theorem 2.5. *Suppose that Hypotheses 1, 2, 3, and 4 hold for some $|c| > 1$. Then the results of Theorem 1.2 remain true.*

3. Periodic Solutions

In this section we state our existence result for small-amplitude periodic solutions to the traveling wave problem $\mathcal{G}_c(\boldsymbol{\rho}, \mu) = 0$ from (1.3.1). We emphasize that the results in this section hold for all $|c| > 1$; we do not need to invoke any of the c-dependent hypotheses here.

We work in the periodic Sobolev spaces

$$H^r_{\text{per}}(\mathbb{R}, \mathbb{C}^m) := \left\{ \mathbf{f} \in L^2_{\text{per}}(\mathbb{R}, \mathbb{C}^m) \;\middle|\; \|\mathbf{f}\|_{H^r_{\text{per}}(\mathbb{R},\mathbb{C}^m)} < \infty \right\}, \tag{3.0.1}$$

where

$$\|\mathbf{f}\|_{H^r_{\text{per}}(\mathbb{R},\mathbb{C}^m)} := \left(\sum_{k=-\infty}^{\infty} (1+k^2)^r |\widehat{\mathbf{f}}(k)|^2 \right)^{1/2}.$$

These are Hilbert spaces with the inner product

$$\langle \mathbf{f}, \mathbf{g} \rangle_{H^r_{\text{per}}(\mathbb{R},\mathbb{C}^m} = \sum_{k=-\infty}^{\infty} (1+k^2)^r (\widehat{\mathbf{f}}(k) \cdot \widehat{\mathbf{g}}(k)). \tag{3.0.2}$$

When $m = 1$, we write just H^r_{per}. We will continue to exploit the symmetries of our traveling wave problem and work on the subspaces

$$E^r_{\text{per}} := \left\{ f \in H^r_{\text{per}} \;\middle|\; f \text{ is even} \right\},$$

$$E^r_{\text{per},0} := \left\{ f \in E^r_{\text{per}} \;\middle|\; \widehat{f}(0) = 0 \right\},$$

and

$$O_{\mathrm{per}}^r := \left\{ f \in H_{\mathrm{per}}^r \;\middle|\; f \text{ is odd} \right\}.$$

Set $\boldsymbol{\rho}(x) = \boldsymbol{\phi}(\omega x)$, where $\boldsymbol{\phi} \in E_{\mathrm{per},0}^2 \times O_{\mathrm{per}}^2 \subseteq H_{\mathrm{per}}^r(\mathbb{R}, \mathbb{C}^2)$ and $\omega \in \mathbb{R}$. Under this scaling, the problem $\mathcal{G}_c(\boldsymbol{\rho}, \mu) = 0$ becomes

$$\underbrace{c^2\omega^2\boldsymbol{\phi}'' + \mathcal{D}_\mu[\omega]\boldsymbol{\phi} + \mathcal{D}_\mu[\omega]\mathcal{Q}(\boldsymbol{\phi}, \boldsymbol{\phi})}_{\boldsymbol{\Phi}_c^\mu(\boldsymbol{\phi}, \omega)} = 0, \tag{3.0.3}$$

where

$$\mathcal{D}_\mu[\omega] := \frac{1}{2}\begin{bmatrix} (2+\mu)(1 - A[\omega]) & \mu\delta[\omega] \\ -\mu\delta[\omega] & (2+\mu)(1 + A[\omega]) \end{bmatrix},$$

with

$$A[\omega] := S^\omega + S^{-\omega} \qquad \text{and} \qquad \delta[\omega] := S^\omega - S^{-\omega}.$$

As in the proofs of periodic solutions for the long wave problems (Theorems 4.1 in [FW18] and 3.1 in [Fav]) and the small mass problem (Theorem 5.1 in [HW17]), we first look for solutions to the linear problem, which is

$$\underbrace{c^2\omega^2\boldsymbol{\phi}'' + \mathcal{D}_\mu[\omega]\boldsymbol{\phi}}_{\Gamma_c^\mu[\omega]\boldsymbol{\phi}} = 0. \tag{3.0.4}$$

Observe that if $\Gamma_c^\mu[\omega]\boldsymbol{\phi} = 0$, then taking the Fourier transform yields

$$c^2\omega^2k^2\widehat{\boldsymbol{\phi}}(k) = \widetilde{\mathcal{D}}_\mu[\omega k]\widehat{\boldsymbol{\phi}}(k), \tag{3.0.5}$$

where

$$\widetilde{\mathcal{D}}_\mu[K] := \begin{bmatrix} (2+\mu)(1 - \cos(K)) & i\mu\sin(K) \\ -i\mu\sin(K) & (2+\mu)(1+\cos(K)) \end{bmatrix}. \tag{3.0.6}$$

Thus $\Gamma_c^\mu[\omega]\boldsymbol{\phi} = 0$ with $\boldsymbol{\phi}$ nonzero if and only if, for some $k \in \mathbb{Z}$, $\widehat{\phi}(k) \neq 0$ is an eigenvector of the matrix $\widetilde{\mathcal{D}}_\mu[\omega k] \in \mathbb{C}^{2\times 2}$ corresponding to the eigenvalue $c^2\omega^2k^2$.

We compute that the eigenvalues of $\widetilde{\mathcal{D}}_\mu[K]$ are $\lambda_\mu^\pm(K)$, where

$$\lambda_\mu^\pm(K) := 2 + \mu \pm \sqrt{\mu^2 + 4(1+\mu)\cos^2(K)}. \tag{3.0.7}$$

These are the same eigenvalues studied in [FW18], [HW17], and [Fav]. See Figure 4 for a sketch of the curves $\lambda_\mu^\pm(K)$ against c^2K^2.

So, $\Gamma_c^\mu[\omega]\boldsymbol{\phi} = 0$ with $\widehat{\boldsymbol{\phi}}(k) \neq 0$ if and only if

$$c^2\omega^2k^2 = \lambda_\mu^-(\omega k) \qquad \text{or} \qquad c^2\omega^2k^2 = \lambda_\mu^+(\omega k)$$

We will prove below in Proposition C.2 that if $|c| > 1$, then the first equality above can never hold, at least over an appropriate c-dependent range of μ, while the second holds precisely at $\omega = \omega_c^\mu/k$, $k \in \mathbb{Z} \setminus \{0\}$, for a certain "critical frequency" $\omega_c^\mu > 0$, which is an $\mathcal{O}_c(\mu)$ perturbation of the frequency ω_c from Section 1.5.1. We elect to take $\omega = \omega_c^\mu$ (i.e., $k = \pm 1$) so that the frequency of our periodic solutions is close to ω_c; this is important in our construction of the micropterons in Section 5.

We now highlight several important properties of ω_c^μ, both contained in and proved as part of Lemma C.1.

Proposition 3.1. **(i)** *For all $|c| > 1$, there is $\mu_\omega(c) > 0$ such that if $|\mu| \leq \mu_\omega(c)$, then there is a unique number $\omega_c^\mu > 0$ such that $c^2(\omega_c^\mu)^2 = \lambda_\mu^+(\omega_c^\mu)$.*

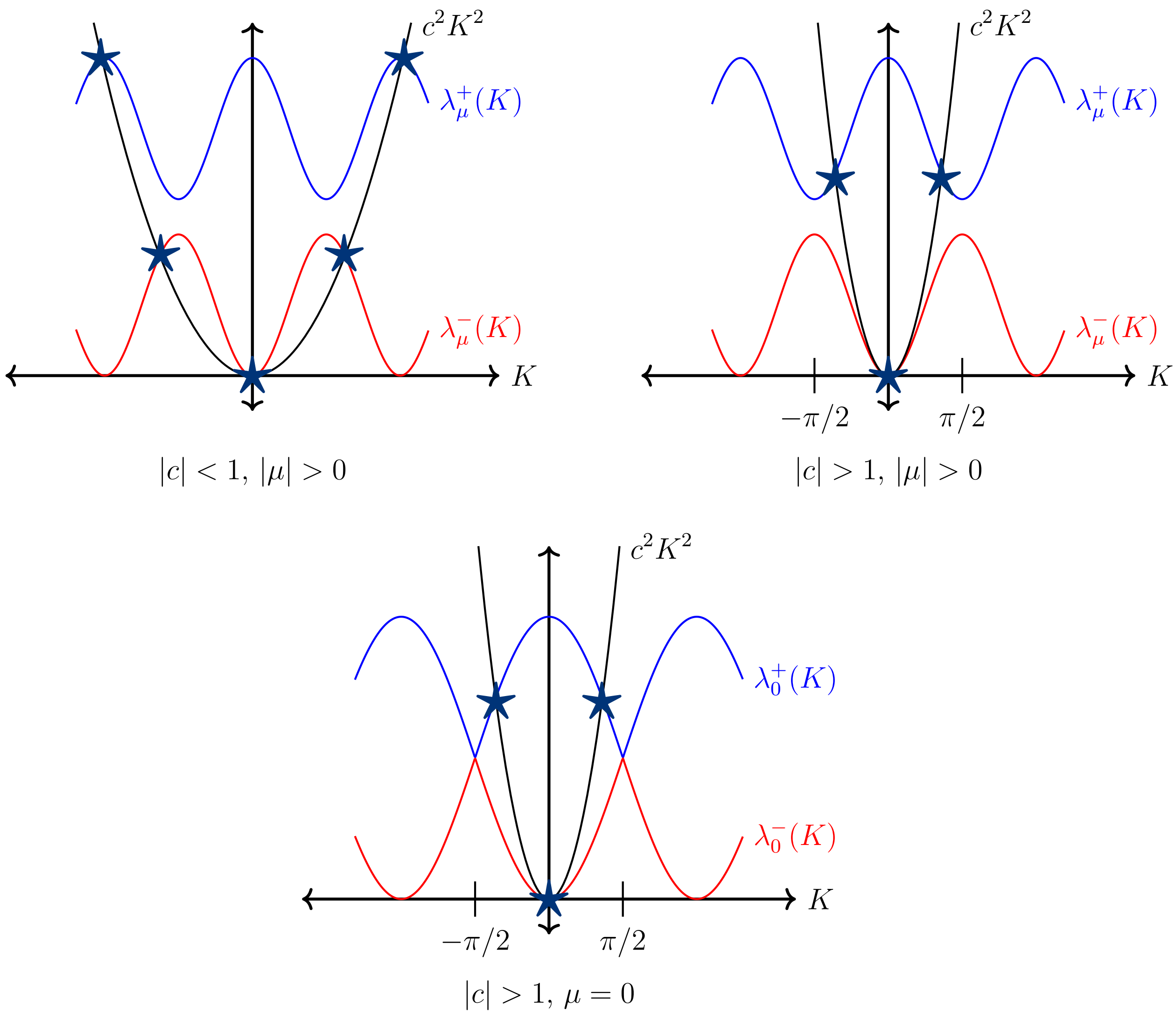

FIGURE 4. Sketches of the graphs of the eigencurves $\lambda_\mu^\pm(K)$ and the parabola c^2K^2 for different cases on $|c|$ and $|\mu|$. When $|c| < 1$, $\lambda_\mu^\pm(K)$ may have more intersections with c^2K^2 than just $K = \pm\omega_c^\mu$. Note that when $\mu = 0$, the curves $\lambda_0^\pm$ intersect at odd multiples of $\pi/2$, but this does not affect the eigenvalue analysis, since the critical frequencies ω_c^μ are contained in $(-\pi/2, \pi/2)$.

(ii) *There are numbers* $0 < A_c < B_c < \pi/2$ *such that* $A_c < \omega_c^\mu < B_c$ *for all* $|\mu| \le \mu_\omega(c)$.

(iii) *If* $|c| \in (1, \sqrt{2}]$, *then* $A_c \ge 1$.

(iv) $\omega_c^\mu - \omega_c = \mathcal{O}_c(\mu)$.

Remark 3.2. *The restriction* $|c| \in (1, \sqrt{2}]$ *will appear in several technical estimates throughout the rest of the paper. This is merely to ensure the convenient lower bound* $\omega_c^\mu > 1$, *which will be useful when we verify the four hypotheses for* $|c| \gtrsim 1$.

Following the methods of our predecessors, we will use this critical frequency ω_c^μ in a modified Crandall-Rabinowitz-Zeidler bifurcation from a simple eigenvalue argument [CR71, Zei95] to construct the exact periodic solutions to the full problem $\mathcal{G}_c(\boldsymbol{\rho}, \mu) = 0$. Here is our result, proved in Appendix C.

Proposition 3.3. *For each $|c| > 1$, there are $\mu_{\text{per}}(c) \in (0, \min\{\mu_\omega(c), 1\})$ and $a_{\text{per}}(c) > 0$ such that for all $|\mu| \le \mu_{\text{per}}(c)$, there are maps*

$$\begin{aligned} \omega_c^\mu[\cdot] &\colon [-a_{\text{per}}(c), a_{\text{per}}(c)] \to \mathbb{R} \\ \psi_{c,1}^\mu[\cdot] &\colon [-a_{\text{per}}(c), a_{\text{per}}(c)] \to \mathcal{C}_{\text{per}}^\infty \cap \{\textit{even functions}\} \\ \psi_{c,2}^\mu[\cdot] &\colon [-a_{\text{per}}(c), a_{\text{per}}(c)] \to \mathcal{C}_{\text{per}}^\infty \cap \{\textit{odd functions}\} \end{aligned}$$

with the following properties.

(i) *If*

$$\boldsymbol{\phi}_c^\mu[a](x) := \begin{pmatrix} \upsilon_c^\mu \cos(\omega_c^\mu[a]x) \\ \sin(\omega_c^\mu[a]x) \end{pmatrix} + \begin{pmatrix} \psi_{c,1}^\mu[a](\omega_c^\mu[a]x) \\ \psi_{c,2}^\mu[a](\omega_c^\mu[a]x) \end{pmatrix}, \tag{3.0.8}$$

where $\upsilon_c^\mu = \mathcal{O}_c(\mu)$ is defined below in (C.1.17)*, then $\mathcal{G}_c(a\boldsymbol{\phi}_c^\mu[a], \mu) = 0$.*

(ii) $\omega_c^\mu[0] = \omega_c^\mu$.

(iii) $\psi_{c,1}^\mu[0] = \psi_{c,2}^\mu[0] = 0$.

(iv) *For each $r \ge 0$, there is $C(c, r) > 0$ such that if $|a|$, $|\grave{a}| \le a_{\text{per}}$ and $|\mu| \le \mu_{\text{per}}(c)$, then*

$$|\omega_c^\mu[a] - \omega_c^\mu[\grave{a}]| + \|\psi_{c,1}^\mu[a] - \psi_{c,1}^\mu[\grave{a}]\|_{W^{r,\infty}} + \|\psi_{c,2}^\mu[a] - \psi_{c,2}^\mu[\grave{a}]\|_{W^{r,\infty}} \le C(c, r)|a - \grave{a}|. \tag{3.0.9}$$

For later use, we isolate two additional estimates on the periodic solutions and their frequencies; the proof follows directly from Proposition 3.3.

Corollary 3.4. *Under the notation of Proposition 3.3, we have*

$$\sup_{\substack{|\mu| \le \mu_{\text{per}}(c) \\ |a| \le a_{\text{per}}(c)}} |\omega_c^\mu[a]| + \|\boldsymbol{\phi}_c^\mu[a]\|_{W^{r,\infty}} < \infty. \tag{3.0.10}$$

and

$$\sup_{|\mu| \le \mu_{\text{per}}(c)} \|\boldsymbol{\psi}_c^\mu[a]\|_{W^{r,\infty}} \le C(c, r)|a|. \tag{3.0.11}$$

We emphasize that our periodic solutions persist for $\mu = 0$, i.e., for the monatomic lattice. Such persistence at the zero limit of the small parameter was impossible in the long wave and small mass limits, as there the analogue of the critical frequency diverged to $+\infty$ as the small parameter approached zero. This cannot happen in our problem, due to the bounds on ω_c^μ in part (ii) in Proposition 3.1.

We also note that the existence of periodic solutions to the monatomic traveling wave problems has already been established by Friesecke and Mikikits-Leitner [FML15] in the long wave limit using a construction inspired by [FP99]. These periodic traveling waves are close to a KdV cnoidal profile, whereas when $\mu = 0$, the expansion (3.0.8) says that the periodic solutions from Proposition 3.3 are, to leading order in the amplitude parameter a, close to $(0, \sin(\omega_c \cdot))$.

4. Analysis of the Operators $\mathcal{L}_c$ and $\mathcal{L}_c^*$

Proposition 2.3 characterized the range of $\mathcal{L}_c$, defined in (1.5.2), as an operator from O_q^{r+2} to O_q^r. In this section we prove that theorem, which we restate below with some additional details.

Theorem 4.1. *Let $|c| > 1$ satisfy Hypotheses 1 and 3.*

(i) *The operator $\mathcal{L}_c$ is injective from O_q^{r+2} to O_q^r for each $r \geq 0$ and $q \geq q_{\mathcal{L}}(c)$.*

(ii) *There is a nonzero odd function $\gamma_c \in W^{2,\infty}$ such that for $f \in O_q^{r+2}$ and $g \in O_q^r$, with $r \geq 0$ and $q \in [q_{\mathcal{L}}(c), 1)$, we have*

$$\mathcal{L}_c f = g \iff \underbrace{\int_{-\infty}^{\infty} g(x)\gamma_c(x)\ dx}_{\iota_c[g]} = 0. \tag{4.0.1}$$

Moreover, $\mathcal{L}_c^*\gamma_c = 0$, where $\mathcal{L}_c^*$ is the L^2-adjoint of $\mathcal{L}_c$ and was defined in (1.5.6).

(iii) *The function γ_c satisfies the limits*

$$\lim_{x\to\infty} |\gamma_c(x) - \sin(\omega_c(x+\vartheta_c))| = \lim_{x\to\infty} |(\gamma_c)'(x) - \omega_c\cos(\omega_c(x+\vartheta_c))| = 0. \tag{4.0.2}$$

(iv) *The functional ι_c is bounded on O_q^r for any q, $r \geq 0$, i.e., there is $C(c,q,r) > 0$ such that*

$$|\iota_c[f]| \leq C(c,q,r)\|f\|_{r,q}, \ \ f \in O_q^r. \tag{4.0.3}$$

Throughout the proof of this theorem, we will use a number of properties of the symbol $\widetilde{\mathcal{B}}_c$ of the operator $\mathcal{B}_c$, which are proved in Appendix D.1.

Proposition 4.2. *For $|c| > 1$, the function*

$$\widetilde{\mathcal{B}}_c(z) := -c^2z^2 + 2 + 2\cos(z) \tag{4.0.4}$$

has the following properties.

(i) *Let $q_{\mathcal{B}} = 1$. Then for $z \in S_{3q_{\mathcal{B}}} := \{z \in \mathbb{C} \mid |\operatorname{Im}(z)| \leq 3q_{\mathcal{B}}\}$, we have $\widetilde{\mathcal{B}}_c(z) = 0$ if and only if $z = \pm\omega_c$, where $\omega_c \in (\sqrt{2}/|c|, \pi/2)$ was previously studied in Section 1.5.1. The zeros at $z = \pm\omega_c$ are simple. Additionally,*

$$\inf_{|c|\in(1,\sqrt{2}]} |(\widetilde{\mathcal{B}}_c)'(\omega_c)| > 0. \tag{4.0.5}$$

(ii) *The function $1/\widetilde{\mathcal{B}}_c$ is meromorphic on the strip $S_{3q_{\mathcal{B}}}$. The only poles of $1/\widetilde{\mathcal{B}}_c$ in $S_{3q_{\mathcal{B}}}$ are $z = \pm\omega_c$, and each of these poles is simple.*

(iii) *There exists $r_{\mathcal{B}}(c) > 0$ such that if $z \in S_{3q_{\mathcal{B}}}$ with $|z| \geq r_{\mathcal{B}}(c)$, then*

$$\frac{1}{|\widetilde{\mathcal{B}}_c(z)|} \leq \frac{2}{|\operatorname{Re}(z)|^2}. \tag{4.0.6}$$

Moreover,

$$r_0 := \sup_{|c|\in(1,\sqrt{2}]} r_{\mathcal{B}}(c) < \infty. \tag{4.0.7}$$

(iv) *There is a constant $C_{\mathcal{B}} > 0$ such that if $0 < |q| \le q_{\mathcal{B}}$, then*

$$\sup_{|c|\in(1,\sqrt{2}]} \left\| \mathfrak{F}^{-1}\left[\frac{1}{\widetilde{\mathcal{B}}_c(-\cdot+iq)}\right]\right\|_{L^1} \le \frac{C_{\mathcal{B}}}{|q|}. \tag{4.0.8}$$

Last, we mention two useful properties of the one-sided exponentially weighted spaces from Definition 2.2. First, if $q > 0$ and $\mathbf{f} \in W_q^{r,\infty}(\mathbb{R},\mathbb{C}^m)$, then $\mathbf{f}$ vanishes at $-\infty$, and if $\mathbf{f} \in W_{-q}^{r,\infty}(\mathbb{R},\mathbb{C}^m)$, then $\mathbf{f}$ vanishes at ∞. Next, the Sobolev embedding and some calculus tell us that the two-sided exponentially weighted spaces H_q^r, defined in (1.4.1), are continuously embedded in $W_{\pm q}^{r-1,\infty}$ for $r \ge 1$.

Now we are ready to prove Theorem 4.1. We distribute the proof over the remainder of this section.

4.1. **Injectivity of $\mathcal{L}_c\colon O_q^{r+2} \to O_q^r$.** Recall from the definition of the H_q^r spaces in (1.4.1) that if $0 \le r_1 \le r_2$ and $0 \le q_1 \le q_2$, then $O_{q_2}^{r_2} \subseteq O_{q_1}^{r_1}$. Since we take $q_{\mathcal{L}}(c) \le q$ and $r \ge 0$, it therefore suffices to prove that $\mathcal{L}_c$ is injective from $O_{q_{\mathcal{L}}(c)}^2$ to $O_{q_{\mathcal{L}}(c)}^0$.

We do this by considering $\mathcal{L}_c$ and $\mathcal{L}_c^*$ as unbounded operators on L^2 with domain $O_{q_{\mathcal{L}}(c)}^2$. For convenience, we recall that

$$\mathcal{L}_c^* f = \underbrace{c^2 f'' + (2+A)f}_{\mathcal{B}_c f} + \underbrace{2\varsigma_c(x)(2+A)f}_{-\Sigma_c^* f}. \tag{4.1.1}$$

Suppose $\mathcal{L}_c^* f = 0$ for some $f \in O_{q_{\mathcal{L}}(c)}^2$. Then

$$f'' = -\frac{(2+A)f + 2\varsigma_c(x)(2+A)f}{c^2}.$$

Since $\varsigma_c \in \cap_{r=0}^{\infty} E_{q_\varsigma(c)}^r$, we can bootstrap from this equality to obtain $f \in \cap_{r=0}^{\infty} O_{q_{\mathcal{L}}(c)}^r$. In particular, we have $f \in O_{q_{\mathcal{L}}(c)}^3 \subseteq W_{-q_{\mathcal{L}}(c)}^{2,\infty}$. Hypothesis 3 then implies that $f = 0$, and so $\mathcal{L}_c^*$ must be injective. That is, 0 is not an eigenvalue of $\mathcal{L}_c^*$, and so 0 is also not an eigenvalue of $\mathcal{L}_c$. Hence $\mathcal{L}_c$ is injective on $O_{q_{\mathcal{L}}(c)}^2$.

4.2. **A solution to $\mathcal{L}_c^* f = 0$.** To obtain the forward implication in (4.0.1), it suffices to find a function γ_c with $\mathcal{L}_c^*\gamma_c = 0$. We will construct a very particular γ_c that will give us the asymptotics (4.0.2). We do this in three steps. First, we find a function f_c satisfying $\mathcal{L}_c^* f_c = 0$ using methods derived from [MP99]. Then the structure of the operator $\mathcal{L}_c^*$ guarantees that $\widetilde{f}_c(x) := f_c(x) - f_c(-x)$ also satisfies $\mathcal{L}_c^* f_c = 0$. Last, since the function ς_c and all the coefficients in $\mathcal{L}_c^*$ are real-valued, the functions $\mathrm{Re}[\widetilde{f}_c]$ and $\mathrm{Im}[\widetilde{f}_c]$ satisfy $\mathcal{L}_c^* \mathrm{Re}[\widetilde{f}_c] = \mathcal{L}_c^* \mathrm{Im}[\widetilde{f}_c] = 0$ as well. We will show that a rescaled version of either $\mathrm{Re}[\widetilde{f}_c]$ or $\mathrm{Im}[\widetilde{f}_c]$ has the sinusoidal asymptotics (4.0.2).

Consider the problem $\mathcal{L}_c^* f = 0$ and make the ansatz

$$f(x) = e^{i\omega_c x} + g(x),$$

where $g \in L_{-q_{\mathcal{L}}(c)}^{\infty}$. Per (i) of Proposition 4.2, we have $\mathcal{B}_c e^{i\omega_c \cdot} = 0$, so $\mathcal{L}_c^* f = 0$ if and only if

$$\mathcal{L}_c^* g = \Sigma_c^* e^{i\omega_c \cdot}. \tag{4.2.1}$$

We use the definition of Σ_c^* in (4.1.1) and the property $e^{q_\varsigma(c)|\cdot|}\varsigma_c \in L^\infty$ to obtain $\Sigma_c^* e^{i\omega_c\cdot} \in L^\infty_{-q_{\mathcal{L}}(c)}$. Let $[\mathcal{L}_c^*]^{-1}$ be the inverse of $\mathcal{L}_c^*$ from $W^{2,\infty}_{-q_{\mathcal{L}}(c)}$ to $L^\infty_{-q_{\mathcal{L}}(c)}$, per Hypothesis 2. If we set

$$g_c := [\mathcal{L}_c^*]^{-1}\Sigma_c^* e^{i\omega_c\cdot} \qquad \text{and} \qquad f_c := e^{i\omega_c\cdot} + g_c \tag{4.2.2}$$

then $\mathcal{L}_c^* f_c = 0$.

4.3. **Asymptotics of f_c at $\pm\infty$.** Since $g_c \in W^{2,\infty}_{-q_{\mathcal{L}}(c)}$ with $q_{\mathcal{L}}(c) > 0$, we know that g_c vanishes at ∞. Thus we have the asymptotics

$$\lim_{x\to\infty} |f_c(x) - e^{i\omega_c x}| = 0. \tag{4.3.1}$$

Now we need the asymptotics of f_c at $-\infty$. For this, we turn to the methods of Mallet-Paret in [MP99]. Specifically, we follow the proof of his Proposition 6.1. We rewrite (4.2.1) as

$$\mathcal{B}_c g_c = \underbrace{\Sigma_c^*(g_c + e^{i\omega_c\cdot})}_{h_c}. \tag{4.3.2}$$

Since $g_c \in W^{2,\infty}_{-q_{\mathcal{L}}(c)}$, we have $e^{q_{\mathcal{L}}(c)\cdot} g_c \in L^\infty$. The Laplace transform $\mathscr{L}_-[g_c]$ is therefore defined and analytic on $\mathrm{Re}(z) < -q_{\mathcal{L}}(c)$; the definition and some essential properties of this Laplace transform are given in Appendix A.2. The bounds on $q_{\mathcal{L}}(c)$ from Hypothesis 3 ensure that we may find $q_\gamma(c) > 0$ with

$$q_{\mathcal{L}}(c) < q_\gamma(c) < \min\{q_\varsigma(c) - q_{\mathcal{L}}(c), q_{\mathcal{B}}\}.$$

Since $e^{q_\varsigma(c)|\cdot|}\varsigma_c$, $e^{q_{\mathcal{L}}(c)\cdot} g_c \in L^\infty$ with $q_{\mathcal{L}}(c) < q_\varsigma(c) - q_{\mathcal{L}}(c)$, it follows that $e^{(q_\varsigma(c) - q_{\mathcal{L}}(c))|\cdot|} h_c \in L^\infty$, and therefore $\mathscr{L}_-[h_c] = \mathscr{L}_-[\mathcal{B}_c g_c]$ is defined and analytic on $|\mathrm{Re}(z)| < (q_\varsigma(c) - q_{\mathcal{L}}(c))$. Elementary properties of the Laplace transform give the bounds

$$\sup_{|\mathrm{Re}(z)| \le q_\gamma(c)} |\mathscr{L}_-[\mathcal{B}_c g_c](z)| < \infty. \tag{4.3.3}$$

Next, we use the formula (A.2.2) for the inverse Laplace transform to write

$$g_c(x) = \frac{1}{2\pi i}\int_{\mathrm{Re}(z) = q_\gamma(c)} \mathscr{L}_-[g_c](-z) e^{xz}\, dz. \tag{4.3.4}$$

Then we apply the Laplace transform $\mathscr{L}_-$ to our advance-delay problem (4.3.2) and find

$$\widetilde{\mathcal{B}}_c(iz)\mathscr{L}_-[g_c](z) = \mathfrak{R}_c(z) + \mathscr{L}_-[h_c](z) = \mathfrak{R}_c(z) + \mathscr{L}_-[\mathcal{B}_c g_c](z), \tag{4.3.5}$$

where the remainder term $\mathfrak{R}_c$ arises from the identities (A.2.3) and (A.2.5) for the Laplace transform under derivatives and shifts, respectively. More precisely, we have

$$\mathfrak{R}_c(z) := 2\int_0^1 \left(g_c(x)e^{z(1-x)} + g_c(-x)e^{z(x-1)}\right) dx - zg_c(0) - g_c'(0) \tag{4.3.6}$$

and we note that $\mathfrak{R}_c$ is entire.

Part (i) of Lemma 4.2 tells us that $\widetilde{\mathcal{B}}_c(iz) \neq 0$ for $0 < |\mathrm{Re}(z)| < q_{\mathcal{B}}$ and so $\widetilde{\mathcal{B}}_c(iz) \neq 0$ for $0 \le |\mathrm{Re}(z)| < q_\gamma(c)$ and $z \neq \pm i\omega_c$. We can therefore solve for $\mathscr{L}_-[g_c](z)$ in (4.3.5) and find

$$\mathscr{L}_-[g_c](z) = \frac{\mathfrak{R}_c(z) + \mathscr{L}_-[\mathcal{B}_c g_c](z)}{\widetilde{\mathcal{B}}_c(iz)}, \ 0 \le |\mathrm{Re}(z)| < q_{\mathcal{B}}, \ z \neq \pm i\omega_c.$$

For $x < 0$, the formula (4.3.4) for g_c then becomes

$$g_c(x) = \frac{1}{2\pi i}\int_{\mathrm{Re}(z)=q_\gamma(c)} e^{-xz}\frac{\mathfrak{I}_c(z)}{\widetilde{\mathcal{B}}_c(iz)}\,dz, \qquad \mathfrak{I}_c(x,z) := \big(\mathfrak{R}_c(-z) + \mathscr{L}_-[\mathcal{B}_c g_c](-z)\big). \tag{4.3.7}$$

This integrand is meromorphic on an open set containing $|\mathrm{Re}(z)| < q_\gamma(c)$; it has simple poles at $z = \pm i\omega_c$ and is analytic elsewhere. Moreover, the quadratic decay of $\widetilde{\mathcal{B}}_c(z)$ for $|z|$ large from part (iii) of Lemma 4.2, the estimate (4.3.3), and the definition of $\mathfrak{R}_c$ in (4.3.6) imply

$$\sup_{|k|\le q_\gamma(c)}\lim_{|q|\to\infty}\left|\frac{e^{-x(k+iq)}\mathfrak{I}_c(k+iq)}{\widetilde{\mathcal{B}}_c(ik-q)}\right| = 0,\ \ x<0.$$

We can therefore shift the contour of the integral in (4.3.4) from $\mathrm{Re}(z) = q_\gamma(c)$ to $\mathrm{Re}(z) = -q_\gamma(c)$ and obtain from the residue theorem that

$$g_c(x) = \underbrace{\frac{1}{2\pi i}\int_{\mathrm{Re}(z)=-q_\gamma(c)}\frac{e^{-xz}\mathfrak{I}_c(x,z)}{\widetilde{\mathcal{B}}_c(-iz)}\,dz}_{r_c(x)} + \alpha_c e^{i\omega_c x} + \beta_c e^{-i\omega_c x}, \tag{4.3.8}$$

where

$$\alpha_c e^{i\omega_c x} = 2\pi i\,\mathrm{Res}\left(\frac{e^{-xz}\mathfrak{I}_c(x,z)}{\widetilde{\mathcal{B}}_c(-iz)}; z = i\omega_c\right) \tag{4.3.9}$$

and

$$\beta_c e^{-i\omega_c x} = 2\pi i\,\mathrm{Res}\left(\frac{e^{-xz}\mathfrak{I}_c(x,z)}{\widetilde{\mathcal{B}}_c(-iz)}; z = -i\omega_c\right). \tag{4.3.10}$$

Since $\widetilde{\mathcal{B}}_c$ has simple zeros at $z = \pm i\omega_c$ by part (i) of Lemma 4.2, and since $\mathfrak{I}_c(x,\cdot)$ and $e^{x\cdot}$ are analytic on $|\mathrm{Re}(z)| \le q_\gamma(c)$, these residues are

$$\mathrm{Res}\left(\frac{\mathfrak{I}_c(x,z)}{\widetilde{\mathcal{B}}_c(-iz)}; z = \pm i\omega_c\right) = \frac{e^{\pm i\omega_c x}\mathfrak{I}_c(\mp i\omega)}{i(\widetilde{\mathcal{B}}_c)'(\pm\omega_c)}.$$

The strategy of the remainder of Mallet-Paret's proof of his Proposition 6.1 in [MP99] shows that both $r_c(x)$ and $r_c'(x)$ from (4.3.8) vanish as $x \to -\infty$. Thus

$$\lim_{x\to-\infty}|f_c(x) - e^{i\omega_c x} - \alpha_c e^{i\omega_c x} - \beta_c e^{-i\omega_c x}| = 0. \tag{4.3.11}$$

Remark 4.3. *The limitation of this approach is that we do not have an explicit formula for g_c, and so we cannot calculate further the residues in* (4.3.9) *and* (4.3.10)*. When $|c| \gtrsim 1$, we can use the Neumann series and results from* [HL07] *to produce an explicit formula for $(\mathcal{B}_c - \Sigma_c^*)^{-1}$; we do this in Sections 6.3.2 and 6.3.3. In turn, this does give a formula for g_c and, ultimately, α_c and β_c.*

On the other hand, we point out that our proof here does not use the fact that ς_c solves the monatomic traveling wave problem; instead, we need only the decay property $e^{q_\varsigma(c)|\cdot|}\varsigma_c \in L^\infty$. The methods of this section could therefore be applied to much more general advance-delay operators than $\mathcal{L}_c^*$; indeed, Mallet-Paret's results in [MP99] *are phrased for a very broad class of such operators.*

4.4. **The phase shift revealed.** With f_c from (4.2.2), the structure of $\mathcal{L}_c^*$ implies that $\widetilde{f}_c(x) := f_c(x) - f_c(-x)$ also satisfies $\mathcal{L}_c^*\widetilde{f}_c = 0$. Setting

$$\mathcal{E}_c(x) := e^{i\omega_c x} - \big(e^{-i\omega_c x} + \alpha_c e^{-i\omega_c x} + \beta_c e^{i\omega_c x}\big), \tag{4.4.1}$$

we have

$$\lim_{x\to\infty}\left|\widetilde{f}_c(x) - \mathcal{E}_c(x)\right| = 0. \tag{4.4.2}$$

This follows by estimating

$$\begin{aligned}&\left|\widetilde{f}_c(x) - \big[e^{i\omega_c x} - \big(e^{-i\omega_c x} + \alpha_c e^{-i\omega_c x} + \beta_c e^{i\omega_c x}\big)\big]\right| \\ &\qquad = \left|f_c(x) - f_c(-x) - \big[e^{i\omega_c x} - \big(e^{-i\omega_c x} + \alpha_c e^{-i\omega_c x} + \beta_c e^{i\omega_c x}\big)\big]\right| \\ &\qquad\qquad \le \left|f_c(x) - e^{i\omega_c x}\right| + \left|f_c(-x) - \big(e^{-i\omega_c x} + \alpha_c e^{-i\omega_c x} + \beta_c e^{i\omega_c x}\big)\right|\end{aligned}$$

and using the limits (4.3.1) and (4.3.11).

Now we claim

$$\mathcal{E}_c(x) \ne 0, \ x \in \mathbb{R}. \tag{4.4.3}$$

We prove this claim in Section 4.6. Then either $\mathrm{Re}[\mathcal{E}_c]$ or $\mathrm{Im}[\mathcal{E}_c]$ does not vanish identically on $\mathbb{R}$. We assume that $\mathrm{Im}[\mathcal{E}_c]$ does not vanish. If it is $\mathrm{Re}[\mathcal{E}_c]$ that does not vanish, then the proof still proceeds along the lines of what follows below.

Set $\breve{f}_c(x) := \mathrm{Im}[\widetilde{f}_c(x)] = \mathrm{Im}[f_c(x) - f_c(-x)]$. Then $\mathcal{L}_c^*\breve{f}_c = 0$ and, from (4.4.2), we have

$$\lim_{x\to\infty}\left|\breve{f}_c(x) - \mathrm{Im}[\mathcal{E}_c(x)]\right| = 0. \tag{4.4.4}$$

Write

$$\alpha_c = \alpha_{c,\mathsf{r}} + i\alpha_{c,\mathsf{i}} \qquad \text{and} \qquad \beta_c = \beta_{c,\mathsf{r}} + i\beta_{c,\mathsf{i}},$$

where $\alpha_{c,\mathsf{r}}$, $\alpha_{c,\mathsf{i}}$, $\beta_{c,\mathsf{r}}$, $\beta_{c,\mathsf{i}} \in \mathbb{R}$. Then

$$\mathrm{Im}[\mathcal{E}_c(x)] = (2 + \alpha_{c,\mathsf{r}} - \beta_{c,\mathsf{r}})\sin(\omega_c x) - (\alpha_{c,\mathsf{i}} + \beta_{c,\mathsf{i}})\cos(\omega_c x). \tag{4.4.5}$$

Now we need the identity

$$A\sin(\omega x) + B\cos(\omega x) = \sqrt{A^2+B^2}\sin\left(\omega x + \arctan\left(\frac{B}{A}\right)\right), \tag{4.4.6}$$

valid for A, B, $\omega \in \mathbb{R}$ with $A \ge 0$ (in the case $A = 0$, we interpret $\arctan(B/0) = \arctan(\pm\infty) = \pm\pi/2$). Since $\mathrm{Im}[\mathcal{E}_c]$ does not vanish identically, at least one of the coefficients $(2 + \alpha_{c,\mathsf{r}} - \beta_{c,\mathsf{r}})$, $(\alpha_{c,\mathsf{i}} + \beta_{c,\mathsf{i}})$ is nonzero. We then apply the identity (4.4.6) directly to (4.4.5) and conclude that

$$\gamma_c := \frac{\mathrm{sgn}(2 + \alpha_{c,\mathsf{r}} - \beta_{c,\mathsf{r}})(2 + \alpha_{c,\mathsf{r}} - \beta_{c,\mathsf{r}})}{\sqrt{(2 + \alpha_{c,\mathsf{r}} - \beta_{c,\mathsf{r}})^2 + (\alpha_{c,\mathsf{i}} + \beta_{c,\mathsf{i}})^2}}\breve{f}_c$$

satisfies $\mathcal{L}_c^*\gamma_c = 0$ and

$$\lim_{x\to\infty}|\gamma_c(x) - \sin(\omega_c(x + \vartheta_c))| = 0, \tag{4.4.7}$$

where

$$\vartheta_c := -\frac{1}{\omega_c}\arctan\left(\frac{\alpha_{c,\mathrm{i}}+\beta_{c,\mathrm{i}}}{2+\alpha_{c,\mathrm{r}}-\beta_{c,\mathrm{r}}}\right).$$

Since $\mathcal{L}_c^*\gamma_c = 0$, we have the forward implication of (4.0.1) in part (ii) of Theorem 4.1, and also (4.4.7) implies the first limit in the asymptotics (4.0.2) from part (iii). Proving the second limit in (4.0.2) is essentially a matter of establishing the limit (4.4.4) with $\breve{f}_c(x)$ replaced by $\partial_x[\breve{f}_c](x)$ and $\Im[\mathcal{E}_c(x)]$ replaced by $\Im[\mathcal{E}_c'(x)]$. The validity of this limit with derivatives, in turn, is a consequence of the two representations for f_c: first, per (4.2.2), as $f_c(x) = e^{i\omega_c x} + g_c(x)$, where g_c and g_c' vanish at ∞, and, next, with g_c replaced by its expression in (4.3.8), in which r_c' decays at $-\infty$.

Last, since f_c and f_c' are asymptotic to bounded functions at $\pm\infty$ by (4.3.1) and (4.3.11), it follows that γ_c, $\gamma_c' \in L^\infty$, which implies (4.0.3). This proves part (iv).

4.5. **Characterization of the range of $\mathcal{L}_c$ as an operator from O_q^{r+2} to O_q^r.** Here we prove the reverse implication in (4.0.1). In Section 1.5.2, we argued[5] that the cokernel of $\mathcal{L}_c$ in O_q^r was one-dimensional, and we characterized the range of $\mathcal{L}_c$ via

$$\mathcal{L}_c f = g,\ f \in O_q^{r+2},\ g \in O_q^r \iff \underbrace{\int_{-\infty}^{\infty} g(x)\widetilde{\mathfrak{z}}_c(x)\ dx}_{\mathfrak{z}_c[g]} = 0 \tag{4.5.1}$$

for some odd function $\widetilde{\mathfrak{z}}_c \in L^1_{\mathrm{loc}}$ with $\operatorname{sech}^q(\cdot)\widetilde{\mathfrak{z}}_c \in L^2$. On the other hand, we constructed in Section 4.4 an odd nontrivial function $\gamma_c \in W^{2,\infty}$ such that

$$\underbrace{\int_{-\infty}^{\infty} (\mathcal{L}_c f)(x)\gamma_c(x)\ dx}_{\iota_c[\mathcal{L}_c f]} = 0$$

for all $f \in O_q^{r+2}$. The functions γ_c and $\widetilde{\mathfrak{z}}_c$ must be linearly dependent, as otherwise the functionals ι_c and $\mathfrak{z}_c$ would be linearly independent and $\mathcal{L}_c$ would have a cokernel of dimension at least 2. The characterization (4.5.1) therefore implies (4.0.1).

4.6. **Proof of the claim** (4.4.3)**.** Fix $q \in [q_{\mathcal{L}}(c), q_{\mathcal{B}})$ and suppose the claim is false, so that $\mathcal{E}_c(x) = 0$ for all x. Then (4.4.2) and the exponential decay of g_c imply that $\widetilde{f}_c(x) = f_c(x) - f_c(-x)$ vanishes exponentially fast at $\pm\infty$, so $\widetilde{f}_c \in W^{2,\infty}_{-q_{\mathcal{L}}(c)}$. But $\mathcal{L}_c^*\widetilde{f}_c = 0$, so Hypothesis 3 forces $\widetilde{f}_c = 0$. That is, f_c must be even.

Now observe that $\mathcal{L}_c^* f_c = \mathcal{L}_c^*\overline{f}_c = 0$, where f_c and its complex conjugate $\overline{f}_c$ are linearly independent since they asymptote, respectively, to $e^{i\omega_c x}$ and $e^{-i\omega_c x}$ at ∞. The functionals

$$\mathfrak{z}_{c,1}[g] := \int_{-\infty}^{\infty} g(x)f_c(x)\ dx \qquad \text{and} \qquad \mathfrak{z}_{c,2}[g] := \int_{-\infty}^{\infty} g(x)\overline{f_c(x)}\ dx,$$

defined for $g \in H_q^0$, are therefore also linearly independent. Moreover, $\mathfrak{z}_{c,1}[\mathcal{L}_c f] = \mathfrak{z}_{c,2}[\mathcal{L}_c f] = 0$ for all $f \in H_q^2$.

[5]This argument used the injectivity of $\mathcal{L}_c$ that was established in Section 1.5.1 for the restricted case $|c| \gtrsim 1$. Now we may rely on part (i) of Theorem 4.1 to obtain injectivity for arbitrary c assuming Hypothesis 3, and the results of Section 1.5.2 remain true.

The methods of Section 1.5.2 can be adapted to show that $\mathcal{L}_c$ has a two-dimensional cokernel in H_q^0 when considered as an operator from H_q^2 to H_q^0. Consequently, if $\mathfrak{z}$ is any functional on H_q^0 such that $\mathfrak{z}[\mathcal{L}_c f] = 0$ for all $f \in H_q^2$, then $\mathfrak{z}$ must be a linear combination of $\mathfrak{z}_{c,1}$ and $\mathfrak{z}_{c,2}$. On the other hand, with $\mathfrak{z}_c$ and $\widetilde{\mathfrak{z}}_c$ from (4.5.1), we already have $\mathfrak{z}_c[\mathcal{L}_c f] = 0$ for all $f \in H_q^2$, and so $\mathfrak{z}_c$ must be a linear combination of $\mathfrak{z}_{c,1}$ and $\mathfrak{z}_{c,2}$. But then the odd function $\widetilde{\mathfrak{z}}_c$ must be a linear combination of f_c and $\overline{f}_c$, and so $\widetilde{\mathfrak{z}}_c$ is even, a contradiction.

5. The Micropteron Fixed Point Problem

5.1. **Beale's ansatz.** We study our problem $\mathcal{G}_c(\boldsymbol{\rho}, \mu) = 0$ from (1.3.1) under Beale's ansatz

$$\boldsymbol{\rho} = \boldsymbol{\varsigma}_c + a\boldsymbol{\phi}_c^{\mu}[a] + \boldsymbol{\eta},$$

where

- $\boldsymbol{\varsigma}_c = (\varsigma_c, 0)$ solves $\mathcal{G}_c(\boldsymbol{\varsigma}_c, 0) = 0$, per Hypothesis 1.
- $\boldsymbol{\phi}_c^{\mu}[a]$ is periodic, $a \in \mathbb{R}$ with $|a| \le a_{\mathrm{per}}(c)$, $|\mu| \le \mu_{\mathrm{per}}(c)$, and $\mathcal{G}_c(a\boldsymbol{\phi}_c^{\mu}[a], \mu) = 0$ by Proposition 3.3.
- $\boldsymbol{\eta} \in E_q^r \times O_q^r$ with $r \ge 2$ and $q > 0$ to be specified later.

We expand $\mathcal{G}_c(\boldsymbol{\varsigma}_c + a\boldsymbol{\varphi}_c^{\mu}[a] + \boldsymbol{\eta}, \mu)$ using the bilinearity (B.0.4) of $\mathcal{Q}$ and the following decomposition of $\mathcal{D}_\mu$ from (1.3.2) as the sum of a diagonal operator and a small perturbation term:

$$\mathcal{D}_\mu = \mathcal{D}_0 + \underbrace{\mu \begin{bmatrix} (2-A)/2 & \delta \\ -\delta & (2+A)/2 \end{bmatrix}}_{\mu\mathring{\mathcal{D}}}.$$

Next, we cancel a number of terms with the existing solutions

$$\mathcal{G}_c(\boldsymbol{\varsigma}_c, 0) = 0 \qquad \text{and} \qquad \mathcal{G}_c(a\boldsymbol{\varphi}_c^{\mu}[a], \mu) = 0.$$

After some further rearrangements, we find that $\mathcal{G}_c(\boldsymbol{\varsigma}_c + a\boldsymbol{\varphi}_c^{\mu}[a] + \boldsymbol{\eta}) = 0$ is equivalent to

(5.1.1)
$$c^2\boldsymbol{\eta}'' + \mathcal{D}_0\boldsymbol{\eta} + 2\mathcal{D}_0\mathcal{Q}(\boldsymbol{\varsigma}_c, \boldsymbol{\eta}) = -\mu\mathring{\mathcal{D}}(\boldsymbol{\varsigma}_c + \mathcal{Q}(\boldsymbol{\varsigma}_c, \boldsymbol{\varsigma}_c)) - \mu\mathring{\mathcal{D}}(\boldsymbol{\eta} - 2\mathcal{Q}(\boldsymbol{\varsigma}_c, \boldsymbol{\eta})) - 2a\mathcal{D}_\mu\mathcal{Q}(\boldsymbol{\varsigma}_c, \boldsymbol{\phi}_c^{\mu}[a])$$
$$- 2a\mathcal{D}_\mu\mathcal{Q}(\boldsymbol{\phi}_c^{\mu}[a], \boldsymbol{\eta}) - \mathcal{D}_\mu\mathcal{Q}(\boldsymbol{\eta}, \boldsymbol{\eta}).$$

Recalling the definitions of $\mathcal{H}_c$ in (1.4.3) and $\mathcal{L}_c$ in (1.5.1), we see that the left side of this equation is just the diagonal operator $\mathrm{diag}(\mathcal{H}_c, \mathcal{L}_c)$ applied to $\boldsymbol{\eta}$. Then we can rewrite (5.1.1) in the componentwise form

(5.1.2)
$$\begin{cases} \mathcal{H}_c\eta_1 = \displaystyle\sum_{k=1}^{5} h_{c,k}^{\mu}(\boldsymbol{\eta}, a) \\ \mathcal{L}_c\eta_2 = \displaystyle\sum_{k=1}^{5} \ell_{c,k}^{\mu}(\boldsymbol{\eta}, a), \end{cases}$$

where the multitude of terms on the right side are

$$
\begin{aligned}
&h_{c,1}^{\mu}(\boldsymbol{\eta},a) := -\mu\mathring{\mathcal{D}}(\boldsymbol{\varsigma}_c + \mathcal{Q}(\boldsymbol{\varsigma}_c,\boldsymbol{\varsigma}_c))\cdot\mathbf{e}_1 && \ell_{c,1}^{\mu}(\boldsymbol{\eta},a) := -\mu\mathring{\mathcal{D}}(\boldsymbol{\varsigma}_c + \mathcal{Q}(\boldsymbol{\varsigma}_c,\boldsymbol{\varsigma}_c))\cdot\mathbf{e}_2 \\
&h_{c,2}^{\mu}(\boldsymbol{\eta},a) := -\mu\mathring{\mathcal{D}}(\boldsymbol{\eta} - 2\mathcal{Q}(\boldsymbol{\varsigma}_c,\boldsymbol{\eta}))\cdot\mathbf{e}_1 && \ell_{c,2}^{\mu}(\boldsymbol{\eta},a) := -\mu\mathring{\mathcal{D}}(\boldsymbol{\eta} - 2\mathcal{Q}(\boldsymbol{\varsigma}_c,\boldsymbol{\eta}))\cdot\mathbf{e}_2 \\
&h_{c,3}^{\mu}(\boldsymbol{\eta},a) := -2a\mathcal{D}_{\mu}\mathcal{Q}(\boldsymbol{\varsigma}_c,\boldsymbol{\phi}_c^{\mu}[a])\cdot\mathbf{e}_1 && \ell_{c,3}^{\mu}(\boldsymbol{\eta},a) := -2a\mathcal{D}_{\mu}\mathcal{Q}(\boldsymbol{\varsigma}_c,\boldsymbol{\phi}_c^{\mu}[a])\cdot\mathbf{e}_2 \\
&h_{c,4}^{\mu}(\boldsymbol{\eta},a) := -2a\mathcal{D}_{\mu}\mathcal{Q}(\boldsymbol{\phi}_c^{\mu}[a],\boldsymbol{\eta})\cdot\mathbf{e}_1 && \ell_{c,4}^{\mu}(\boldsymbol{\eta},a) := -2a\mathcal{D}_{\mu}\mathcal{Q}(\boldsymbol{\phi}_c^{\mu}[a],\boldsymbol{\eta})\cdot\mathbf{e}_2 \\
&h_{c,5}^{\mu}(\boldsymbol{\eta},a) := -\mathcal{D}_{\mu}\mathcal{Q}(\boldsymbol{\eta},\boldsymbol{\eta})\cdot\mathbf{e}_1 && \ell_{c,5}^{\mu}(\boldsymbol{\eta},a) := -\mathcal{D}_{\mu}\mathcal{Q}(\boldsymbol{\eta},\boldsymbol{\eta})\cdot\mathbf{e}_2.
\end{aligned}
\tag{5.1.3}
$$

There are, indeed, many terms here, but the salient features are that $h_{c,k}^{\mu}(\boldsymbol{\eta},a) \in E_{q,0}^r$ and $\ell_{c,k}^{\mu}(\boldsymbol{\eta},a) \in O_q^r$ for $\boldsymbol{\eta} \in E_q^r \times O_q^r$ and $a \in \mathbb{R}$ and that most of these terms are "small." For example, $h_{c,1}^{\mu}$ is $\mathcal{O}_c(\mu)$, $h_{c,2}^{\mu}$ is, very roughly, of the form $\mu\boldsymbol{\eta}$, and $h_{c,4}^{\mu}$ and $h_{c,5}^{\mu}$ are quadratic in $\boldsymbol{\eta}$ and a. The terms $h_{c,3}^{\mu}$ and $\ell_{c,3}^{\mu}$ are more complicated and merit more precise analysis later. We will use the explicit algebraic structure of the terms and their smallness frequently in our subsequent proofs.

5.2. **Construction of the equation for η_1.** We first extend Hypothesis 2 to allow $\mathcal{H}_c$ to be invertible over a broader range of exponentially localized spaces. The proof of the next proposition is in Appendix F.2.

Proposition 5.1. *Assume Hypotheses 1 and 2. There exist $q_{\mathcal{H}}^{\star}(c)$, $q_{\mathcal{H}}^{\star\star}(c)$ with $q_{\mathcal{H}}(c) \le q_{\mathcal{H}}^{\star}(c) < q_{\mathcal{H}}^{\star\star}(c) < \min\{1, q_{\varsigma}(c)\}$ such that for any $q \in [q_{\mathcal{H}}^{\star}(c), q_{\mathcal{H}}^{\star\star}(c)]$ and $r \ge 0$, the operator $\mathcal{H}_c$ is invertible from E_q^{r+2} to $E_{q,0}^r$.*

From now on we fix $q \in (\max\{q_{\mathcal{L}}(c), q_{\mathcal{H}}^{\star}(c)\}, \min\{q_{\varsigma}(c), q_{\mathcal{H}}^{\star\star}(c), 1\})$. Since $q_{\mathcal{H}}^{\star}(c) < \min\{q_{\varsigma}(c), q_{\mathcal{H}}^{\star\star}(c), 1\}$ and $q_{\mathcal{L}}(c) < \min\{q_{\varsigma}(c), q_{\mathcal{H}}(c), 1\} < q_{\mathcal{H}}^{\star\star}(c)$ by Hypothesis 3, this interval is nonempty. Moreover, since $q \in (q_{\mathcal{H}}^{\star}(c), q_{\mathcal{H}}^{\star\star}(c))$, Proposition 5.1 tells us that $\mathcal{H}_c$ is invertible from E_q^{r+2} to $E_{q,0}^r$ for any $r \ge 0$. We then invert $\mathcal{H}_c$ in the first equation in (5.1.2) to obtain a fixed point equation for η_1:

$$
\eta_1 = -\mathcal{H}_c^{-1}\sum_{j=1}^{6} h_{c,k}^{\mu}(\boldsymbol{\eta},a) =: \mathcal{N}_{c,1}^{\mu}(\boldsymbol{\eta},a). \tag{5.2.1}
$$

5.3. **Construction of the fixed point equations for a.** From the system (5.1.2), the unknowns η_2 and a must satisfy

$$
\mathcal{L}_c\eta_2 = \sum_{k=1}^{5}\ell_{c,k}^{\mu}(\boldsymbol{\eta},a) \tag{5.3.1}
$$

We formally differentiate the right side of (5.3.1) to isolate a term containing a factor of a:

$$
\frac{\partial}{\partial a}\left[\sum_{k=1}^{5}\ell_{c,k}^{\mu}(\boldsymbol{\eta},a)\right]\Big|_{\boldsymbol{\eta}=0,a=0} = \frac{\partial}{\partial a}[\ell_{c,3}^{\mu}(\boldsymbol{\eta},a)]\big|_{\boldsymbol{\eta}=0,a=0} = -2\mathcal{D}_{\mu}\mathcal{Q}(\boldsymbol{\varsigma}_c,\boldsymbol{\phi}_c^{\mu}[0])\cdot\mathbf{e}_2 =: -\chi_c^{\mu}. \tag{5.3.2}
$$

We calculate this formal derivative by recalling the definitions of $\ell_{c,1}^{\mu},\ldots,\ell_{c,5}^{\mu}$ in (5.1.3) and observing that all the terms except $\ell_{c,3}^{\mu}$ are either constant in a or quadratic in $\boldsymbol{\eta}$ and a, in which case their derivatives with respect to a at $\boldsymbol{\eta} = 0$ and $a = 0$ are zero. We expect, therefore, that the term

$$
\ell_{c,3}^{\mu}(\boldsymbol{\eta},a) + a\chi_c^{\mu}
$$

will be roughly quadratic in a. We could then write

$$\sum_{k=1}^{5} \ell_{c,k}^{\mu}(\boldsymbol{\eta}, a) = -a\chi_c^{\mu} + \underbrace{\sum_{\substack{k=1 \\ k\neq 3}}^{5} \ell_{c,k}^{\mu}(\boldsymbol{\eta}) + \left(\ell_{c,3}^{\mu}(\boldsymbol{\eta}, a) + a\chi_c^{\mu}\right)}_{\text{these terms are "small"}}.$$

However, it turns out to be more convenient not to work with χ_c^{μ} but instead with

$$\chi_c := \chi_c^0 = 2\mathcal{D}_0\mathcal{Q}(\boldsymbol{\varsigma}_c, \boldsymbol{\phi}_c^0[0]) \cdot \mathbf{e}_2 = (2+A)(\varsigma_c \sin(\omega_c \cdot)). \tag{5.3.3}$$

We now set

$$\widetilde{\ell}_{c,k}^{\mu}(\boldsymbol{\eta}, a) := \begin{cases} \ell_{c,k}^{\mu}(\boldsymbol{\eta}, a), k \neq 3 \\ \ell_{c,3}^{\mu}(\boldsymbol{\eta}, a) + a\chi_c, \ k = 3, \end{cases} \tag{5.3.4}$$

so that (5.3.1) is equivalent to

$$\mathcal{L}_c\eta_2 + a\chi_c = \sum_{k=1}^{5} \widetilde{\ell}_{c,k}^{\mu}(\boldsymbol{\eta}, a). \tag{5.3.5}$$

Then we apply the functional ι_c, defined in (4.0.1), to both sides of (5.3.5) to find

$$a\iota_c[\chi_c] = \sum_{k=1}^{5} \iota_c[\widetilde{\ell}_{c,k}^{\mu}(\boldsymbol{\eta}, a)]. \tag{5.3.6}$$

The asymptotics (4.0.2) on γ_c give us an explicit formula for $\iota_c[\chi_c]$, which we prove in Appendix F.3:

$$\iota_c[\chi_c] = (2c^2\omega_c - \sin(\omega_c))\sin(\omega_c\vartheta_c) \tag{5.3.7}$$

Since $|c| > 1$, we have

$$2c^2\omega_c - \sin(\omega_c) > \omega_c - \sin(\omega_c) > 0. \tag{5.3.8}$$

We assumed in Hypothesis 4 that $\sin(\omega_c\vartheta_c) \neq 0$, and so we have $\iota_c[\chi_c] \neq 0$. Then (5.3.6) is equivalent to

$$a = \frac{1}{\iota_c[\chi_c]} \sum_{k=1}^{5} \iota_c[\widetilde{\ell}_{c,k}^{\mu}(\boldsymbol{\eta}, a)] =: \mathcal{N}_{c,3}^{\mu}(\boldsymbol{\eta}, a). \tag{5.3.9}$$

5.4. **Construction of the fixed point equation for η_2.** We substitute the fixed point equation (5.3.9) for a into (5.3.5) to find

$$\mathcal{L}_c\eta_2 = \sum_{k=1}^{5} \widetilde{\ell}_{c,k}^{\mu}(\boldsymbol{\eta}, a) - a\chi_c = \sum_{k=1}^{5} \widetilde{\ell}_{c,k}^{\mu}(\boldsymbol{\eta}, a) - \frac{1}{\iota_c[\chi_c]}\iota_c\left[\sum_{k=1}^{6} \widetilde{\ell}_{c,k}^{\mu}(\boldsymbol{\eta}, a)\right]\chi_c. \tag{5.4.1}$$

If we set

$$\mathcal{P}_c f := f - \frac{\iota_c[f]}{\iota_c[\chi_c]}\chi_c, \tag{5.4.2}$$

then (5.4.1) can be written more succinctly as

$$\mathcal{L}_c\eta_2 = \mathcal{P}_c \sum_{k=1}^{5} \widetilde{\ell}_{c,k}^{\mu}(\boldsymbol{\eta}, a). \tag{5.4.3}$$

It is obvious from the definition of χ_c in (5.3.3) that $\chi_c \in \cap_{r=0}^{\infty} O_{q_\varsigma(c)}^r \subseteq \cap_{r=0}^{\infty} O_q^r$, and so $\mathcal{P}_c f \in O_q^r$ for any $f \in O_q^r$. Also, a straightforward calculation shows that $\iota_c[\mathcal{P}_c f] = 0$.

Now, recall that in Section 5.2 we specified q so that $q \in (q_{\mathcal{L}}(c), 1)$. Then part (ii) of Theorem 4.1 implies that $\mathcal{P}_c f$ is in the range of $\mathcal{L}_c$. Conversely, if $\mathcal{L}_c f = g$, then $\iota_c[g] = 0$ and so $\mathcal{P}_c g = g$. That is, $\mathcal{P}_c[O_q^r] = \mathcal{L}_c[O_q^{r+2}]$. The injectivity of $\mathcal{L}_c$ on O_q^r for $r \geq 1$, established as part (i) of Theorem 4.1, then implies that $\mathcal{L}_c$ is bijective from O_q^{r+2} to $\mathcal{P}_c[O_q^r]$. The functional ι_c is continuous on O_q^r by part (iv) of Theorem 4.1, and the subspace

$$\mathcal{P}_c[O_q^r] = \mathcal{L}_c[O_q^{r+2}] = \iota_c^{-1}(\{0\}) \cap O_q^r$$

is closed in O_q^r. Hence the inverse of $\mathcal{L}_c$ from $\mathcal{P}_c[O_q^r]$ to O_q^{r+2} is a bounded operator, which we denote by $\mathcal{L}_c^{-1}$.

We are now able to rewrite (5.4.3) as a fixed point equation for η_2:

$$\eta_2 = \mathcal{L}_c^{-1} \mathcal{P}_c \sum_{k=1}^{5} \widetilde{\ell}_{c,k}^{\mu}(\boldsymbol{\eta}, a) =: \mathcal{N}_{c,2}^{\mu}(\boldsymbol{\eta}, a). \tag{5.4.4}$$

With our previously constructed equations (5.2.1) for η_1 and (5.3.9) for a, this gives us a fixed point problem for our three unknowns from Beale's ansatz:

$$(\boldsymbol{\eta}, a) = (\mathcal{N}_{c,1}^{\mu}(\boldsymbol{\eta}, a), \mathcal{N}_{c,2}^{\mu}(\boldsymbol{\eta}, a), \mathcal{N}_{c,3}^{\mu}(\boldsymbol{\eta}, a)) =: \boldsymbol{\mathcal{N}}_c^{\mu}(\boldsymbol{\eta}, a). \tag{5.4.5}$$

5.5. **Solution of the full fixed point problem.** We will solve the fixed point problem (5.4.5) using the following lemma, which was stated and proved as Lemma 4.10 in [JW].

Lemma 5.2. *Let $\mathcal{X}_0$ and $\mathcal{X}_1$ be reflexive Banach spaces with $\mathcal{X}_1 \subseteq \mathcal{X}_0$. For $r > 0$, let $\mathfrak{B}(r) := \{x \in \mathcal{X}_1 \mid \|x\|_{\mathcal{X}_1} \leq r\}$. Suppose that for some $r_0 > 0$, there is a map $\mathcal{N} \colon \mathfrak{B}(r_0) \to \mathcal{X}_1$ with the following properties.*

(i) $\|x\|_{\mathcal{X}_1} \leq r_0 \Longrightarrow \|\mathcal{N}(x)\|_{\mathcal{X}_1} \leq r_0$.

(ii) *There exists $\alpha \in (0,1)$ such that*

$$\|x\|_{\mathcal{X}_0}, \|\grave{x}\|_{\mathcal{X}_1} \leq r_0 \Longrightarrow \|\mathcal{N}(x) - \mathcal{N}(\grave{x})\|_{\mathcal{X}_0} \leq \alpha \|x - \grave{x}\|_{\mathcal{X}_0}.$$

Then there exists a unique $x_\star \in \mathcal{X}_1$ such that $\|x_\star\|_{\mathcal{X}_1} \leq r_0$ and $x_\star = \mathcal{N}(x_\star)$.

To invoke this lemma, we first need to specify the underlying Banach spaces. With $q_{\mathcal{H}}^{\star}(c)$ and $q_{\mathcal{H}}^{\star\star}(c)$ from Proposition 5.1, we fix $q_\star(c) \in (\max\{q_{\mathcal{L}}(c), q_{\mathcal{H}}^{\star}(c)\}, \min\{q_\varsigma(c), q_{\mathcal{H}}^{\star\star}(c), 1\})$ and $\overline{q}_\star(c) \in (q_{\mathcal{H}}(c), q_\star(c))$. Let

$$\mathcal{X}^r := \begin{cases} E_{\overline{q}_\star(c)}^1 \times O_{\overline{q}_\star(c)}^1 \times \mathbb{R}, & r = 0 \\ \\ E_{q_\star(c)}^r \times O_{q_\star(c)}^r \times \mathbb{R}, & r \geq 1 \end{cases}$$

and, for $r \geq 1$ and $\tau > 0$, set

$$\mathcal{U}_{\tau,\mu}^r := \left\{(\boldsymbol{\eta}, a) \in \mathcal{X}^r \;\middle|\; \|\boldsymbol{\eta}\|_{r,q_\star(c)} + |a| \leq \tau|\mu|\right\}.$$

The spaces $\mathcal{X}^r$ are Hilbert spaces and therefore they are reflexive, and the sets $\mathcal{U}_{\tau,\mu}^r$ are the balls of radius $\tau|\mu|$ centered at the origin in $\mathcal{X}^r$.

The next proposition shows that $\boldsymbol{\mathcal{N}}_c^{\mu}$ satisfies the estimates from Lemma 5.2. Its proof is in Appendix G.2.

Proposition 5.3. *Assume that $|c| > 1$ satisfies Hypotheses 1, 2, 3, and 4. There exist $\mu_\star(c)$, $\tau_c > 0$ such that if $|\mu| \le \mu_\star(c)$, then the following hold.*

(i) $(\boldsymbol{\eta}, a) \in \mathcal{U}^1_{\tau_c,\mu} \Longrightarrow \boldsymbol{\mathcal{N}}^\mu_c(\boldsymbol{\eta}, a) \in \mathcal{U}^1_{\tau_c,\mu}$.

(ii) $(\boldsymbol{\eta}, a), (\grave{\boldsymbol{\eta}}, \grave{a}) \in \mathcal{U}^1_{\tau_c,\mu} \Longrightarrow \|\boldsymbol{\mathcal{N}}^\mu_c(\boldsymbol{\eta}, a) - \boldsymbol{\mathcal{N}}^\mu_c(\grave{\boldsymbol{\eta}}, \grave{a})\|_{\mathcal{X}^0} \le \dfrac{1}{2}\|(\boldsymbol{\eta}, a) - (\grave{\boldsymbol{\eta}}, \grave{a})\|_{\mathcal{X}^0}$.

(iii) *For any $\tau > 0$, there is $\widetilde{\tau} > 0$ such that*

$$(\boldsymbol{\eta}, a) \in \mathcal{U}^1_{\tau_c,\mu} \cap \mathcal{U}^r_{\tau,\mu} \Longrightarrow \boldsymbol{\mathcal{N}}^\mu_c(\boldsymbol{\eta}, a) \in \mathcal{U}^{r+1}_{\widetilde{\tau},\mu}.$$

Lemma 5.2 therefore applies to produce a unique $(\boldsymbol{\eta}^\mu_c, a^\mu_c) \in \mathcal{U}^1_{\tau_c,\mu}$ such that $(\boldsymbol{\eta}^\mu_c, a^\mu_c) = \boldsymbol{\mathcal{N}}^\mu_c(\boldsymbol{\eta}^\mu_c, a^\mu_c)$. We then bootstrap with part (iii) of Proposition 5.3 to conclude that $\boldsymbol{\eta}^\mu_c$ is smooth, and so we obtain our main result.

Theorem 5.4. *Assume that $|c| > 1$ satisfies Hypotheses 1, 2, 3, and 4 and take $\mu_\star(c)$ and τ_c from Proposition 5.3. Then for each $|\mu| \le \mu_\star(c)$, there exists a unique $(\boldsymbol{\eta}^\mu_c, a^\mu_c) \in \mathcal{U}^1_{\tau_c,\mu}$ such that $(\boldsymbol{\eta}^\mu_c, a^\mu_c) = \boldsymbol{\mathcal{N}}^\mu_c(\boldsymbol{\eta}^\mu_c, a^\mu_c)$. Moreover, $\boldsymbol{\eta}^\mu_c \in \cap_{r=0}^\infty E^r_{q_\star(c)} \times O^r_{q_\star(c)}$ and, for each $r \ge 0$, there is $C(c, r) > 0$ such that*

$$\|\boldsymbol{\eta}^\mu_c\|_{r,q_\star(c)} + |a^\mu_c| \le C(c, r)|\mu|. \tag{5.5.1}$$

Remark 5.5. *The estimate* (5.5.1) *shows that the amplitude of the ripple is $a^\mu_c = \mathcal{O}_c(\mu)$, which is not necessarily small beyond all orders of μ. Recall from* (5.3.9) *that, roughly, $a^\mu_c = \iota_c[\mathcal{V}^\mu_c(\boldsymbol{\eta}^\mu_c, a^\mu_c)]$, where $\mathcal{V}^\mu_c$ maps $O^r_q \times \mathbb{R}$ to O^r_q. Per* (4.0.3)*, we have an estimate of the form*

$$|a^\mu_c| = |\iota_c[\mathcal{V}^\mu_c(\boldsymbol{\eta}^\mu_c, a^\mu_c)]| \le C(c, r)\|\mathcal{V}^\mu_c(\boldsymbol{\eta}^\mu_c, a^\mu_c)\|_{r,q_\star(c)}.$$

The analogues of ι_c in the lattice nanopteron problems [FW18], [HW17], *and* [Fav] *all depended on μ and, if we denote one of these functionals by $\widetilde{\iota}_\mu$, roughly had an estimate of the form*

$$|\widetilde{\iota}_\mu[f]| \le C(q, r)|\mu|^r\|f\|_{r,q}$$

for $f \in H^r_q$. The proof of this estimate parallels the Riemann-Lebesgue lemma and closely relied on the fact that the analogue in those problems of the critical frequency ω^μ_c was roughly $\mathcal{O}(\mu^{-1})$. In turn, this "high frequency" estimate hinged on the existence of a singular perturbation in the problem. Our equal mass problem is not singularly perturbed, our critical frequency ω^μ_c remains bounded as $\mu \to 0$, and our ripple is not necessarily small beyond all orders of μ.

6. The Proof of Theorem 2.4: Verification of the Hypotheses for $|c| \gtrsim 1$

6.1. Verification of Hypothesis 1.

We extract the following result from Theorem 1.1 in [FP99].

Theorem 6.1 (Friesecke & Pego)**.** *There exist $\epsilon_{\mathsf{FP}} > 0$, $q_{\mathsf{FP}} \in (0, 1)$ such that if $\epsilon \in (0, \epsilon_{\mathsf{FP}})$, then there exists a unique positive function $\varsigma_{c_\epsilon} \in \cap_{r=1}^\infty E^r_{\epsilon q_{\mathsf{FP}}}$ with the following properties.*

(i) *If*

$$c_\epsilon := \left(1 + \frac{\epsilon^2}{24}\right)^{1/2}, \tag{6.1.1}$$

then

$$c_\epsilon^2 \varsigma_{c_\epsilon}'' + (2-A)(\varsigma_{c_\epsilon} + \varsigma_{c_\epsilon}^2) = 0.$$

(ii) *If*

$$\sigma(x) := \frac{1}{4}\operatorname{sech}^2\left(\frac{x}{2}\right), \tag{6.1.2}$$

then for each $r \geq 0$*, there is a constant* $C(r) > 0$ *such that*

$$\left\| \frac{1}{\epsilon^2}\varsigma_{c_\epsilon}\left(\frac{\cdot}{\epsilon}\right) - \sigma \right\|_{H^r} \leq C(r)\epsilon^2 \tag{6.1.3}$$

for all $0 < \epsilon < \epsilon_{\mathsf{FP}}$.

We set

$$q_\varsigma(c_\epsilon) := \epsilon q_{\mathsf{FP}} \tag{6.1.4}$$

in Hypothesis 1. We remark that Friesecke and Pego label the decay rate for their profile ς_c as $b_0(c)$; part (c) of their Theorem 1.1 and their detailed proof in Lemma 3.1 reveal that, with c_ϵ from (6.1.1), they have $b_0(c_\epsilon) = \mathcal{O}(\epsilon)$. It is convenient for us to make explicit this order-ϵ dependence and rewrite this decay rate as $b_0(c_\epsilon) = \epsilon q_{\mathsf{FP}}$.

6.2. **Verification of Hypothesis 2.** As we mentioned in Section 1.4 when we met the operator $\mathcal{H}_c$, the invertibility of $\mathcal{H}_c$ for $|c| \approx 1$ arises from Proposition 3.1 in [HW17]. Here is that proposition.

Proposition 6.2 (Hoffman & Wright)**.** *There exists* $\epsilon_{\mathsf{HWr}} \in (0, \epsilon_{\mathsf{FP}}]$ *such that for* $0 < \epsilon < \epsilon_{\mathsf{HWr}}$*,* $r \geq 0$*, and* $0 < q < \epsilon q_{\mathsf{FP}}$*, the operator* $\mathcal{H}_{c_\epsilon}$*, defined in* (1.4.3)*, is invertible from* E_q^{r+2} *to* $E_{q,0}^r$.

For the precise value of the decay rate in Hypothesis 2, we will set

$$q_{\mathcal{H}}(c_\epsilon) := \frac{\epsilon q_{\mathsf{FP}}}{2}. \tag{6.2.1}$$

Recalling the definition of $q_\varsigma(c_\epsilon)$ in (6.1.4) and that we took $q_{\mathsf{FP}} < 1$ in Theorem 6.1, we have $q_{\mathcal{H}}(c_\epsilon) < \min\{1, q_\varsigma(c_\epsilon)\}$.

We wish to point out that the crux of the proof of Proposition 6.2 by Hoffman and Wright is a clever factorization of $\mathcal{H}_{c_\epsilon}$ as a product of two operators, one of which is invertible from E_q^{r+2} to $E_{q,0}^r$ due to Fourier multiplier theory proved by Beale in [Bea80] (which later appears as Lemma A.3 in this paper), and the other of which is ultimately a small perturbation of an operator that Friesecke and Pego [FP99] prove is invertible from E^r to E^r. Hoffman and Wright apply operator conjugation (cf. Appendix D of [Fav18]) to extend the invertibility of this second operator from E_q^r to E_q^r. It is interesting to note that this perturbation argument is the only time that Hoffman and Wright need to assume that their wave speed is particularly close to 1; for the rest of their paper, the small parameter $|c| - 1$ does not play an explicit role, as it does for us in the concrete verification of our four main hypotheses.

We also note that we are writing ϵq_{FP} for the decay rate of our Friesecke-Pego solitary wave profile ς_{c_ϵ}; Hoffman and Wright use the notation b_c for the decay rate of their Friesecke-Pego profile, which they denote by σ_c. In turn, this b_c is equal to what Friesecke and Pego call $b_0(c)$.

6.3. **Verification of Hypothesis 3 in the case** $|c| \gtrsim 1$**.**

6.3.1. *Preliminary remarks.* We will chose our decay rates q to depend in a very precise way on ϵ. First, let

$$\mathscr{b} := \min\left\{\frac{1}{8}, \frac{\mathscr{a}}{4}, \frac{q_{\mathsf{FP}}}{4}\right\}, \tag{6.3.1}$$

where the constant $\mathscr{a}$ is defined below in Lemma D.1. Next, let

$$\epsilon_{\mathcal{B}} := \min\left\{\frac{\epsilon_{\mathsf{HWa}}}{2\mathscr{b}}, \frac{1}{2\mathscr{b}}, 1\right\}, \tag{6.3.2}$$

where the threshold ϵ_{HWa} comes from Lemma D.1. Last, for $0 < \epsilon < \epsilon_{\mathcal{B}}$, define

$$q_\epsilon := \mathscr{b}\epsilon. \tag{6.3.3}$$

These definitions ensure the useful bound

$$0 < q_\epsilon < \min\left\{\frac{\epsilon}{4}, \frac{\mathscr{a}\epsilon}{2}, \frac{\epsilon q_{\mathsf{FP}}}{4}, 1\right\} \tag{6.3.4}$$

and also, per (6.1.1), $1 < |c_\epsilon| \leq \sqrt{2}$.

6.3.2. *Inversion of the operator $\mathcal{B}_{c_\epsilon}$.* Here is our precise statement about the invertibility of $\mathcal{B}_{c_\epsilon}$. Its proof relies on Theorem E.1, due to Hupkes and Verduyn Lunel [HL07].

Proposition 6.3. *For $0 < \epsilon < \epsilon_{\mathcal{B}}$, the operator $\mathcal{B}_{c_\epsilon}$ is invertible from $W^{2,\infty}_{-q_\epsilon}$ to $L^\infty_{-q_\epsilon}$ and from $W^{2,\infty}_{q_\epsilon}$ to $L^\infty_{q_\epsilon}$. These spaces were defined in Definition 2.2. We denote the inverse from $L^\infty_{-q_\epsilon}$ to $W^{2,\infty}_{-q_\epsilon}$ by $[\mathcal{B}^-_{c_\epsilon}]^{-1}$ and from $L^\infty_{q_\epsilon}$ to $W^{2,\infty}_{q_\epsilon}$ by $[\mathcal{B}^+_{c_\epsilon}]^{-1}$. These operators have the estimates*

$$\|[\mathcal{B}^\pm_{c_\epsilon}]^{-1}\|_{\mathbf{B}(L^\infty_{\pm q_\epsilon}, W^{2,\infty}_{\pm q_\epsilon})} = \mathcal{O}(\epsilon^{-1}). \tag{6.3.5}$$

Proof. We convert the problem $\mathcal{B}_{c_\epsilon} f = g$ with $f \in W^{2,\infty}_{-q_\epsilon}$, $g \in L^\infty_{-q_\epsilon}$ to an equivalent first-order system. Let

$$\mathcal{A}_c := \underbrace{\begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}}_{A_0(c)} + \underbrace{\begin{bmatrix} 0 & 0 \\ -2/c^2 & 0 \end{bmatrix}}_{A_1(c)} S^1 + \begin{bmatrix} 0 & 0 \\ -2/c^2 & 0 \end{bmatrix} S^{-1}, \tag{6.3.6}$$

Then $\mathcal{B}_{c_\epsilon} f = g$ is equivalent to

$$\underbrace{\partial_x[\mathbf{f}] - \mathcal{A}_{c_\epsilon}\mathbf{f}}_{\Lambda_\epsilon \mathbf{f}} = \mathbf{g}, \qquad \mathbf{f} := \begin{pmatrix} f \\ f' \end{pmatrix}, \qquad \mathbf{g} := \begin{pmatrix} 0 \\ g \end{pmatrix}. \tag{6.3.7}$$

Now set

$$\Delta_c(z) := z\mathbb{1} - (A_0(c) + A_1(c)e^z + A_1(c)e^{-z}) = \begin{bmatrix} z & -1 \\ \frac{1}{c^2}\big(2 + 2\cosh(z)\big) & z \end{bmatrix} \tag{6.3.8}$$

and observe that

$$\det(\Delta_c(z)) = z^2 + \frac{1}{c^2}\big(2 + 2\cosh(z)\big) = \frac{1}{c^2}\widetilde{\mathcal{B}}_c(iz), \tag{6.3.9}$$

where $\widetilde{\mathcal{B}}_c$ is defined in (4.0.4).

Using the terminology of Theorem E.1, the determinant of the characteristic equation of (6.3.7) is $\widetilde{\mathcal{B}}_{c_\epsilon}(iz)/c_\epsilon^2$. By (6.3.4), we have $0 < q_\epsilon < 1$, and so Proposition 4.2 implies that

$\widetilde{\mathcal{B}}_{c_\epsilon}(iz) \neq 0$ for $0 < |\operatorname{Re}(z)| < q_\epsilon$. We have therefore satisfied the hypotheses of Theorem E.1 for the system (6.3.7), and so we conclude that Λ_ϵ is invertible from $W^{1,\infty}_{\pm q_\epsilon}$ to $L^\infty_{\pm q_\epsilon}$.

We will refer to this inverse as $(\Lambda_\epsilon^\pm)^{-1}$. More precisely, given $\mathbf{g} \in L^\infty_{q_\epsilon}$, let $(\Lambda_\epsilon^+)^{-1}\mathbf{g}$ be the unique element of $W^{1,\infty}_{q_\epsilon}$ such that $\Lambda_\epsilon[(\Lambda_\epsilon^+)^{-1}\mathbf{g}] = \mathbf{g}$. Likewise, given $\mathbf{g} \in L^\infty_{-q_\epsilon}$, let $(\Lambda_\epsilon^-)^{-1}\mathbf{g}$ be the unique element of $W^{1,\infty}_{-q_\epsilon}$ such that $\Lambda_\epsilon[(\Lambda_\epsilon^-)^{-1}\mathbf{g}] = \mathbf{g}$. If we set

$$\delta_\epsilon := \frac{\mathscr{b}\epsilon}{2},$$

where $\mathscr{b}$ was defined in (6.3.1), then (E.1.5) gives a formula for $(\Lambda_\epsilon^\pm)^{-1}\mathbf{g}$:

$$(6.3.10) \quad (\Lambda_\epsilon^\pm)^{-1}\mathbf{g})(x) = \frac{1}{2\pi i}\int_{\pm q_\epsilon+\delta_\epsilon-i\infty}^{\pm q_\epsilon+\delta_\epsilon+i\infty} e^{xz}\Delta_{c_\epsilon}(z)^{-1}\mathscr{L}_+[\mathbf{g}](z)\,dz + \frac{1}{2\pi i}\int_{\pm q_\epsilon-\delta_\epsilon-i\infty}^{\pm q_\epsilon-\delta_\epsilon+i\infty} e^{xz}\Delta_{c_\epsilon}(z)^{-1}\mathscr{L}_-[\mathbf{g}](z)\,dz.$$

The function Δ_{c_ϵ} was defined in (6.3.8).

Now, recall that we have converted the one-dimensional problem $\mathcal{B}_{c_\epsilon}f = g$ to the vectorized equation $\Lambda_\epsilon\mathbf{f} = \mathbf{g}$ with $\mathbf{f} = (f, f')$ and $\mathbf{g} = (0, g)$, and so we are really interested in solving $\Lambda_\epsilon\mathbf{f} = \mathbf{g}$ for the first component of $\mathbf{f}$. We therefore take the dot product of (6.3.10) with the vector $\mathbf{e}_1 = (1, 0)$ and use the definition of $\Delta_{c_\epsilon}(z)$ in (6.3.8) to find

$$(6.3.11) \quad [\mathcal{B}_{c_\epsilon}^\pm]^{-1}g := ((\Lambda_\epsilon^\pm)^{-1}\mathbf{g})\cdot\mathbf{e}_1 = \frac{1}{2\pi i}\int_{\pm q_\epsilon+\delta_\epsilon-i\infty}^{\pm q_\epsilon+\delta_\epsilon+i\infty} \mathcal{K}_\epsilon(x,z)\mathscr{L}_+[g](z)\,dz + \frac{1}{2\pi i}\int_{\pm q_\epsilon-\delta_\epsilon-i\infty}^{\pm q_\epsilon-\delta_\epsilon+i\infty} \mathcal{K}_\epsilon(x,z)\mathscr{L}_-[g](z)\,dz,$$

where

$$(6.3.12) \qquad \mathcal{K}_\epsilon(x,z) := \frac{e^{xz}}{\widetilde{\mathcal{B}}_{c_\epsilon}(iz)}.$$

As above, given $g \in L^\infty_{\pm q_\epsilon}$, the function $[\mathcal{B}_{c_\epsilon}^\pm]^{-1}g$ is the unique element of $W^{1,\infty}_{\pm q_\epsilon}$ to satisfy $\mathcal{B}_{c_\epsilon}[[\mathcal{B}_{c_\epsilon}^\pm]^{-1}g] = g$. Note that, in fact, $[\mathcal{B}_{c_\epsilon}^\pm]^{-1}g \in W^{2,\infty}_{\pm q_\epsilon}$, since $\partial_x^2[[\mathcal{B}_{c_\epsilon}^\pm]^{-1}g] = \partial_x[(\Lambda_\epsilon^\pm)^{-1}\mathbf{g}\cdot\mathbf{e}_2] \in L^\infty_{\pm q_\epsilon}$.

Last, we prove the operator norm estimate (6.3.5). Part (i) of Theorem E.1 and some matrix-vector arithmetic imply the existence of a function $\mathscr{G}_\epsilon \in \cap_{p=1}^\infty L^p$ such that

$$(6.3.13) \qquad ([\mathcal{B}_{c_\epsilon}^\pm]^{-1}g)(x) = e^{q_\epsilon x}\int_{-\infty}^\infty e^{-q_\epsilon s}\mathscr{G}_\epsilon(x-s)g(s)\,ds,$$

where

$$\widehat{\mathscr{G}}_\epsilon(k) = \frac{1}{\widetilde{\mathcal{B}}_{c_\epsilon}(-k+iq_\epsilon)}.$$

We have

$$\|[\mathcal{B}_{c_\epsilon}^\pm]^{-1}g\|_{W^{2,\infty}_{\pm q_\epsilon}} = \|[\mathcal{B}_{c_\epsilon}^\pm]^{-1}g\|_{L^\infty_{\pm q_\epsilon}} + \|\partial_x^2[[\mathcal{B}_{c_\epsilon}^\pm]^{-1}g]\|_{L^\infty_{\pm q_\epsilon}}.$$

We calculate

$$\partial_x^2[[\mathcal{B}_{c_\epsilon}^{\pm}]^{-1}g] = g - \frac{(2+A)[\mathcal{B}_{c_\epsilon}^{\pm}]^{-1}g}{c^2},$$

and so

$$\|\partial_x^2[[\mathcal{B}_{c_\epsilon}^{\pm}]^{-1}g]\|_{L^\infty_{\pm q_\epsilon}} \le \|g\|_{L^\infty_{\pm q_\epsilon}} + C\|[\mathcal{B}_{c_\epsilon}^{\pm}]^{-1}g\|_{L^\infty_{\pm q_\epsilon}}.$$

Next, we estimate from (6.3.13) that

$$\|[\mathcal{B}_{c_\epsilon}^{\pm}]^{-1}\|_{L^\infty_{\pm q_\epsilon}} \le \|\mathscr{G}_\epsilon\|_{L^1} = \left\|\mathfrak{F}^{-1}\left[\frac{1}{\widetilde{\mathcal{B}}_{c_\epsilon}(-\cdot + iq_\epsilon)}\right]\right\|_{L^1} \le \frac{C_{\mathcal{B}}}{q_\epsilon} = \mathcal{O}(\epsilon^{-1}).$$

The last inequality comes from (4.0.8). □

6.3.3. *Inversion of $\mathcal{B}_{c_\epsilon} - \Sigma_{c_\epsilon}^*$ with the Neumann series.* Now consider the operator $\mathcal{B}_{c_\epsilon} - \Sigma_{c_\epsilon}^*$ from $W^{2,\infty}_{-q_\epsilon}$ to $L^\infty_{-q_\epsilon}$. For simplicity, let

$$\mathcal{M} := 2(2+A) \qquad \text{and} \qquad \widetilde{\mathcal{M}}(k) = 2(2+2\cos(k)), \tag{6.3.14}$$

so

$$\Sigma_{c_\epsilon}^* f = \varsigma_{c_\epsilon}(x)\mathcal{M}f.$$

By part (v) of Proposition D.2 we estimate, for $f \in W^{2,\infty}_{-q_\epsilon}$,

$$\|\Sigma_{c_\epsilon}^* f\|_{L^\infty_{-q_\epsilon}} \le \|e^{q_\epsilon\cdot}\varsigma_{c_\epsilon}\mathcal{M}f\|_{L^\infty} \le C\epsilon^2\|e^{q_\epsilon\cdot}\mathcal{M}f\|_{L^\infty} = C\epsilon^2\|\mathcal{M}f\|_{L^\infty_{-q_\epsilon}} \le C\epsilon^2\|f\|_{W^{2,\infty}_{-q_\epsilon}}.$$

That is,

$$\|\Sigma_{c_\epsilon}^*\|_{\mathbf{B}(W^{2,\infty}_{-q_\epsilon}, L^\infty_{-q_\epsilon})} = \mathcal{O}(\epsilon^2).$$

We may therefore invert $\mathcal{B}_{c_\epsilon} - \Sigma_{c_\epsilon}^*$ from $W^{2,\infty}_{-q_\epsilon}$ to $L^\infty_{-q_\epsilon}$ using the Neumann series. This verifies Hypothesis 3 for $|c| \gtrsim 1$. Specifically, we set $q_{\mathcal{L}}(c_\epsilon) := q_\epsilon$ from (6.3.3) and note from (6.3.4) that $q_\epsilon < \min\{q_\varsigma(c_\epsilon)/2, q_{\mathcal{H}}(c_\epsilon), 1\}$.

6.4. **Verification of Hypothesis 4.** To compute the phase shift ϑ_{c_ϵ} from Theorem 4.1, we at first follow our methods from Section 4.2 of a particular solution to $(\mathcal{B}_{c_\epsilon} - \Sigma_{c_\epsilon}^*)f = 0$. However, now we can use the Neumann series to get an explicit formula for the inverse of this operator from $W^{2,\infty}_{-q_\epsilon}$ to $L^\infty_{-q_\epsilon}$, namely,

$$\big[(\mathcal{B}_{c_\epsilon} - \Sigma_{c_\epsilon}^*)\big]^{-1} = \big[\mathcal{B}_{c_\epsilon}\big(\mathbb{1} - [\mathcal{B}_{c_\epsilon}^-]^{-1}\Sigma_{c_\epsilon}^*\big)\big]^{-1} = \big(\mathbb{1} - [\mathcal{B}_{c_\epsilon}^-]^{-1}\Sigma_{c_\epsilon}^*\big)^{-1}[\mathcal{B}_{c_\epsilon}^-]^{-1} = \sum_{k=0}^\infty \big([\mathcal{B}_{c_\epsilon}^-]^{-1}\Sigma_{c_\epsilon}^*\big)^k[\mathcal{B}_{c_\epsilon}^-]^{-1}.$$

As we did in Section 4.2, make the ansatz $f(x) = e^{i\omega_{c_\epsilon}x} + g(x)$, where $g \in W^{2,\infty}_{-q_\epsilon}$. Then $(\mathcal{B}_{c_\epsilon} - \Sigma_{c_\epsilon}^*)f = 0$ if and only if

$$(\mathcal{B}_{c_\epsilon} - \Sigma_{c_\epsilon}^*)g = \Sigma_{c_\epsilon}^* e^{i\omega_{c_\epsilon}\cdot}. \tag{6.4.1}$$

The Neumann series implies

$$g_\epsilon := \big[(\mathcal{B}_{c_\epsilon} - \Sigma_{c_\epsilon}^*)\big]^{-1}\Sigma_{c_\epsilon}^* e^{i\omega_{c_\epsilon}\cdot} = \sum_{k=0}^\infty \big([\mathcal{B}_{c_\epsilon}^-]^{-1}\Sigma_{c_\epsilon}^*\big)^{k+1} e^{i\omega_{c_\epsilon}\cdot} \tag{6.4.2}$$

and if we set, as before,

$$f_\epsilon(x) := e^{i\omega_{c_\epsilon}x} + g_\epsilon(x),$$

then

$$(\mathcal{B}_{c_\epsilon} - \Sigma_{c_\epsilon}^*)f_\epsilon = 0 \qquad \text{and} \qquad \lim_{x\to\infty} |f_\epsilon(x) - e^{i\omega_{c_\epsilon}x}| = 0. \tag{6.4.3}$$

Now we need asymptotics on f_ϵ as $x \to -\infty$. To develop these limits, we exploit a formula that relates $\mathcal{B}_{c_\epsilon}^-$ to $\mathcal{B}_{c_\epsilon}^+$. The proof is in Appendix E.2. Roughly speaking, this is motivated by our change of contours that converted the inverse Laplace transform (4.3.7) to (4.3.8).

Proposition 6.4. *There exists $\epsilon_{\mathsf{res}} \in (0, \epsilon_{\mathcal{B}}]$ such that for $0 < \epsilon < \epsilon_{\mathsf{res}}$ and $h \in L_{-q_\epsilon}^\infty$, we have $\varsigma_{c_\epsilon} h \in L_{q_\epsilon}^\infty$ and*

$$[\mathcal{B}_{c_\epsilon}^-]^{-1}[\varsigma_{c_\epsilon} h](x) = [\mathcal{B}_{c_\epsilon}^+]^{-1}[\varsigma_{c_\epsilon} h](x) + \alpha_\epsilon[h]e^{i\omega_{c_\epsilon}x} + \beta_\epsilon[h]e^{-i\omega_{c_\epsilon}x}, \tag{6.4.4}$$

where the linear functionals α_ϵ and β_ϵ are defined as

$$\alpha_\epsilon[h] := -i\left(\frac{\mathscr{L}_+[\varsigma_{c_\epsilon} h](i\omega_{c_\epsilon}) + \mathscr{L}_-[\varsigma_{c_\epsilon} h](i\omega_{c_\epsilon})}{(\widetilde{\mathcal{B}}_\epsilon)'(\omega_{c_\epsilon})}\right) \tag{6.4.5}$$

and

$$\beta_\epsilon[h] := i\left(\frac{\mathscr{L}_+[\varsigma_{c_\epsilon} h](-i\omega_{c_\epsilon}) + \mathscr{L}_-[\varsigma_{c_\epsilon} h](-i\omega_{c_\epsilon})}{(\widetilde{\mathcal{B}}_\epsilon)'(-\omega_{c_\epsilon})}\right). \tag{6.4.6}$$

The functionals α_ϵ and β_ϵ satisfy the estimate

$$\max\{|\alpha_\epsilon[h]|, |\beta_\epsilon[h]|\} \le C\epsilon\|h\|_{L_{-q_\epsilon}^\infty}. \tag{6.4.7}$$

Recall that f_ϵ has the form $f_\epsilon(x) = e^{i\omega_{c_\epsilon}x} + g_\epsilon(x)$, where, from the definition of g_ϵ above in (6.4.2),

$$g_\epsilon = \sum_{k=0}^\infty \left([\mathcal{B}_{c_\epsilon}^-]^{-1}\Sigma_{c_\epsilon}^*\right)^{k+1} e^{i\omega_{c_\epsilon}\cdot} = [\mathcal{B}_{c_\epsilon}^-]^{-1}[\varsigma_{c_\epsilon}\mathcal{M}h_\epsilon], \qquad h_\epsilon := \sum_{k=0}^\infty \left([\mathcal{B}_{c_\epsilon}^-]^{-1}\Sigma_{c_\epsilon}^*\right)^{k} e^{i\omega_{c_\epsilon}\cdot}. \tag{6.4.8}$$

We use Proposition 6.4 to rewrite g_ϵ as

$$g_\epsilon(x) = [\mathcal{B}_{c_\epsilon}^+]^{-1}[\varsigma_{c_\epsilon}\mathcal{M}h_\epsilon](x) + \alpha_\epsilon[\mathcal{M}h_\epsilon]e^{i\omega_{c_\epsilon}x} + \beta_\epsilon[\mathcal{M}h_\epsilon]e^{-i\omega_{c_\epsilon}x}. \tag{6.4.9}$$

This is the analogue of (4.3.8).

Since the image of $[\mathcal{B}_{c_\epsilon}^+]^{-1}$ consists of functions that decay to zero at $-\infty$, we have, as in (4.3.11),

$$\lim_{x\to-\infty} \left|f_\epsilon(x) - \left(e^{i\omega_{c_\epsilon}x} + \alpha_\epsilon\left[\mathcal{M}h_\epsilon\right]e^{i\omega_{c_\epsilon}x} + \beta_\epsilon\left[\mathcal{M}h_\epsilon\right]e^{-i\omega_{c_\epsilon}x}\right)\right| = 0. \tag{6.4.10}$$

It is possible to obtain very precise estimates on the coefficients on $e^{\pm i\omega_{c_\epsilon}x}$ in (6.4.10); the proof is in Appendix E.3.

Proposition 6.5. *There exists $\epsilon_\theta \in (0, \epsilon_{\mathcal{B}}]$ such that for all $0 < \epsilon < \epsilon_\theta$, there are numbers $\theta_{\epsilon,0}^+ = \mathcal{O}(1)$, $\theta_\epsilon^+ = \mathcal{O}(1)$, and $\theta_\epsilon^- = \mathcal{O}(1)$ with*

$$\alpha_\epsilon[\mathcal{M}g_\epsilon] = i\epsilon\theta_{\epsilon,0}^+ + \epsilon^2\theta_\epsilon^+ \qquad \text{and} \qquad \beta_\epsilon[\mathcal{M}g_\epsilon] = \epsilon^2\theta_\epsilon^-.$$

Moreover, $\theta_{\epsilon,0}^+ \in \mathbb{R}$, and there are constants C_1, $C_2 > 0$ such that

$$0 < C_1 < \theta_{\epsilon,0}^+ < C_2 < \infty$$

for all such ϵ.

Set $\breve{f}_\epsilon(x) := \mathrm{Im}[f_\epsilon(x) - f_\epsilon(-x)]$ and write

$$\theta_\epsilon^+ = \theta_{\epsilon,\mathsf{r}}^+ + i\theta_{\epsilon,\mathsf{i}}^+ \qquad \text{and} \qquad \theta_\epsilon^- = \theta_{\epsilon,\mathsf{r}}^- + i\theta_{\epsilon,\mathsf{i}}^-$$

with $\theta_{\epsilon,\mathsf{r}}^\pm$, $\theta_{\epsilon,\mathsf{i}}^\pm \in \mathbb{R}$. The limits (6.4.3) and (6.4.10) imply

$$\lim_{x\to\infty} \left| \breve{f}_\epsilon(x) - \left[(2 + \epsilon^2(\theta_{\epsilon,\mathsf{r}}^+ - \theta_{\epsilon,\mathsf{r}}^-)) \sin(\omega_{c_\epsilon} x) - \epsilon(\theta_{\epsilon,0}^+ + \epsilon\theta_{\epsilon,\mathsf{i}}^+ + \epsilon\theta_{\epsilon,\mathsf{i}}^-) \cos(\omega_{c_\epsilon} x)\right] \right| = 0.$$

As in Section 4.4, then, the identity (4.4.6) implies that a rescaled version of $\breve{f}_\epsilon$, which we call γ_ϵ, satisfies both $(\mathcal{B}_{c_\epsilon} - \Sigma_{c_\epsilon}^*)\gamma_\epsilon = 0$ and

$$\lim_{x\to\infty} |\gamma_\epsilon(x) - \sin(\omega_{c_\epsilon}(x + \vartheta_{c_\epsilon}))| = 0,$$

where

$$\vartheta_{c_\epsilon} := -\frac{1}{\omega_{c_\epsilon}} \arctan\left(\epsilon\left(\frac{\theta_{\epsilon,0}^+ + \epsilon\theta_{\epsilon,\mathsf{i}}^+ + \epsilon\theta_{\epsilon,\mathsf{i}}^-}{2 + \epsilon^2(\theta_{\epsilon,\mathsf{r}}^+ - \theta_{\epsilon,\mathsf{r}}^-)}\right)\right). \tag{6.4.11}$$

Since $\theta_{\epsilon,0}^+$ is nonzero and $\mathcal{O}(1)$, it follows from properties of the arctangent that we can write

$$\vartheta_{c_\epsilon} = \epsilon\vartheta_{\epsilon,0},$$

where, for some constant $C_\vartheta > 0$, we have

$$0 < \frac{1}{C_\vartheta} < |\vartheta_{\epsilon,0}| < C_\vartheta < \infty$$

for all $0 < \epsilon < \epsilon_\theta$.

At last, we may verify Hypothesis 4 for $|c| \gtrsim 1$. By part (ii) of Proposition 3.1, we have $A_{c_\epsilon} < \omega_{c_\epsilon} < B_{c_\epsilon} < \pi/2$ for all $|\mu| \le \mu_{\mathrm{per}}(c_\epsilon)$. From part (iii) of that proposition, we deduce the additional bound $1 < A_{c_\epsilon}$ for $0 < \epsilon < \epsilon_{\mathcal{B}}$. Then

$$0 < \frac{\epsilon\omega_{c_\epsilon}}{C_\vartheta} < \epsilon\omega_{c_\epsilon}|\vartheta_{cep,0}| < C_\vartheta\epsilon\omega_{c_\epsilon} < \pi$$

for ϵ small enough, and so

$$\sin(\omega_{c_\epsilon}\vartheta_{c_\epsilon}) = \sin(\epsilon\omega_{c_\epsilon}\vartheta_{\epsilon,0}) > 0.$$

Appendix A. Transform Analysis

A.1. **Fourier analysis.** If $\mathbf{f} \in L^1(\mathbb{R}, \mathbb{C}^m)$, we set

$$\mathfrak{F}[\mathbf{f}](k) = \widehat{\mathbf{f}}(k) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathbf{f}(x)e^{-ikx}\, dx$$

and

$$\mathfrak{F}^{-1}[\mathbf{f}](x) = \check{\mathbf{f}}(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathbf{f}(k)e^{ikx}\, dx.$$

If $\mathbf{f} \in L^2_{\mathrm{per}}(\mathbb{R}, \mathbb{C}^m)$, we define

$$\mathfrak{F}[\mathbf{f}](k) = \widehat{\mathbf{f}}(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathbf{f}(x)e^{-ikx}\, dx,$$

and we have

$$\mathbf{f}(x) = \sum_{k=-\infty}^{\infty} \widehat{\mathbf{f}}(k)e^{ikx}.$$

In either case, with $(S^d\mathbf{f})(x) := f(x+d)$, we have the identity

$$\widehat{S^d\mathbf{f}}(k) = e^{ikd}\widehat{\mathbf{f}}(k).$$

A.1.1. *Fourier multipliers.* We first work on the periodic Sobolev spaces $H^r_{\mathrm{per}}(\mathbb{R},\mathbb{C}^m)$ from (3.0.1). Take r, $s \geq 0$ and suppose that $\widetilde{\mathcal{M}}\colon \mathbb{R} \to \mathbb{C}^{m\times m}$ is measurable with

$$\sup_{k\in\mathbb{R}} \frac{\|\widetilde{\mathcal{M}}(k)\|}{(1+k^2)^{(r-s)/2}} < \infty.$$

Then Fourier multiplier $\mathcal{M}$ with symbol $\widetilde{\mathcal{M}}$, defined by

$$(\mathcal{M}\mathbf{f})(x) := \sum_{k=-\infty}^{\infty} e^{ikx}\widetilde{\mathcal{M}}(k)\widehat{\mathbf{f}}(k),$$

i.e., by

$$\widehat{\mathcal{M}\mathbf{f}}(k) = \widetilde{\mathcal{M}}(k)\widehat{\mathbf{f}}(k), \tag{A.1.1}$$

is a bounded operator from $H^r_{\mathrm{per}}(\mathbb{R},\mathbb{C}^m)$ to $H^s_{\mathrm{per}}(\mathbb{R},\mathbb{C}^m)$.

We will need some calculus on "scaled" Fourier multipliers. Let $\mathcal{M}$ be the Fourier multiplier with symbol $\widetilde{\mathcal{M}}$. For $\omega \in \mathbb{R}$, define $\mathcal{M}[\omega]$ to be the Fourier multiplier with symbol $\widetilde{\mathcal{M}}(\omega k)$, i.e.,

$$\widehat{\mathcal{M}[\omega]f}(k) := \widetilde{\mathcal{M}}(\omega k)\widehat{f}(k).$$

Last, for a function $\mathbf{g}\colon \mathbb{R} \to \mathbb{C}^m$, let

$$\mathrm{Lip}(\mathbf{g}) := \sup_{\substack{x,\grave{x}\in\mathbb{R} \\ x\neq\grave{x}}} \left|\frac{\mathbf{g}(x)-\mathbf{g}(\grave{x})}{x-\grave{x}}\right|. \tag{A.1.2}$$

The methods of Appendix D.3 of [Fav18] prove the next lemma.

Lemma A.1. *Let $\widetilde{\mathcal{M}}\colon \mathbb{R} \to \mathbb{C}^{m\times m}$ be measurable.*

(i) *Suppose* $\mathrm{Lip}(\widetilde{\mathcal{M}}) < \infty$. *Then*

$$\|(\mathcal{M}[\omega]-\mathcal{M}[\grave{\omega}])\mathbf{f}\|_{H^r_{\mathrm{per}}(\mathbb{R},\mathbb{C}^m)} \leq \mathrm{Lip}(\widetilde{\mathcal{M}})\|\mathbf{f}\|_{H^{r+1}_{\mathrm{per}}(\mathbb{R},\mathbb{C}^m)}|\omega-\grave{\omega}|.$$

(ii) *Suppose $\widetilde{\mathcal{M}}$ is differentiable with* $\mathrm{Lip}(\widetilde{\mathcal{M}}') < \infty$. *Fix $\omega_0 \in \mathbb{R}$ and let $\partial_\omega\mathcal{M}[\omega_0]$ be the Fourier multiplier with symbol $k\widetilde{\mathcal{M}}'(\omega_0 k)$, i.e.,*

$$\mathfrak{F}[\partial_\omega\mathcal{M}[\omega_0]f](k) = k\widetilde{\mathcal{M}}'(\omega_0 k)\widehat{f}(k).$$

Then

$$\|(\mathcal{M}[\omega_0+\omega]-\mathcal{M}[\omega_0]-\omega\partial_\omega\mathcal{M}[\omega_0])\mathbf{f}\|_{H^r_{\mathrm{per}}(\mathbb{R},\mathbb{C}^m)} \leq \mathrm{Lip}(\widetilde{\mathcal{M}}')\omega^2\|\mathbf{f}\|_{H^{r+2}_{\mathrm{per}}(\mathbb{R},\mathbb{C}^m)}.$$

Next, we discuss some properties of the adjoint of a Fourier multiplier on periodic Sobolev spaces.

Lemma A.2. *Suppose that $\mathcal{M} \in \mathbf{B}(H^r_{\mathrm{per}}(\mathbb{R},\mathbb{C}^m), H^s_{\mathrm{per}}(\mathbb{R},\mathbb{C}^m))$ is a Fourier multiplier with symbol $\widetilde{\mathcal{M}}\colon \mathbb{R} \to \mathbb{C}^{m\times m}$.*

(i) *The adjoint operator* $\mathcal{M}^* \in \mathbf{B}(H^s_{\mathrm{per}}(\mathbb{R},\mathbb{C}^m), H^r_{\mathrm{per}}(\mathbb{R},\mathbb{C}^m))$, *i.e., the operator* $\mathcal{M}^*$ *satisfying*

$$\langle \mathcal{M}\mathbf{f},\mathbf{g}\rangle_{H^s_{\mathrm{per}}(\mathbb{R},\mathbb{C}^m)} = \langle \mathbf{f},\mathcal{M}^*\mathbf{g}\rangle_{H^r_{\mathrm{per}}(\mathbb{R},\mathbb{C}^m)}, \ \mathbf{f}\in H^r_{\mathrm{per}}(\mathbb{R},\mathbb{C}^m), \ \mathbf{g}\in H^s_{\mathrm{per}}(\mathbb{R},\mathbb{C}^m),$$

is given by

$$\widehat{\mathcal{M}^*\mathbf{g}}(k) = \frac{1}{(1+k^2)^{r-s}}\widetilde{\mathcal{M}}(k)^*\widehat{\mathbf{g}}(k), \tag{A.1.3}$$

where $\widetilde{\mathcal{M}}(k)^* \in \mathbb{C}^{m\times m}$ *is the conjugate transpose of* $\widetilde{\mathcal{M}}(k) \in \mathbb{C}^{m\times m}$.

(ii) *Suppose that* $\mathcal{M} \in \mathbf{B}(H^r_{\mathrm{per}}(\mathbb{R},\mathbb{C}^m), H^s_{\mathrm{per}}(\mathbb{R},\mathbb{C}^m))$ *is a Fourier multiplier with symbol* $\widetilde{\mathcal{M}}\colon \mathbb{R}\to\mathbb{C}^{m\times m}$. *Suppose as well that* $\mathcal{M}$ *has a one-dimensional kernel spanned by* $\boldsymbol{\nu}$ *with* $\widehat{\boldsymbol{\nu}}(k) = 0$ *for all but finitely many* k *and that* $\widetilde{\mathcal{M}}(k)$ *is self-adjoint for all* $k\in\mathbb{R}$. *Then the adjoint* $\mathcal{M}^*$ *from part (i) has a one-dimensional kernel in* $H^r_{\mathrm{per}}(\mathbb{R},\mathbb{C}^m)$ *spanned by* $\boldsymbol{\nu}^*$, *whose Fourier coefficients are*

$$\widehat{\boldsymbol{\nu}^*}(k) := (1+k^2)^{(r-s)/2}\widehat{\boldsymbol{\nu}}(k). \tag{A.1.4}$$

Last, we state a slight generalization of a result for Fourier multipliers on the exponentially weighted spaces H^r_q from (1.4.1). In this case, a Fourier multiplier on H^r_q (or H^r) is, of course, defined as before by (A.1.1). For $0 < q < \grave{q}$, write

$$S_{q,\grave{q}} := \{z\in\mathbb{C} \mid q \le |\operatorname{Im}(z)| \le \grave{q}\}.$$

Lemma A.3 (Beale)**.** *Let* $0 < q_0 \le q_1 < q_2$ *and suppose that* $\widetilde{\mathcal{M}}\colon\mathbb{R}\to\mathbb{C}$ *is a measurable function with the following properties.*

($\mathcal{M}$1) *The function* $\widetilde{\mathcal{M}}$ *is analytic on the strips* S_{0,q_1} *and* S_{q_1,q_2}.

($\mathcal{M}$2) *The function* $\widetilde{\mathcal{M}}$ *has finitely many zeros in* $\mathbb{R}$, *all of which are simple. Denote the collection of these zeros by* $\mathfrak{P}_{\mathcal{M}}$.

($\mathcal{M}$3) *There exist* C, $z_0 > 0$ *and* $s \ge 0$ *such that if* $z\in S_{0,q_1}\cup S_{q_1,q_2}$ *with* $|z|\ge z_0$, *then*

$$C|\operatorname{Re}(z)|^s \le |\widetilde{\mathcal{M}}(z)|. \tag{A.1.5}$$

Now let $\mathcal{M}$ *be the Fourier multiplier with symbol* $\widetilde{\mathcal{M}}$. *There exist* $q_\star$, $q_{\star\star} > 0$ *with* $q_1 \le q_\star < q_{\star\star} \le q_2$ *such that if* $q\in[q_\star,q_{\star\star}]$, *then, for any* $r\ge 0$, $\mathcal{M}$ *is invertible from* H^{r+s}_q *to the subspace*

$$\mathfrak{D}^r_{\mathcal{M},q} := \left\{f\in H^r_q \;\middle|\; z\in\mathfrak{P}_{\mathcal{M}} \Longrightarrow \widehat{f}(z) = 0\right\}$$

and, for $f\in\mathfrak{D}^r_{\mathcal{M},q}$,

$$\|\mathcal{M}^{-1}f\|_{r+s,q} \le \left(\sup_{k\in\mathbb{R}} \frac{(1+k^2)^{s/2}}{|\widetilde{\mathcal{M}}(k\pm iq)|}\right)\|f\|_{r,s}.$$

There are two additional special cases.

(i) *If* $q_1 = q_2$, *then the result above is true for all* $0 < q \le q_2$.

(ii) *If* $\mathfrak{P}_{\mathcal{M}} = \{0\}$ *and 0 is a double zero of* $\widetilde{\mathcal{M}}$, *then the result above is still true if we replace* H^r_q *and* H^{r+s}_q *by* E^r_q *and* E^{r+s}_q *throughout.*

This lemma was proved by Beale as Lemma 3 in [Bea80] in the particular case (i).

A.2. **The Laplace transform.** Suppose $\mathbf{f} \in L^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{C}^m)$ with $e^{-a\cdot}\mathbf{f} \in L^\infty(\mathbb{R}, \mathbb{C}^m)$. We set

$$\mathscr{L}_+[\mathbf{f}](z) := \int_0^\infty \mathbf{f}(s)e^{-sz}\ ds,\ \mathrm{Re}(z) > a$$

and

$$\mathscr{L}_-[\mathbf{f}](z) := \int_0^\infty \mathbf{f}(-s)e^{sz}\ ds,\ \mathrm{Re}(z) < a.$$

These Laplace transforms have the following useful properties. First, $\mathscr{L}_+[\mathbf{f}]$ is analytic on $\mathrm{Re}(z) > a$ and $\mathscr{L}_-[\mathbf{f}]$ is analytic on $\mathrm{Re}(z) < a$. In particular, if $e^{-a|\cdot|}\mathbf{f} \in L^\infty(\mathbb{R}, \mathbb{C}^m)$ with $a > 0$, then $\mathscr{L}_+[\mathbf{f}]$ and $\mathscr{L}_-[\mathbf{g}]$ are defined and analytic on the common strip $|\mathrm{Re}(z)| < a$. Next, we have the inverse formulas

$$\frac{\mathbf{f}(x^+) + \mathbf{f}(x^-)}{2} = \frac{1}{2\pi i}\int_{\mathrm{Re}(z)=b} \mathscr{L}_+[\mathbf{f}](z)e^{xz}\ dz,\ x > 0,\ b > a \tag{A.2.1}$$

and

$$\frac{\mathbf{f}(x^+) + \mathbf{f}(x^-)}{2} = \frac{1}{2\pi i}\int_{\mathrm{Re}(z)=b} \mathscr{L}_-[\mathbf{f}](-z)e^{-xz}\ dz,\ x < 0,\ b > -a, \tag{A.2.2}$$

where

$$\mathbf{f}(x^\pm) := \lim_{t \to x^\pm} \mathbf{f}(t).$$

If $\mathbf{f}$ is differentiable, then

$$\mathscr{L}_\pm[\mathbf{f}'](z) = \mp\mathbf{f}(0) + z\mathscr{L}_\pm[\mathbf{f}](z). \tag{A.2.3}$$

Last, the Laplace transform interacts with shift operators via the identities

$$\mathscr{L}_+[S^d\mathbf{f}](z) = e^{zd}\mathscr{L}_+[\mathbf{f}](z) - e^{zd}\int_0^d f(s)e^{-sz}\ ds \tag{A.2.4}$$

and

$$\mathscr{L}_-[S^d\mathbf{f}](z) = e^{zd}\mathscr{L}_-[\mathbf{f}](z) + e^{zd}\int_{-d}^0 \mathbf{f}(-s)e^{sz}\ ds. \tag{A.2.5}$$

Appendix B. Derivation of the Traveling Wave Problem (1.3.1)

Under the traveling wave ansatz (1.1.3), the original equations of motion (1.1.2) for the diatomic FPUT lattice convert to the system

$$\begin{cases} c^2p_1'' = -(1+w)(p_1 + p_1^2) + (wS^1 + S^{-1})(p_2 + p_2)^2 \\ \\ c^2p_2'' = (wS^{-1} + S^1)(p_1 + p_1^2) - (1+w)(p_2 + p_2^2). \end{cases} \tag{B.0.1}$$

Here we define $w := 1/m$. With $\mathbf{p} = (p_1, p_2)$, (B.0.1) compresses into

$$c^2\mathbf{p}'' + L_w\mathbf{p} + L_w\mathbf{p}^{.2} = 0, \qquad L_w := \begin{bmatrix} (1+w) & -(wS^1 + S^{-1}) \\ -(wS^{-1} + S^1) & (1+w) \end{bmatrix}. \tag{B.0.2}$$

For $\mathbf{f} = (f_1, f_2)$ and $\mathbf{g} = (g_1, g_2)$, we use the notation

$$\mathbf{f}^{.2} := \begin{pmatrix} f_1^2 \\ f_2^2 \end{pmatrix} \qquad \text{and} \qquad \mathbf{f}.\mathbf{g} := \begin{pmatrix} f_1g_1 \\ g_2g_2 \end{pmatrix}.$$

We note that (B.0.2) is the same system that was derived in equation (2.4) in [FW18].

Set

$$J := \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix},$$

so $J^{-1} = J/2$. If we substitute $\mathbf{p} = J\boldsymbol{\rho}$ with $\boldsymbol{\rho} = (\rho_1, \rho_2)$, then (B.0.2) becomes

$$c^2 J\boldsymbol{\rho}'' + L_w J\boldsymbol{\rho} + L_w (J\boldsymbol{\rho})^{.2} = 0.$$

Multiplying through by J^{-1}, this is equivalent to

$$c^2 \boldsymbol{\rho}'' + \frac{1}{2} J L_w J\boldsymbol{\rho} + \frac{1}{2} J L_w (J\boldsymbol{\rho})^{.2} = 0. \tag{B.0.3}$$

Setting $w = 1 + \mu$ for $|\mu| < 1$ and

$$\mathcal{D}_\mu = \frac{1}{2} J L_{1+\mu} J \qquad \text{and} \qquad \mathcal{Q}(\boldsymbol{\rho}, \grave{\boldsymbol{\rho}}) := (J\boldsymbol{\rho}).(J\grave{\boldsymbol{\rho}})$$

we see that (B.0.3) is equivalent to (1.3.1). One easily checks that for functions $\boldsymbol{\rho}$, $\grave{\boldsymbol{\rho}}$, and $\breve{\boldsymbol{\rho}}$ and scalars a, we have

(B.0.4)

$$\mathcal{Q}(a\boldsymbol{\rho}, \grave{\boldsymbol{\rho}}) = a\mathcal{Q}(\boldsymbol{\rho}, \grave{\boldsymbol{\rho}}), \qquad \mathcal{Q}(\boldsymbol{\rho}, \grave{\boldsymbol{\rho}}) = \mathcal{Q}(\grave{\boldsymbol{\rho}}, \boldsymbol{\rho}) \qquad \text{and} \qquad \mathcal{Q}(\boldsymbol{\rho} + \grave{\boldsymbol{\rho}}, \breve{\boldsymbol{\rho}}) = \mathcal{Q}(\boldsymbol{\rho}, \breve{\boldsymbol{\rho}}) + \mathcal{Q}(\grave{\boldsymbol{\rho}}, \breve{\boldsymbol{\rho}}),$$

and so $\mathcal{Q}$ is indeed symmetric and bilinear.

Last, we discuss the even-odd symmetries of $\mathcal{G}_c$. First, observe that if f is even, then Af is even and δf is odd, while if f is odd, then Af is odd and δf is even. So, if ρ_1 is even and ρ_2 is odd and $\boldsymbol{\rho} = (\rho_1, \rho_2)$, then $(\mathcal{D}_\mu \boldsymbol{\rho}) \cdot \mathbf{e}_1$ is even and $(\mathcal{D}_\mu \boldsymbol{\rho}) \cdot \mathbf{e}_2$ is odd, where $\mathbf{e}_1 = (1, 0)$ and $\mathbf{e}_2 = (0, 1)$. Likewise, if ρ_1 and $\grave{\rho}_1$ are even and ρ_2 and $\grave{\rho}_2$ are odd, then $\mathcal{Q}(\boldsymbol{\rho}, \grave{\boldsymbol{\rho}}) \cdot \mathbf{e}_1$ is even and $\mathcal{Q}(\boldsymbol{\rho}, \grave{\boldsymbol{\rho}}) \cdot \mathbf{e}_2$ is odd. We conclude that $\mathcal{G}_c(\boldsymbol{\rho}, \mu) \cdot \mathbf{e}_1$ is even and $\mathcal{G}_c(\boldsymbol{\rho}, \mu) \cdot \mathbf{e}_2$ is odd when ρ_1 is even and ρ_2 is odd.

Next, if f is integrable on $\mathbb{R}$ or $2P$-periodic and integrable on $[-P, P]$, then $(2 - A)f$ and δf are "mean-zero" in the sense that

$$\int_{-\infty}^{\infty} \big((2 - A)f\big)(x)\, dx = \int_{-P}^{P} \big((2 - A)f\big)(x)\, dx = \int_{-\infty}^{\infty} (\delta f)(x)\, dx = \int_{-P}^{P} (\delta f)(x)\, dx = 0.$$

Thanks to the structure of $\mathcal{D}_\mu$, all terms in $\mathcal{G}_c(\boldsymbol{\rho}, \mu) \cdot \mathbf{e}_1$ contain either a factor of $2 - A$ or δ, and so $\mathcal{G}_c(\boldsymbol{\rho}, \mu) \cdot \mathbf{e}_1$ is always mean-zero. We conclude that the symmetries in (1.3.4) hold.

Appendix C. Existence of Periodic Solutions

C.1. **Linear analysis.** We begin with two propositions that study the linearization $\Gamma_c^\mu[\omega]$ defined in (3.0.4). The first of these contains all the technical details needed to prove Proposition 3.1.

Lemma C.1. *For each* $|c| > 1$, *there is* $\mathrm{M}(c) > 0$ *with the following properties.*

(i) *The functions* $\lambda_\mu^\pm$ *defined in* (3.0.7) *are even, real-valued, and bounded on* $\mathbb{R}$ *with*

$$0 \le \lambda_\mu^-(K) \le \begin{cases} 2(1+\mu), & \mu \in (-1, 0] \\ 2, & \mu \in [0, 1) \end{cases} \tag{C.1.1}$$

and

$$\begin{cases} 2, & \mu \in (-1,0] \\ 2(1+\mu), & \mu \in [0,1) \end{cases} \le \lambda_\mu^+(K) \le 2 + 2(1+\mu) \tag{C.1.2}$$

for all $K \in \mathbb{R}$. For $\mu \neq 0$, $\lambda_\mu^\pm$ are differentiable on $\mathbb{R}$, while $\lambda_0^\pm$ are continuous on $\mathbb{R}$ and differentiable except at the points $K = (2j+1)\pi/2$, $j \in \mathbb{Z}$.

(ii) *For $|\mu| < \mathrm{M}(c)$, the functions*

$$K \mapsto \Lambda_c^\pm(K,\mu) := c^2K^2 - \lambda_\mu^\pm(K)$$

are strictly increasing on $(0,\infty)$.

(iii) *$\Lambda_c^-(K,\mu) = 0$ if and only if $K = 0$.*

(iv) *For all $|\mu| \le \mathrm{M}(c)$, there is a unique $\omega_c^\mu > 0$ such that $\Lambda_c^+(\omega_c^\mu, \mu) = 0$. Moreover, there are numbers $0 < A_c < B_c < \pi/2$ such that $A_c < \omega_c^\mu < B_c$ for each such μ. This root ω_c^μ also satisfies the cruder bounds*

$$\sqrt{\frac{2}{c^2(1+\mu)}} \le \omega_c^\mu \le \sqrt{\frac{2(2+\mu)}{c^2(1+\mu)}}. \tag{C.1.3}$$

(v) *$\omega_c^\mu - \omega_c = \mathcal{O}_c(\mu)$.*

Proof. **(i)** We recall that the eigenvalue curves $\lambda_\mu^\pm$ were defined in (3.0.7). Restricting $|\mu| \le 1$, we have

$$|\mu| \le \sqrt{\mu^2 + 4(1+\mu)\cos^2(K)} \le \sqrt{\mu^2 + 4(1+\mu)} = \sqrt{(2+\mu)^2} = 2 + \mu. \tag{C.1.4}$$

Then

$$0 \le \lambda_\mu^-(K) \le 2 + \mu - |\mu| = \begin{cases} 2(1+\mu), & \mu \in (-1,0) \\ 2, & \mu \in [0,1). \end{cases}$$

This proves (C.1.1).

Next, we use (C.1.4) to bound

$$2 + \mu + |\mu| \le \lambda_\mu^+(K) \le 2(2+\mu),$$

where

$$2 + \mu + |\mu| = \begin{cases} 2, & \mu \in (-1,0) \\ 2(1+\mu), & \mu \in [0,1). \end{cases}$$

This proves (C.1.2).

(ii) We prove this separately for the cases $\mu \in (-1,0)$, $\mu \in (0,1)$, and $\mu = 0$. First, when $\mu \neq 0$, the derivatives $(\lambda_\mu^\pm)'$ satisfy

$$|(\lambda_\mu^\pm)'(K)| = 4(1+\mu)\left|\frac{\sin(K)\cos(K)}{\sqrt{\mu^2 + 4(1+\mu)\cos^2(K)}}\right|. \tag{C.1.5}$$

The case $\mu \in (-1, 0]$. We rewrite (C.1.5) as

$$|(\lambda_\mu^\pm)'(K)| = 2\left(\frac{2(1+\mu)}{2+\mu}\right)|\sin(K)|\sqrt{\frac{1-\sin^2(K)}{1-\dfrac{4(1+\mu)}{(2+\mu)^2}\sin^2(K)}}.$$

We can check that

$$0 < \frac{2(1+\mu)}{2+\mu} \le 1 \qquad \text{and} \qquad 0 < \frac{4(1+\mu)}{(2+\mu)^2} \le 1$$

for $\mu \in (-1, 0]$ and we also have the estimate

$$\sup_{0 \le s \le 1} \frac{1-s}{1-rs} \le 1$$

when $0 < r < 1$. Combining these estimates, we find

$$|(\lambda_\mu^\pm)'(K)| \le 2|\sin(K)| \le 2|K|. \tag{C.1.6}$$

Taking $K > 0$, we have

$$\partial_K[\Lambda_c^\pm(K,\mu)] = 2c^2K - (\lambda_\mu^\pm)'(K) \ge 2c^2K - 2K = 2(c^2-1)K > 0.$$

The case $\mu \in (0, 1)$. We claim

$$\sup_{0 \le s \le 1} \frac{s}{\sqrt{\mu^2 + 4(1+\mu)s^2}} \le \frac{1}{2} \tag{C.1.7}$$

when $\mu \in (0, 1)$. This inequality is equivalent to

$$0 \le \mu^2 + 4\mu s^2$$

for all $s \in [0, 1]$, and this clearly holds for $\mu \in (0, 1)$. Hence (C.1.7) is true, and we use that estimate on (C.1.5) to find

$$|(\lambda_\mu^\pm)'(K)| = 4(1+\mu)|\sin(K)|\frac{|\cos(K)|}{\sqrt{\mu^2 + 4(1+\mu)\cos^2(K)}} \le 4(1+\mu)|K|\frac{1}{2} = 2(1+\mu)|K|. \tag{C.1.8}$$

Now we estimate

$$\partial_K[\Lambda_c^\pm(K,\mu)] = 2c^2K - (\lambda_\mu^\pm)'(K) > 2c^2K - 2(1+\mu)K = 2(c^2-(1+\mu))K > 0.$$

This last inequality holds if $c^2-(1+\mu) > 0$, and we can ensure this by taking $|\mu| < (c^2-1)/4$. So, we take our threshold for μ to be

$$\mathrm{M}(c) := \min\left\{1, \frac{c^2-1}{4}\right\} \tag{C.1.9}$$

and assume in the following that $|\mu| \le \mathrm{M}(c)$. Note that $\mathrm{M}(c) \to 0^+$ as $|c| \to 1^+$, and so the μ-interval over which we work necessarily shrinks as $|c| \to 1^+$.

The case $\mu = 0$. We have

$$\Lambda_c^\pm(K, 0) = c^2K^2 - 2 \mp 2|\cos(K)|,$$

and so the maps $\Lambda_c^\pm(\cdot, 0)$ are differentiable on all intervals of the form

$$\left(\frac{(2j+1)\pi}{2}, \frac{(2j+3)\pi}{2}\right), \tag{C.1.10}$$

with derivative equal to

$$2c^2K \pm 2\sin(K),$$

with the $\pm$ sign depending on the particular interval. Since

$$2c^2K \pm 2\sin(K) > 0$$

for all $K > 0$, and so we conclude that the maps $\Lambda_c^{\pm}(\cdot, 0)$ are strictly increasing on all intervals of the form (C.1.10). Since the maps $\Lambda_c^{\pm}(\cdot, 0)$ are continuous, it follows that they are strictly increasing on all of $(0, \infty)$ and therefore have at most one real root.

(iii) This follows from part (ii), the calculation $\Lambda_c^-(0, \mu) = 0$, and the evenness of $\Lambda_c^-(\cdot, \mu)$.

(iv) Using the bounds (C.1.2) on λ_μ^+, it is straightforward to show that $\Lambda_c^+(K, \mu) < 0$ for $K < \sqrt{2/c^2(1+\mu)}$ and $\Lambda_c^+(K, \mu) > 0$ for $K > \sqrt{2(2+\mu)/c^2(1+\mu)}$. Consequently, there exists $\omega_c^\mu \in [\sqrt{2/c^2(1+\mu)}, \sqrt{2(2+\mu)/c^2(1+\mu)}]$ such that $\Lambda_\mu^+(\omega_c^\mu, \mu) = 0$. This ω_c^μ is necessarily unique because $\Lambda_c^+(\cdot, \mu)$ is strictly increasing on $(0, \infty)$.

We now want sharper, uniform bounds of the form $0 < A_c < \omega_c^\mu < B_c < \pi/2$, which the estimates on ω_c^μ in the preceding paragraph do not necessarily give. Let

$$A_c := \frac{\sqrt{2}}{|c|}. \tag{C.1.11}$$

Since $|c| > 1$, we have $A_c < \sqrt{2} < \pi/2$. The bound (C.1.2) implies $-\lambda_\mu^+(K) \leq -2$ for all K, and so if $0 < K \leq A_c$, then

$$\Lambda_c^+(K, \mu) < 2 - \lambda_\mu^+(K) \leq 0.$$

In particular, $\Lambda_c^+(A_c, \mu) < 0$.

We use (C.1.2) again to estimate

$$\Lambda_c^+\left(\frac{\pi}{2}, \mu\right) = \frac{c^2\pi^2}{4} - (2 + \mu + |\mu|) \geq 2(c^2 - (1 + |\mu|)) \geq 2(c^2 - (1 + \mathrm{M}(c)) > 0 \tag{C.1.12}$$

by the definition of $\mathrm{M}(c)$ in (C.1.9).

All together, we have the chain of inequalities

$$\Lambda_c^+(A_c, \mu) \leq 0 < c^2 - (1 + \mathrm{M}(c)) < \Lambda_c^+\left(\frac{\pi}{2}, \mu\right),$$

and so the intermediate value theorem produces $B_c^\mu \in (A_c, \pi/2)$ such that

$$\Lambda_c^+(B_c^\mu, \mu) = c^2 - (1 + \mathrm{M}(c)) > 0. \tag{C.1.13}$$

A second application of the intermediate value theorem then yields $\omega_c^\mu \in (A_c, B_c^\mu)$ such that $\Lambda_c^+(\omega_c^\mu, \mu) = 0$. This is, of course, the same ω_c^μ that we initially found using the cruder bounds above.

Now let

$$B_c := \sup_{|\mu| \leq \mathrm{M}(c)} B_c^\mu.$$

We want to show $A_c < B_c < \pi/2$. The first inequality is obvious since $A_c < B_c^\mu$. If $B_c = \pi/2$, then we may take a sequence $(\mu_n(c))$ in $[-\mathrm{M}(c), \mathrm{M}(c)]$ such that

$$\lim_{n\to\infty} B_c^{\mu_n(c)} = \frac{\pi}{2} \qquad \text{and} \qquad \lim_{n\to\infty} \mu_n(c) = \overline{\mu}(c)$$

for some $\overline{\mu}(c) \in [-\mathrm{M}(c), \mathrm{M}(c)]$. In that case, (C.1.13) and the continuity of Λ_c^+ on $\mathbb{R} \times [-\mathrm{M}(c), \mathrm{M}(c)]$ imply

$$c^2 - (1 + \mathrm{M}(c)) = \lim_{n \to \infty} \Lambda_c^+(B_c^{\mu_n(c)}, \mu_n(c)) = \Lambda_c^+\left(\frac{\pi}{2}, \overline{\mu}(c)\right) \geq 2(c^2 - (1 + \mathrm{M}(c)))$$

by (C.1.12). This is, of course, a contradiction.

(v) Now we show that $\omega_c^\mu - \omega_c = \mathcal{O}_c(\mu)$. If $\omega_c = \omega_c^\mu$, then there is nothing to prove, so assume $\omega_c \neq \omega_c^\mu$. Recall that ω_c satisfies

$$c^2\omega_c^2 - (2 + 2\cos(\omega_c)) = 0$$

and ω_c^μ satisfies

$$c^2(\omega_c^\mu)^2 - \lambda_\mu^+(\omega_c^\mu) = 0.$$

Subtracting these two equalities and using the definition of λ_μ^+ in (3.0.7), we obtain

$$c^2(\omega_c - \omega_c^\mu) + \mu - 2\cos(\omega_c) + \sqrt{\mu^2 + 4(1+\mu)\cos^2(\omega_c^\mu)} = 0.$$

Taylor-expanding the square root and using the uniform bound $0 < A_c < \omega_c^\mu < B_c$ from part (iv), this rearranges to

$$(\omega_c - \omega_c^\mu)\left(c^2(\omega_c + \omega_c^\mu) - 2\left(\frac{\cos(\omega_c) - \cos(\omega_c^\mu)}{\omega_c - \omega_c^\mu}\right)\right) = \mathcal{O}_c(\mu). \tag{C.1.14}$$

Since the cosine has Lipschitz constant equal to 1, we have

$$\sup_{|\mu| \leq \mathrm{M}(c)} 2\left|\frac{\cos(\omega_c) - \cos(\omega_c^\mu)}{\omega_c - \omega_c^\mu}\right| \leq 2.$$

From part (iv), specifically (C.1.11), we have $\sqrt{2}/|c| \leq \omega_c^\mu$. Likewise, since $\widetilde{\mathcal{B}}_c(\omega_c) = 0$ with $\widetilde{\mathcal{B}}_c$ defined in (1.5.3), one can extract the inequaliy $\sqrt{2}/|c| < \omega_c$. Specifically, the proof is in Appendix D.1 in the context of the proof of part (i) of Proposition 4.2. Thus

$$c^2(\omega_c + \omega_c^\mu) \geq 2\sqrt{2}|c| > 2$$

whenever $|c| > 1$, and so we have

$$\inf_{|\mu| \leq \mathrm{M}(c)} \left|c^2(\omega_c + \omega_c^\mu) - 2\left(\frac{\cos(\omega_c) - \cos(\omega_c^\mu)}{\omega_c - \omega_c^\mu}\right)\right| > 0.$$

We conclude from (C.1.14) that $\omega_c - \omega_c^\mu = \mathcal{O}_c(\mu)$. □

The next lemma contains the remaining details that we need for the quantitative bifurcation argument that we will carry out in the following sections. To phrase this lemma, we need the definitions of the periodic Sobolev spaces H_{per}^r, E_{per}^r, and O_{per}^r from the start of Section 3. In this appendix only, we abbreviate

$$\|\mathbf{f}\|_r := \|\mathbf{f}\|_{H_{\mathrm{per}}^r \times H_{\mathrm{per}}^r}.$$

We also need to recall the definition of the operator $\Gamma_c^\mu[\gamma_c^\mu]$ from (3.0.4).

Lemma C.2. *For each $|c| > 1$, there is $\mu_{\mathrm{per}}(c) \in (0, \mathrm{M}(c)]$ such that the following hold.*

(i) *There exists* $\upsilon_c^\mu \in \mathbb{R}$ *such that if*

$$\boldsymbol{\nu}_c^\mu := \begin{pmatrix} \upsilon_c^\mu \cos(\cdot) \\ \sin(\cdot) \end{pmatrix}, \tag{C.1.15}$$

then the kernel of $\Gamma_c^\mu[\omega_c^\mu]$ *in* $E_{\text{per},0}^r \times O_{\text{per}}^r$ *is spanned by* $\boldsymbol{\nu}_c^\mu$ *for all* $r \geq 2$. *Moreover, there is a constant* $C_\upsilon(c) > 0$ *such that* $|\upsilon_c^\mu| \leq C_\upsilon(c)\mu$.

(ii) *Given* $\mathbf{f} \in E_{\text{per},0}^{r+2} \times O_{\text{per}}^{r+2}$ *and* $\mathbf{g} \in E_{\text{per},0}^r \times O_{\text{per}}^r$, *we have* $\Gamma_c^\mu[\omega_c^\mu]\mathbf{f} = \mathbf{g}$ *if and only if* $\langle \mathbf{g}, \boldsymbol{\nu}_c^\mu \rangle_0 = 0$.

(iii) *For each* $r \geq 0$, *there is a constant* $C(c,r) > 0$ *such that if* $\mathbf{f} \in E_{\text{per},0}^{r+2} \times \mathcal{O}_{\text{per}}^{r+2}$ *and* $\mathbf{g} \in E_{\text{per},0}^r \times O_{\text{per}}^r$ *with*

$$\Gamma_c^\mu[\omega_c^\mu]\mathbf{f} = \mathbf{g} \qquad \text{and} \qquad \langle \mathbf{f}, \boldsymbol{\nu}_c^\mu \rangle_0 = \langle \mathbf{g}, \boldsymbol{\nu}_c^\mu \rangle_0 = 0,$$

then $\|\mathbf{f}\|_{r+2} \leq C(c,r)\|\mathbf{g}\|_r$. *In particular, given* $\mathbf{g} \in E_{\text{per},0}^r \times O_{\text{per}}^r$ *satisfying* $\langle \mathbf{g}, \boldsymbol{\nu}_c^\mu \rangle_0 = 0$, *there is a unique* $\mathbf{f} \in E_{\text{per},0}^{r+2} \times O_{\text{per}}^{r+2}$ *with* $\Gamma_c^\mu[\omega_c^\mu]\mathbf{f} = \mathbf{g}$ *and* $\langle \mathbf{f}, \boldsymbol{\nu}_c^\mu \rangle_0 = 0$, *and, for this* $\mathbf{f}$, *we write* $\mathbf{f} = \Gamma_c^\mu[\omega_c^\mu]^{-1}\mathbf{g}$.

(iv) *There is a constant* $C(c) > 0$ *such that* $|\langle \partial_\omega \Gamma_c^\mu[\omega_c^\mu]\boldsymbol{\nu}_c^\mu, \boldsymbol{\nu}_c^\mu \rangle_0| \geq C(c)$.

Proof. **(i)** Suppose $\Gamma_c^\mu[\omega_c^\mu]\boldsymbol{\phi} = 0$, for some nonzero $\boldsymbol{\phi} = (\phi_1, \phi_2) \in E_{\text{per},0}^r \times O_{\text{per}}^r$. Let $k \in \mathbb{Z}$ such that $\widehat{\boldsymbol{\phi}}(k) \neq 0$. Then $\widehat{\boldsymbol{\phi}}(k)$ is an eigenvector of $\widetilde{\mathcal{D}}_\mu[\omega_c^\mu k]$, and so it follows that

$$c^2(\omega_c^\mu k)^2 - \lambda_\mu^-(\omega_c^\mu k) = 0 \qquad \text{or} \qquad c^2(\omega_c^\mu k)^2 - \lambda_\mu^+(\omega_c^\mu k) = 0. \tag{C.1.16}$$

In the first case, part (iii) of Lemma C.1 tells us that $\omega_c^\mu k = 0$, and consequently $k = 0$. But we know $\widehat{\phi}_1(0) = 0$ since $\phi_1 \in E_{\text{per},0}^r$, and we have $\widehat{\phi}_2(0) = 0$ since ϕ_2 is odd. Then $\widehat{\boldsymbol{\phi}}(0) = 0$, a contradiction.

It therefore must be the case that the second equality in (C.1.16) holds, and so $\omega_c^\mu k = \pm\omega_c^\mu$ by part (iv) of Lemma C.1. Hence $k = \pm 1$ and $\widehat{\boldsymbol{\phi}}(k) = 0$ for $|k| \neq 1$. So, with $e_k(x) := e^{ikx}$, we may write

$$\boldsymbol{\phi} = \widehat{\boldsymbol{\phi}}(-1)e_{-1} + \widehat{\boldsymbol{\phi}}(1)e_1 = \begin{pmatrix} \widehat{\phi}_1(-1)e_{-1} + \widehat{\phi}_1(1)e_1 \\ \widehat{\phi}_2(-1)e_{-1} + \widehat{\phi}_2(1)e_1 \end{pmatrix} = \begin{pmatrix} \widehat{\phi}_1(1)(e_{-1} + e_1) \\ \widehat{\phi}_2(1)(e_1 - e_{-1}) \end{pmatrix} = \begin{pmatrix} 2\widehat{\phi}_1(1)\cos(\cdot) \\ 2i\widehat{\phi}_2(1)\sin(\cdot) \end{pmatrix}.$$

Here we have used the assumption that ϕ_1 is even and ϕ_2 is odd.

We know that $\widehat{\boldsymbol{\phi}}(1)$ is an eigenvector of $\widetilde{\mathcal{D}}_\mu[\omega_c^\mu]$ corresponding to the eigenvalue $\lambda_\mu^+(\omega_c^\mu)$. Using the definition of this matrix in (3.0.6), we conclude there is $a \in \mathbb{C}$ such that

$$\widehat{\boldsymbol{\phi}}(1) = a\begin{pmatrix} i\upsilon_c^\mu \\ 1 \end{pmatrix}, \qquad \upsilon_c^\mu := \frac{\mu \sin(\omega_c^\mu)}{\lambda_\mu^+(\omega_c^\mu) - (2+\mu)(1 - \cos(\omega_c^\mu))}, \tag{C.1.17}$$

provided that the term in the denominator of υ_c^μ is nonzero. It is, in fact, bounded below away from zero. If we assume

$$|\mu| \leq \min\left\{\frac{1}{2}, \mathrm{M}(c)\right\} =: \mathrm{M}_1(c), \tag{C.1.18}$$

then

$$\lambda_\mu^+(\omega_c^\mu) - (2+\mu)(1 - \cos(\omega_c^\mu)) = \sqrt{\mu^2 + 4(1+\mu)\cos^2(\omega_c^\mu)} + (2+\mu)\cos(\omega_c^\mu) \geq \sqrt{2}\cos(B_c)$$

This gives the desired estimate $|\upsilon_c^\mu| \leq C_\upsilon(c)|\mu|$.

We conclude that if $\Gamma_c^\mu[\omega_c^\mu]\boldsymbol{\phi} = 0$, then

$$\boldsymbol{\phi} = \begin{pmatrix} 2\widehat{\phi}_1(1)\cos(\cdot) \\ 2i\widehat{\phi}_2(1)\sin(\cdot) \end{pmatrix} \qquad \text{and} \qquad \widehat{\boldsymbol{\phi}}(1) = a\begin{pmatrix} i\upsilon_c^\mu \\ 1 \end{pmatrix},$$

thus

$$\boldsymbol{\phi} = 2ai\begin{pmatrix} \upsilon_c^\mu\cos(\cdot) \\ \sin(\cdot) \end{pmatrix},$$

and so the kernel of $\Gamma_c^\mu[\omega_c^\mu]$ is spanned by the vector $\boldsymbol{\nu}_c^\mu$.

(ii) Recall that the symbol of $\Gamma_c^\mu[\omega_c^\mu]$ is

$$-c^2(\omega_c^\mu k)^2 + \widetilde{\mathcal{D}}_\mu[\omega_c^\mu k],$$

and this is a self-adjoint matrix in $\mathbb{C}^{2\times 2}$ by the definition of $\widetilde{\mathcal{D}}_\mu$ in (3.0.6). Moreover, by part (i), the kernel of $\Gamma_c^\mu[\omega_c^\mu]$ is one-dimensional and spanned by $\boldsymbol{\nu}_c^\mu$ from (C.1.15), where $\widehat{\boldsymbol{\nu}_c^\mu}(k) = 0$ for $k \neq \pm 1$. If we denote by $\Gamma_c^\mu[\omega_c^\mu]^*$ the adjoint of $\Gamma_c^\mu[\omega_c^\mu]$ from $E_{\text{per},0}^2 \times O_{\text{per}}^2$ to $E_{\text{per},0}^0 \times O_{\text{per}}^0$, Lemma A.2 tells us that the kernel of $\Gamma_c^\mu[\omega_c^\mu]^*$ is spanned by the function $(\boldsymbol{\nu}_c^\mu)^*$ whose Fourier coefficients are

$$\widehat{(\boldsymbol{\nu}_c^\mu)^*}(k) = (1+k^2)^{(2-0)/2}\widehat{\boldsymbol{\nu}_c^\mu}(k) = \begin{cases} 2\widehat{\boldsymbol{\nu}_c^\mu}(k), & |k| = 1 \\ 0, & |k| \neq 1. \end{cases}$$

That is, the kernel of $\Gamma_c^\mu[\omega_c^\mu]^*$ is just spanned by $\boldsymbol{\nu}_c^\mu$.

Now we show

$$\Gamma_c^\mu[\omega_c^\mu]\mathbf{f} = \mathbf{g},\ \mathbf{f} \in E_{\text{per},0}^{r+2} \times O_{\text{per}}^{r+2},\ \mathbf{g} \in E_{\text{per},0}^r \times O_{\text{per}}^r \iff \langle \mathbf{g}, \boldsymbol{\nu}_c^\mu \rangle_0 = 0. \tag{C.1.19}$$

The forward implication is clear from the containment $E_{\text{per},0}^r \times O_{\text{per}}^r \subseteq E_{\text{per},0}^0 \times O_{\text{per}}^0$. For the reverse, take $\mathbf{g} \in E_{\text{per},0}^r \times O_{\text{per}}^r$ and suppose $\langle \mathbf{g}, \boldsymbol{\nu}_c^\mu \rangle_0 = 0$. Classical functional analysis tells us there is $\mathbf{f} \in E_{\text{per},0}^2 \times O_{\text{per}}^2$ such that $\Gamma_c^\mu[\omega_c^\mu]\mathbf{f} = \mathbf{g}$. We then use the structure of $\Gamma_c^\mu[\omega_c^\mu]$ to bootstrap until we have $\mathbf{f} \in E_{\text{per},0}^{r+2} \times O_{\text{per}}^{r+2}$.

(iii) If $\Gamma_c^\mu[\omega_c^\mu]\mathbf{f} = \mathbf{g}$, then for $k \in \mathbb{Z} \setminus \{0\}$ we have

$$\left(\mathbb{1} - \frac{1}{c^2(\omega_c^\mu)^2k^2}\widetilde{\mathcal{D}}_\mu[\omega_c^\mu k]\right)\widehat{\mathbf{f}}(k) = -\frac{1}{c^2(\omega_c^\mu)^2k^2}\widehat{\mathbf{g}}(k). \tag{C.1.20}$$

Here $\mathbb{1}$ is the 2×2 identity matrix, and we may ignore the case $k = 0$ since $\widehat{\mathbf{f}}(0) = \widehat{\mathbf{g}}(0) = 0$. In this part of the proof, we will denote the ∞-norm of a matrix $A \in \mathbb{C}^{2\times 2}$ by $\|A\|$ and the 2-norm of a vector $\mathbf{v} \in \mathbb{C}^2$ by $|\mathbf{v}|$.

Our goal is to show that

$$\sup_{|k|\geq 2}\left\|\frac{1}{c^2(\omega_c^\mu)^2k^2}\widetilde{\mathcal{D}}_\mu[\omega_c^\mu k]\right\| < 1, \tag{C.1.21}$$

in which case the matrix on the left in (C.1.20) is invertible by a Neumann series argument, and, moreover, this inverse is uniformly bounded in k. This will show $|\widehat{\mathbf{f}}(k)| \leq C(c)|\widehat{\mathbf{g}}(k)|$ for $|k| \geq 2$, and then we will handle the case $|k| = 1$ separately.

We begin with the case $|k| \geq 2$. The estimate (C.1.3) yields

$$\frac{1}{c^2(\omega_c^\mu)^2} \leq \frac{1+\mu}{2}, \tag{C.1.22}$$

while the definition of $\widetilde{\mathcal{D}}_\mu[K]$ in (3.0.6) implies

$$\left\|\widetilde{\mathcal{D}}_\mu[\omega_c^\mu k]\right\| \le 2(2+\mu). \tag{C.1.23}$$

We combine (C.1.22) and (C.1.23) and assume

$$|\mu| \le \min\left\{\frac{1}{10}, \mathrm{M}_1(c)\right\} =: \mathrm{M}_2(c) \tag{C.1.24}$$

to find

$$\left\|\frac{1}{c^2(\omega_c^\mu)^2k^2}\widetilde{\mathcal{D}}_\mu[\omega_c^\mu k]\right\| \le \frac{(1+\mu)(2+\mu)}{k^2} \le \frac{3}{k^2} \le \frac{3}{4}$$

for $|k| \ge 2$. This proves (C.1.21).

Now we show $|\widehat{\mathbf{f}}(\pm 1)| \le C_c|\widehat{\mathbf{g}}(\pm 1)|$. It is here that we need the additional hypothesis in part (iii) that $\langle \mathbf{f}, \boldsymbol{\nu}_c^\mu\rangle_0 = \langle \mathbf{g}, \boldsymbol{\nu}_c^\mu\rangle_0 = 0$, which implies

$$\widehat{f}_2(1) = -i\upsilon_c^\mu \widehat{f}_1(1) \qquad \text{and} \qquad \widehat{g}_2(1) = -i\upsilon_c^\mu\widehat{g}_1(1). \tag{C.1.25}$$

This allows to rearrange the equality $\mathfrak{F}[\Gamma_c^\mu[\omega_c^\mu]\mathbf{f}](1)\cdot \mathbf{e}_1 = \widehat{\mathbf{g}}(1)\cdot\mathbf{e}_1$ into

$$\big(-c^2(\omega_c^\mu)^2 + (2+\mu)(1-\cos(\omega_c^\mu)) - i\upsilon_c^\mu\mu\sin(\omega_c^\mu)\big)\widehat{f}_1(1) = \widehat{g}_1(1).$$

Some straightforward calculus proves

$$cK^2 - 2(1-\cos(K)) > K^2 - 2(1-\cos(K)) > 2\cos(1) - 1 > 0$$

for $K > 1$, so it follows that

$$|-c^2(\omega_c^\mu)^2 + (2+\mu)(1-\cos(\omega_c^\mu)) - i\upsilon_c^\mu\mu\sin(\omega_c^\mu)| \ge \frac{2\cos(1)-1}{2},$$

provided that

$$|\mu| < \min\left\{\frac{2\cos(1)-1}{8}, \frac{2\cos(1)-1}{4C_\upsilon(c)}, \mathrm{M}_2(c)\right\} =: \mathrm{M}_3(c). \tag{C.1.26}$$

This shows

$$|\widehat{f}_1(1)| \le C(c)|\widehat{g}_1(1)|.$$

If $\upsilon_c^\mu = 0$, then (C.1.25) provides $\widehat{f}_2(1) = 0 = \widehat{g}_2(1)$, and there is nothing more to prove. Otherwise, we have

$$\frac{1}{|\upsilon_c^\mu|}|\widehat{f}_2(1)| = |\widehat{f}_1(1)| \le C(c)|\widehat{g}_1(1)| = \frac{1}{|\upsilon_c^\mu|}|\widehat{g}_2(1)|,$$

and so $|\widehat{f}_2(1)| \le C(c)|\widehat{g}_2(1)|$. The estimates for $k = -1$ follow by the even-odd symmetry of $\mathbf{f}$ and $\mathbf{g}$.

Last, if $\mathbf{g} \in E^r_{\mathrm{per},0} \times O^r_{\mathrm{per}}$ with $\langle \mathbf{g}, \boldsymbol{\nu}_c^\mu\rangle_0 = 0$, then part (ii) gives $\mathbf{f} \in E^{r+2}_{\mathrm{per},0} \times O^{r+2}_{\mathrm{per}}$ such that $\Gamma_c^\mu[\omega_c^\mu]\mathbf{f} = \mathbf{g}$. Requiring $\langle \mathbf{f}, \boldsymbol{\nu}_c^\mu\rangle_0 = 0$ is enough to make $\mathbf{f}$ unique; the proof is a straightforward exercise in Hilbert space theory.

(iv) A lengthy calculation using patient matrix-vector multiplication along with the identity

$$\mathfrak{F}[\partial_\omega\Gamma_c^\mu[\omega_c^\mu]\boldsymbol{\nu}_c^\mu](k) = -2c^2\omega_c^\mu k^2\widehat{\boldsymbol{\nu}_c^\mu}(k) + k\widetilde{\mathcal{D}}_\mu'(\omega_c^\mu k)\widehat{\boldsymbol{\nu}_c^\mu}(k),$$

the definition of $\boldsymbol{\nu}_c^\mu$ in (C.1.15), the fact that $\widehat{\boldsymbol{\nu}_c^\mu}(k) = 0$ for $|k| \ne 1$, and the definition of $\widetilde{\mathcal{D}}_\mu$ in (3.0.6) and its corresponding componentwise derivative $\widetilde{\mathcal{D}}_\mu'$ yields

$$\langle \partial_\omega \Gamma_c^\mu[\omega_c^\mu]\boldsymbol{\nu}_c^\mu, \boldsymbol{\nu}_c^\mu\rangle_0 = \underbrace{-(c^2\omega_c^\mu + \sin(\omega_c^\mu))}_{I} + \underbrace{\mu\sin(\omega_c^\mu)\big(((\upsilon_c^\mu)^2 - 1) + 4\upsilon_c^\mu + \mu\cos(\omega_c^\mu)\big)}_{II}$$

$$+ \underbrace{2\sin(\omega_c^\mu)\big((\upsilon_c^\mu)^2 + 4\upsilon_c^\mu + \mu\cos(\omega_c^\mu)\big)}_{III}.$$

Since $\upsilon_c^\mu = \mathcal{O}_c(\mu)$, it follows that $II = \mathcal{O}_c(\mu)$ and $III = \mathcal{O}_c(\mu)$. And since $0 < A_c < \omega_c^\mu < B_c$, we have

$$c^2\omega_c^\mu + \sin(\omega_c^\mu) > c^2 A_c + \sin(A_c).$$

It follows that for $|\mu|$ small, say, $|\mu| \le \mu_{\mathrm{per}}(c) \le \mathrm{M}_3(c)$, we have

$$|\,\langle \partial_\omega \Gamma_c^\mu[\omega_c^\mu]\boldsymbol{\nu}_c^\mu, \boldsymbol{\nu}_c^\mu\rangle_0\,| \ge \frac{c^2 A_c + \sin(A_c)}{2} > 0. \qquad \square$$

C.2. **Bifurcation from a simple eigenvalue.** We are now ready to solve the periodic traveling wave problem $\boldsymbol{\Phi}_c^\mu(\boldsymbol{\phi},\omega) = 0$ posed in (3.0.3). We follow [HW17] and [CR71] and make the revised ansatz

$$\boldsymbol{\phi} = a\boldsymbol{\nu}_c^\mu + a\boldsymbol{\psi} \qquad \text{and} \qquad \omega = \omega_c^\mu + \xi.$$

where a, $\xi \in \mathbb{R}$ and $\langle \boldsymbol{\nu}_c^\mu, \boldsymbol{\psi}\rangle_0 = 0$. Then our problem $\boldsymbol{\Phi}_c^\mu(\boldsymbol{\phi},\omega) = 0$ becomes

$$\boldsymbol{\Phi}_c^\mu(a\boldsymbol{\nu}_c^\mu + a\boldsymbol{\psi}, \omega_c^\mu + \xi) = 0.$$

After some considerable rearranging, we find that $\boldsymbol{\psi}$ and a must satisfy

$$\Gamma_c^\mu[\omega_c^\mu]\boldsymbol{\psi} = \mathcal{R}_{c,1}^\mu(\xi) + \mathcal{R}_{c,2}^\mu(\xi) + \mathcal{R}_{c,3}^\mu(\boldsymbol{\psi},\xi) + \mathcal{R}_{c,4}^\mu(\boldsymbol{\psi},\xi) + a\mathcal{R}_{c,5}^\mu(\boldsymbol{\psi},\xi), \tag{C.2.1}$$

where

$$\mathcal{R}_{c,1}^\mu(\xi) := -2c^2\omega_c^\mu\xi(\boldsymbol{\nu}_c^\mu)''$$

$$\mathcal{R}_{c,2}^\mu(\xi) := -(\mathcal{D}_\mu[\omega_c^\mu + \xi] - \mathcal{D}_\mu[\omega_c^\mu])\boldsymbol{\nu}_c^\mu$$

$$\mathcal{R}_{c,3}^\mu(\boldsymbol{\psi},\xi) := -c^2[\xi^2(\boldsymbol{\nu}_c^\mu)'' + (2\omega_c^\mu + \xi)\xi\boldsymbol{\psi}''] \tag{C.2.2}$$

$$\mathcal{R}_{c,4}^\mu(\boldsymbol{\psi},\xi) := -(\mathcal{D}_\mu[\omega_c^\mu + \xi] - \mathcal{D}_\mu[\omega_c^\mu])\boldsymbol{\psi}$$

$$\mathcal{R}_{c,5}^\mu(\boldsymbol{\psi},\xi) := -\mathcal{D}_\mu[\omega_c^\mu + \xi]\mathcal{Q}(\boldsymbol{\nu}_c^\mu + \boldsymbol{\psi}, \boldsymbol{\nu}_c^\mu + \boldsymbol{\psi}).$$

This is roughly the same system that Hoffman and Wright study when they construct periodic solutions for the small mass problem; specifically, its analogue appears in equation (B.9) in [HW17]. Our goal, like theirs, is to rewrite (C.2.1) as a fixed point argument in the unknowns $\boldsymbol{\psi}$ and ξ on the space $E_{\mathrm{per},0}^2 \times O_{\mathrm{per}}^2 \times \mathbb{R}$ with μ as a small parameter and c fixed. We intend to use the following lemma, proved in [FW18] as Lemma C.1.

Lemma C.3. *Let $\mathcal{X}$ be a Banach space and for $r > 0$ let $\mathfrak{B}(r) = \{x \in \mathcal{X} : \|x\| \le r\}$.*

(i) *For $|\mu| \le \mu_0$ let $F_\mu \colon \mathcal{X} \times \mathbb{R} \to \mathcal{X}$ be maps with the property that for some C_1, a_1, $r_1 > 0$, if $x, y \in \mathfrak{B}(r_1)$ and $|a| \le a_1$, then*

$$\sup_{|\mu| \le \mu_0} \|F_\mu(x,a)\| \le C_1\big(|a| + |a|\|x\| + \|x\|^2\big) \tag{C.2.3}$$

$$\sup_{|\mu| \le \mu_0} \|F_\mu(x,a) - F_c(y,a)\| \le C_1\big(|a| + \|x\| + \|y\|\big)\|x-y\| \tag{C.2.4}$$

Then there exist $a_0 \in (0, a_1], r_0 \in (0, r_1]$ such that for each $|\mu| \le \mu_0$ and $|a| \le a_0$, there is a unique $x_\mu^a \in \mathfrak{B}(r_0)$ such that $F_\mu(x_\mu^a, a) = x_\mu^a$.

(ii) *Suppose as well that the maps $F_\mu(\cdot, a)$ are Lipschitz on $\mathfrak{B}(r_0)$ uniformly in a and μ, i.e., there is $L_1 > 0$ such that*

$$\sup_{\substack{|\mu| \le \mu_0 \\ \|x\| \le r_0}} \|F_\mu(x,a) - F_\mu(x, \grave{a})\| \le L_1|a - \grave{a}| \tag{C.2.5}$$

for all $|a|$, $|\grave{a}| \le a_0$. Then the mappings $[-a_0, a_0] \to \mathcal{X} \colon a \mapsto x_\mu^a$ are also uniformly Lipschitz; that is, there is $L_0 > 0$ such that

$$\sup_{|\mu| \le \mu_0} \|x_\mu^a - x_\mu^{\grave{a}}\| \le L_0|a - \grave{a}| \tag{C.2.6}$$

for all $|a|$, $|\grave{a}| \le a_1$.

Unlike Hoffman and Wright, we cannot invoke this lemma immediately as the terms $\mathcal{R}_{c,1}^\mu$ and $\mathcal{R}_{c,2}^\mu$ do not have quite the right structure. Namely, they contain linear terms in ξ that have no companion factor of μ, and so these terms will not be small enough to achieve the estimates (C.2.3) and (C.2.4). Due to a different scaling inherent to the small mass limit, this was not an issue for Hoffman and Wright.

Therefore, we begin to modify the problem (C.2.1) as follows. Let

$$\varpi_c^\mu \boldsymbol{\psi} := \frac{\langle \boldsymbol{\psi}, \boldsymbol{\nu}_c^\mu \rangle_0}{\|\boldsymbol{\nu}_c^\mu\|_0^2} \boldsymbol{\nu}_c^\mu \qquad \text{and} \qquad \Pi_c^\mu := \mathbb{1} - \varpi_c^\mu. \tag{C.2.7}$$

The definition of $\boldsymbol{\nu}_c^\mu$ in (C.1.15) ensures that the denominator of ϖ_c^μ is bounded away from zero uniformly in μ. Then (C.2.1) holds if and only if both

$$\varpi_c^\mu \Gamma_c^\mu[\omega_c^\mu]\boldsymbol{\psi} = \varpi_c^\mu[\mathcal{R}_{c,1}^\mu(\xi) + \mathcal{R}_{c,2}^\mu(\xi) + \mathcal{R}_{c,3}^\mu(\boldsymbol{\psi}, \xi) + \mathcal{R}_{c,4}^\mu(\boldsymbol{\psi}, \xi) + a\mathcal{R}_{c,5}^\mu(\boldsymbol{\psi}, \xi)] \tag{C.2.8}$$

and

$$\Pi_c^\mu \Gamma_c^\mu[\omega_c^\mu]\boldsymbol{\psi} = \Pi_c^\mu[\mathcal{R}_{c,1}^\mu(\xi) + \mathcal{R}_{c,2}^\mu(\xi) + \mathcal{R}_{c,3}^\mu(\boldsymbol{\psi}, \xi) + \mathcal{R}_{c,4}^\mu(\boldsymbol{\psi}, \xi) + a\mathcal{R}_{c,5}^\mu(\boldsymbol{\psi}, \xi)]. \tag{C.2.9}$$

Since $\langle \Gamma_c^\mu[\omega_c^\mu]\boldsymbol{\psi}, \boldsymbol{\nu}_c^\mu \rangle_0 = 0$ by (C.1.19), the first equation (C.2.8) is equivalent to

$$-\varpi_c^\mu[\mathcal{R}_{c,1}^\mu(\xi, , w) + \mathcal{R}_{c,2}^\mu(\xi)] = \varpi_c^\mu[\mathcal{R}_{c,3}^\mu(\boldsymbol{\psi}, \xi) + \mathcal{R}_{c,4}^\mu(\boldsymbol{\psi}, \xi) + a\mathcal{R}_{c,5}^\mu(\boldsymbol{\psi}, \xi)]$$

Observe that

(C.2.10)
$$-\mathcal{R}_{c,1}^\mu(\xi) - \mathcal{R}_{c,2}^\mu(\xi) = \underbrace{\xi(2c^2\omega_c^\mu(\boldsymbol{\nu}_c^\mu)'' + \partial_\omega \mathcal{D}_\mu[\omega_c^\mu]\boldsymbol{\nu}_c^\mu)}_{\xi\partial_\omega\Gamma_c^\mu[\omega_c^\mu]\boldsymbol{\nu}_c^\mu} + \underbrace{(\mathcal{D}_\mu[\omega_c^\mu + \xi] - \mathcal{D}_\mu[\omega_c^\mu] - \xi\partial_\omega\mathcal{D}_\mu[\omega_c^\mu])\boldsymbol{\nu}_c^\mu}_{\mathcal{R}_{c,6}^\mu(\xi)},$$

where the derivative $\partial_\omega \Gamma_c^\mu[\omega_c^\mu]$ of a scaled Fourier multiplier is discussed in Appendix A.1.1. Consequently, (C.2.8) is really equivalent to

$$\xi = \underbrace{\frac{1}{\langle \partial_\omega \Gamma_c^\mu[\omega_c^\mu]\boldsymbol{\nu}_c^\mu, \boldsymbol{\nu}_c^\mu\rangle_0} \varpi_c^\mu[\mathcal{R}_{c,3}^\mu(\boldsymbol{\psi},\xi) + \mathcal{R}_{c,4}^\mu(\boldsymbol{\psi},\xi) + a\mathcal{R}_{c,5}^\mu(\boldsymbol{\psi},\xi) + \mathcal{R}_{c,6}^\mu(\xi)]}_{\Xi_c^\mu(\boldsymbol{\psi},\xi,a)}, \tag{C.2.11}$$

and part (iv) of Proposition C.2 guarantees that $|\langle \partial_\omega \Gamma_c^\mu[\omega_c^\mu]\boldsymbol{\nu}_c^\mu, \boldsymbol{\nu}_c^\mu\rangle_0|$ is bounded away[6] from zero uniformly in μ.

Now we construct the fixed point equation for $\boldsymbol{\psi}$. By definition of Π_c^μ in (C.2.7) and part (ii) of Proposition C.2, we have

$$\Pi_c^\mu \Gamma_c^\mu[\omega_c^\mu]\boldsymbol{\psi} = \Gamma_c^\mu[\omega_c^\mu]\boldsymbol{\psi},$$

so that (C.2.9) is equivalent to

$$\Gamma_c^\mu[\omega_c^\mu]\boldsymbol{\psi} = \Pi_c^\mu[\mathcal{R}_{c,1}^\mu(\xi) + \mathcal{R}_{c,2}^\mu(\xi) + \mathcal{R}_{c,3}^\mu(\boldsymbol{\psi},\xi) + \mathcal{R}_{c,4}^\mu(\boldsymbol{\psi},\xi) + a\mathcal{R}_{c,5}^\mu(\boldsymbol{\psi},\xi)]. \tag{C.2.12}$$

Next, it is obvious from the definition of Π_c^μ that $\langle \Pi_c^\mu \mathbf{f}, \boldsymbol{\nu}_c^\mu\rangle_0 = 0$. Consequently, there exists a solution $\boldsymbol{\psi}$ to (C.2.12), but, since $\Gamma_c^\mu[\omega_c^\mu]$ has a nontrivial kernel by part (i) of Proposition C.2, this solution is not unique. Part (iii) of that proposition, however, allows us to force uniqueness by taking $\boldsymbol{\psi}$ to be the solution of (C.2.12) that also satisfies $\langle \boldsymbol{\psi}, \boldsymbol{\nu}_c^\mu\rangle_0 = 0$. Hence the $\boldsymbol{\psi}$ that we seek must satisfy

$$\boldsymbol{\psi} = \Gamma_c^\mu[\omega_c^\mu]^{-1}\Pi_c^\mu[\mathcal{R}_{c,1}^\mu(\xi) + \mathcal{R}_{c,2}^\mu(\xi) + \mathcal{R}_{c,3}^\mu(\boldsymbol{\psi},\xi) + \mathcal{R}_{c,4}^\mu(\boldsymbol{\psi},\xi) + a\mathcal{R}_{c,5}^\mu(\boldsymbol{\psi},\xi)].$$

As with our construction of the fixed-point equation for ξ, the terms $\mathcal{R}_{c,1}^\mu$ and $\mathcal{R}_{c,2}^\mu$ are too large in ξ. However, by (C.2.11) we may replace ξ by $\Xi_c^\mu(\boldsymbol{\psi},\xi,a)$, which will turn out to correct this overshoot. So, our equation for $\boldsymbol{\psi}$ is

$$\boldsymbol{\psi} = \underbrace{\Gamma_c^\mu[\omega_c^\mu]^{-1}\Pi_c^\mu[\mathcal{R}_{c,1}^\mu(\Xi_c^\mu(\boldsymbol{\psi},\xi,a)) + \mathcal{R}_{c,2}^\mu(\Xi_c^\mu(\boldsymbol{\psi},\xi,a)) + \mathcal{R}_{c,3}^\mu(\boldsymbol{\psi},\xi) + \mathcal{R}_{c,4}^\mu(\boldsymbol{\psi},\xi) + a\mathcal{R}_{c,5}^\mu(\boldsymbol{\psi},\xi)]}_{\boldsymbol{\Psi}_c^\mu(\boldsymbol{\psi},\xi,a)}. \tag{C.2.13}$$

It is worthwhile pointing out now that

$$\|\Gamma_c^\mu[\omega_c^\mu]^{-1}\Pi_c^\mu\boldsymbol{\psi}\|_{r+2} \le C(c,r)\|\boldsymbol{\psi}\|_r \tag{C.2.14}$$

by the estimate in part (iii) from Lemma C.2.

C.3. **Solution of the fixed-point problem.** The fixed-point problem

$$\begin{cases} \boldsymbol{\psi} = \boldsymbol{\Psi}_c^\mu(\boldsymbol{\psi},\xi,a) \\ \xi = \Xi_c^\mu(\boldsymbol{\psi},\xi,a) \end{cases}$$

is now in a form amenable to Lemma C.3. One first shows that Ξ_c^μ satisfies the estimates (C.2.3), (C.2.4), and (C.2.5) using, chiefly, the calculus on Fourier multipliers from Lemma A.1 and the techniques of Appendices B of [HW17] and C of [FW18]. Then one establishes the same estimates on $\boldsymbol{\Psi}_c^\mu$, using the existing estimates on Ξ_c^μ along the way. Since the techniques so closely resemble those of [HW17] and [FW18], we omit the details.

[6] One can think of the condition $|\langle \partial_\omega \Gamma_c^\mu[\omega_c^\mu]\boldsymbol{\nu}_c^\mu, \boldsymbol{\nu}_c^\mu\rangle_0| > 0$ as, ultimately, a quantitative version of the "bifurcation condition" from the original theorem of Crandall and Rabinowitz, namely, part (d) of Theorem 1.7 in [CR71], when their Banach spaces are specified to be Hilbert spaces.

Appendix D. Function Theory

D.1. **Proof of Proposition 4.2.**

D.1.1. *The proof of part (i).* Observe that

$$\widetilde{\mathcal{B}}_c(0) = 4 > 0 \qquad \text{and} \qquad \widetilde{\mathcal{B}}_c\left(\frac{\pi}{2}\right) = -\frac{c^2\pi^2}{4} + 2 < -\frac{8}{4} + 2 = 0.$$

The intermediate value theorem furnishes $\omega_c \in (0, \pi/2)$ such that $\widetilde{\mathcal{B}}_c(\omega_c) = 0$. Since $0 < \omega_c < \pi/2$, we can rewrite the relation $\widetilde{\mathcal{B}}_c(\omega_c) = 0$ as

$$\omega_c = \sqrt{\frac{2 + 2\cos(\omega_c)}{c^2}} > \sqrt{\frac{2}{c^2}} = \frac{\sqrt{2}}{|c|},$$

and so we have the refined bounds $\sqrt{2}/|c| < \omega_c < \pi/2$. Next, for $k \in \mathbb{R}$, we calculate

$$|(\widetilde{\mathcal{B}}_c)'(k)| = 2|c^2k - \sin(k)| > 0$$

since $c^2 > 1$. This shows that the zeros at $z = \pm\omega_c$ are simple and, moreover, unique in $\mathbb{R}$. The bound (4.0.5) follows by estimating

$$|(\widetilde{\mathcal{B}}_c)'(\omega_c)| \geq 2(c^2A_c - \sin(A_c)) \geq 2(1 - \sin(1)).$$

Now we prove that $z = \pm\omega_c$ are the only zeros of $\widetilde{\mathcal{B}}_c$ on a suitably narrow strip in $\mathbb{C}$. For k, $q \in \mathbb{R}$, we compute

$$\widetilde{\mathcal{B}}_c(k + iq) = \underbrace{-c^2(k^2 - q^2) + 2 + (e^{-q} + e^q)\cos(k)}_{\mathcal{R}_c(k,q)} + i\underbrace{-2c^2kq + (e^{-q} - e^q)\sin(k)}_{\mathcal{I}_c(k,q)}.$$

We claim for a suitable $q_{\mathcal{B}} > 0$, if $|q| \leq 3q_{\mathcal{B}}$, then $\mathcal{I}_c(k, q) = 0$ if and only if $k = 0$. But

$$\mathcal{R}_c(0, q) = q^2 + e^q + e^{-q} > 0$$

for all $q \in \mathbb{R}$, and so, if our claim is true, then $\widetilde{\mathcal{B}}_c(z) \neq 0$ for $|\operatorname{Im}(z)| \leq 3q_{\mathcal{B}}$.

So, we just need to prove the claim about $\mathcal{I}_c$. Observe that, for $q \neq 0$, $\mathcal{I}_c(k, q) = 0$ if and only if

$$\frac{2c^2q}{e^{-q} - e^q} = \operatorname{sinc}(k) := \frac{\sin(k)}{k}. \tag{D.1.1}$$

Elementary calculus tells us

$$\operatorname{sinc}(k) > -\frac{1}{4}, \ k \in \mathbb{R} \qquad \text{and} \qquad 2\frac{q}{e^{-q} - e^q} < -\frac{1}{4}, \ |q| \leq 3.$$

So, we set $q_{\mathcal{B}} = 1$ and find that for $|q| \leq 3q_{\mathcal{B}}$ and any $k \in \mathbb{R}$

$$\frac{2c^2q}{e^{-q} - e^q} < \frac{2q}{e^{-q} - e^q} < -\frac{1}{4} < \operatorname{sinc}(k).$$

Hence (D.1.1) cannot hold for $|q| \leq 3q_{\mathcal{B}}$.

With part (i) established, part (ii) follows at once.

D.1.2. *The proof of part (iii).* The reverse triangle inequality implies

$$|\widetilde{\mathcal{B}}_c(z)| \ge c^2|z|^2 - |2 + 2\cos(z)|$$

We estimate

$$|2 + 2\cos(z)| \le 2 + 2e^{2q_{\mathcal{B}}} \le 2 + 2e^2$$

for $z \in S_{2q_{\mathcal{B}}}$, since we assumed $q_{\mathcal{B}} = 1$. Hence

$$|\widetilde{\mathcal{B}}_c(z)| \ge c^2|z|^2 - (2 + 2e^2).$$

Now take

$$|z| \ge \sqrt{\frac{2 + 2e^2}{c^2 - 1/2}} =: r_{\mathcal{B}}(c)$$

to find that

$$\frac{1}{2}|z|^2 \le c^2|z|^2 - (2 + 2e^2) \le |\widetilde{\mathcal{B}}_c(z)|.$$

D.1.3. *The proof of part (iv).* Residue theory tells us that

$$\frac{1}{\widetilde{\mathcal{B}}_c(z)} = \frac{1}{(\widetilde{\mathcal{B}}_c)'(\omega_c)(z - \omega_c)} + \frac{1}{(\widetilde{\mathcal{B}}_c)'(-\omega_c)(z + \omega_c)} + \mathfrak{R}_c(z), \ z \in S_{3q_{\mathcal{B}}} \setminus \{\pm\omega_c\}, \tag{D.1.2}$$

where $\mathfrak{R}_c$ is analytic on $S_{3q_{\mathcal{B}}}$.

We need two elementary Fourier transforms. Let α, $\beta \in \mathbb{C}$ with $\mathrm{Re}(\alpha) > 0$ and $\mathrm{Re}(\beta) < 0$ and define

$$\mathsf{e}_\alpha(x) := \begin{cases} e^{\alpha x}, \ x < 0 \\ 0, \ x \ge 0 \end{cases} \qquad \text{and} \qquad \mathsf{e}^\beta(x) := \begin{cases} 0, \ x < 0 \\ e^{\beta x}, \ x \ge 0. \end{cases}$$

Then

$$\widehat{\mathsf{e}}_\alpha(k) = \frac{1}{\sqrt{2\pi}(\alpha - ik)} \qquad \text{and} \qquad \widehat{\mathsf{e}^\beta}(k) = -\frac{1}{\sqrt{2\pi}(\beta - ik)}.$$

Since

$$\frac{1}{-k + iq \pm \omega_c} = \frac{i\sqrt{2\pi}}{\sqrt{2\pi}((-q \pm i\omega_c) - ik)},$$

we have

$$\mathfrak{F}^{-1}\left[\frac{1}{-\cdot + iq \pm \omega_c}\right](x) = \begin{cases} i\sqrt{2\pi}e^{(-q \pm i\omega_c)x}, \ x < 0 \\ 0, \ x \ge 0 \end{cases} \tag{D.1.3}$$

if $q < 0$ and

$$\mathfrak{F}^{-1}\left[\frac{1}{-\cdot + iq \pm \omega_c}\right](x) = \begin{cases} 0, \ x < 0 \\ i\sqrt{2\pi}e^{(-q \pm i\omega_c)x}, \ x \ge 0 \end{cases} \tag{D.1.4}$$

if $q > 0$. In either case, using the decomposition (D.1.2) and the transforms (D.1.3) and (D.1.4), we bound

$$\left\|\mathfrak{F}^{-1}\left[\frac{1}{\widetilde{\mathcal{B}}_c(-\cdot + iq)}\right]\right\|_{L^1} \le \sqrt{2\pi}\left(\frac{1}{|q| + i\omega_c} + \frac{1}{|q| - i\omega_c}\right) + \|\mathfrak{F}^{-1}[\mathfrak{R}_c(-\cdot + iq)\|_{L^1}, \tag{D.1.5}$$

where

$$\text{(D.1.6)}\quad \mathfrak{F}^{-1}[\mathfrak{R}_c(-\cdot+iq)](x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}\mathfrak{R}_c(-k+iq)\,dk = -\frac{e^{-qx}}{\sqrt{2\pi}}\int_{\mathrm{Im}(z)=q} e^{-ixz}\mathfrak{R}_c(z)\,dz.$$

We claim that

$$\text{(D.1.7)}\qquad \sup_{\substack{1<|c|\le\sqrt{2}\\ 0<|q|\le q_{\mathcal{B}}}} \|\mathfrak{F}^{-1}[\mathfrak{R}_c(-\cdot+iq)]\|_{L^1(\mathbb{R}_\pm)} < \infty.$$

If this estimate holds, then (D.1.5) will establish the desired estimate (4.0.8). We first prove (D.1.7) for the $L^1(\mathbb{R}_-)$ case and then comment briefly on how to proceed with the similar $L^1(\mathbb{R}_+)$ estimate. Observe that $\mathfrak{R}_c$ vanishes as $|\mathrm{Re}(z)| \to \infty$, since the other three functions in (D.1.2), where $\mathfrak{R}_c$ is defined implicitly, all vanish as $|\mathrm{Re}(z)| \to \infty$. Moreover, $\mathfrak{R}_c$ is analytic on the strip $S_{3q_{\mathcal{B}}}$. We can therefore shift the integration contour in (D.1.6) from $\mathrm{Im}(z) = q$ to $\mathrm{Im}(z) = 2q_{\mathcal{B}}$ and obtain

$$\int_{\mathrm{Im}(z)=q} e^{-ixz}\mathfrak{R}_c(z)\,dz = e^{2q_{\mathcal{B}}x}\int_{-\infty}^{\infty} e^{ikx}\mathfrak{R}_c(k+2iq_{\mathcal{B}})\,dk.$$

From (D.1.6), we conclude

$$\text{(D.1.8)}\qquad \mathfrak{F}^{-1}[\mathfrak{R}_c(-\cdot+iq)](x) = -\frac{e^{(2q_{\mathcal{B}}-q)x}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}\mathfrak{R}_c(k+2iq_{\mathcal{B}})\,dk,$$

Since $2q_{\mathcal{B}} - q > 0$, we will have (D.1.7) in the $L^1(\mathbb{R}_-)$ case if we can show

$$\text{(D.1.9)}\qquad \sup_{\substack{1<|c|\le\sqrt{2}\\ x\in\mathbb{R}_-}} \left|\int_{-\infty}^{\infty} e^{ikx}\mathfrak{R}_c(k+2iq_{\mathcal{B}})\,dk\right| < \infty.$$

The integral $\int_{-\infty}^{\infty} e^{ikx}\mathfrak{R}_c(-k+2iq_{\mathcal{B}})\,dk$ in (D.1.9) is the inverse Fourier transform of $\mathfrak{R}_c(-\cdot+2iq_{\mathcal{B}})$, which we may estimate using the implicit definition of $\mathfrak{R}_c$ in (D.1.2): We have

$$|\mathfrak{F}^{-1}[\mathfrak{R}_c(-\cdot+2iq_{\mathcal{B}})](x)| \le \underbrace{\left|\mathfrak{F}^{-1}\left[\frac{1}{\widetilde{\mathcal{B}}_c(-\cdot+2iq_{\mathcal{B}})}\right](x)\right|}_{I} + \underbrace{\frac{1}{|(\widetilde{\mathcal{B}}_c)'(\omega_c)|}\left|\mathfrak{F}^{-1}\left[\frac{1}{(-\cdot+2iq_{\mathcal{B}}-\omega_c)}\right](x)\right|}_{II}$$

$$+ \underbrace{\frac{1}{|(\widetilde{\mathcal{B}}_c)'(-\omega_c)|}\left|\mathfrak{F}^{-1}\left[\frac{1}{(-\cdot+2iq_{\mathcal{B}}+\omega_c)}\right](x)\right|}_{III}.$$

The quantities II and III are uniformly bounded for $1 < |c| \le \sqrt{2}$ due to the calculations of the specific Fourier transforms in (D.1.3) and (D.1.4) and the bounds (4.0.5). We obtain a uniform bound for I over $1 < |c| \le \sqrt{2}$ by estimating the integral $\int_{-\infty}^{\infty}(1/|\widetilde{\mathcal{B}}_c(-k+2iq_{\mathcal{B}})|)\,dk$ over the intervals $(-\infty, r_0]$ and $[r_0, \infty)$, where r_0 is defined in (4.0.7), with the quadratic estimate (4.0.6) and then over the interval $[-r_0, r_0]$ just by bounding $1/|\widetilde{\mathcal{B}}_c(-k+2iq_{\mathcal{B}})|$ uniformly in c for $|k| \le r_0$. This proves (D.1.9).

To study $\|\mathfrak{F}^{-1}[\mathfrak{R}_c(-\cdot+iq)\|_{L^1(\mathbb{R}_+)}$, we repeat the work above, except we shift the contour in (D.1.6) to $\mathrm{Im}(z) = -2q_{\mathcal{B}}$, so that

$$\int_{-\infty}^{\infty} e^{ikx}\mathfrak{R}_c(-k+iq)\ dk = -e^{-(2q_{\mathcal{B}}+q)x}\int_{-\infty}^{\infty} e^{-ikx}\mathfrak{R}_c(k-2iq_{\mathcal{B}})\ dk.$$

Then we obtain an estimate analogous to (D.1.9), which in turn proves (D.1.7) for $L^1(\mathbb{R}_+)$.

D.2. **Additional estimates on the Friesecke-Pego solitary wave ς_{c_ϵ}.** We deduce the following lemma from Lemmas 3.1 and 3.2 in [HW08].

Lemma D.1 (Hoffman & Wayne)**.** *There exist $\epsilon_{\mathsf{HWa}} \in (0, \epsilon_{\mathsf{FP}}]$ and a, $C > 0$ such that*

$$\left\| e^{(a/\epsilon)\cdot}\left(\frac{1}{\epsilon^2}\varsigma_{c_\epsilon}\left(\frac{\cdot}{\epsilon}\right) - \sigma\right)\right\|_{H^1} \le C\epsilon^2$$

for $0 < \epsilon < \epsilon_{\mathsf{HWa}}$.

Now we prove a bevy of estimates on ς_{c_ϵ}, all of which say either that ς_{c_ϵ} is "small" in a certain norm or that ς_{c_ϵ} and $\epsilon^2\sigma(\epsilon\cdot)$, where σ was defined in (6.1.2), are "close" in some norm.

Proposition D.2. *There is $C > 0$ such that the following estimates hold for all $\epsilon \in (0, \epsilon_{\mathsf{HWa}})$.*

(i) $\|e^{a|\cdot|}(\varsigma_{c_\epsilon} - \epsilon^2\sigma(\epsilon\cdot))\|_{L^2} \le C\epsilon^{7/2}$.

(ii) $\|\varsigma_{c_\epsilon} - \epsilon^2\sigma(\epsilon\cdot)\|_{L^1} \le C\epsilon^{7/2}$.

(iii) $\|\varsigma_{c_\epsilon}\|_{L^1} \le C\epsilon$.

(iv) $\|\varsigma_{c_\epsilon} - \epsilon^2\sigma(\epsilon\cdot)\|_{L^\infty} \le C\epsilon^2$.

(v) $\|\varsigma_{c_\epsilon}\|_{L^\infty} \le C\epsilon^2$.

(vi) $\|e^{q\cdot}\varsigma_{c_\epsilon}\|_{L^\infty} \le C\epsilon^2$ *for $0 < q < \min\{\epsilon/2, a\}$.*

Proof. **(i)** We use the evenness of the integrand and substitute $u = \epsilon x$ to find

$$\|e^{a|\cdot|}(\varsigma_{c_\epsilon}-\epsilon^2\sigma(\epsilon\cdot))\|_{L^2}^2 = \int_{-\infty}^{\infty} e^{2a|x|}|\varsigma_{c_\epsilon}(x)-\epsilon^2\sigma(\epsilon x)|^2\ dx = 2\epsilon^3\int_0^{\infty} e^{2au/\epsilon}\left|\frac{1}{\epsilon^2}\varsigma_{c_\epsilon}\left(\frac{u}{\epsilon}\right) - \sigma(u)\right|^2\ du$$

$$\le 2\epsilon^3\int_{-\infty}^{\infty} e^{2au/\epsilon}\left|\frac{1}{\epsilon^2}\varsigma_{c_\epsilon}\left(\frac{u}{\epsilon}\right) - \sigma(u)\right|^2\ du = 2\epsilon^3\left\|e^{(a/\epsilon)\cdot}\left(\frac{1}{\epsilon^2}\varsigma_{c_\epsilon}\left(\frac{\cdot}{\epsilon}\right) - \sigma\right)\right\|_{L^2}^2 \le C\epsilon^7.$$

(ii) The Cauchy-Schwarz inequality implies

$$\|\varsigma_{c_\epsilon} - \epsilon^2\sigma(\epsilon\cdot)\|_{L^1} = \int_{-\infty}^{\infty}|\varsigma_{c_\epsilon}(x) - \epsilon^2\sigma(\epsilon x)|\ dx = \int_{-\infty}^{\infty} e^{-a|x|}\big(e^{a|x|}|\varsigma_{c_\epsilon}(x) - \epsilon^2\sigma(\epsilon x)|\big)\ dx$$

$$\le \left(\int_{-\infty}^{\infty} e^{-2a|x|}\ dx\right)^{1/2}\left(\int_{-\infty}^{\infty} e^{2a|x|}|\varsigma_{c_\epsilon}(x) - \epsilon^2\sigma(\epsilon x)|^2\ dx\right)^{1/2}.$$

The first integral on the second line is just a constant (depending on a), and the second integral is $\mathcal{O}(\epsilon^{7/2})$ by part (i).

(iii) This follows directly from part (ii) and the calculation $\|\sigma(\epsilon\cdot)\|_{L^1} = \mathcal{O}(\epsilon^{-1})$.

(iv) The Sobolev embedding and the Friesecke-Pego estimate (6.1.3) imply

$$\left\|\frac{1}{\epsilon^2}\varsigma_{c_\epsilon}\left(\frac{\cdot}{\epsilon}\right) - \sigma\right\|_{L^\infty} \le C\epsilon^2,$$

and then rescaling gives the desired estimate.

(v) This follows directly from part (iv) and the calculation $\|\sigma(\epsilon\cdot)\|_{L^\infty} = \mathcal{O}(1)$.

(vi) First, we have

$$\|e^{q\cdot}\varsigma_{c_\epsilon}\|_{L^\infty} \le \|e^{q\cdot}\varsigma_{c_\epsilon}\|_{L^\infty(\mathbb{R}_-)} + \|e^{q\cdot}\varsigma_{c_\epsilon}\|_{L^\infty(\mathbb{R}_+)}.$$

Since $q > 0$, part (v) gives

$$\|e^{q\cdot}\varsigma_{c_\epsilon}\|_{L^\infty(\mathbb{R}_-)} \le \|\varsigma_{c_\epsilon}\|_{L^\infty(\mathbb{R}_-)} \le \|\varsigma_{c_\epsilon}\|_{L^\infty} \le C\epsilon^2.$$

Next,

$$\|e^{q\cdot}\varsigma_{c_\epsilon}\|_{L^\infty(\mathbb{R}_+)} \le \epsilon^2\|e^{q\cdot}\sigma(\epsilon\cdot)\|_{L^\infty(\mathbb{R}_+)} + \|e^{q\cdot}(\varsigma_{c_\epsilon} - \epsilon^2\sigma(\epsilon\cdot))\|_{L^\infty(\mathbb{R}_+)}. \tag{D.2.1}$$

Since $0 < q < \epsilon/2$ and $x \ge 0$, we have

$$|e^{qx}\sigma(\epsilon x)| \le Ce^{qx}e^{-\epsilon x} \le Ce^{-\epsilon x/2} \le C,$$

hence $\|e^{q\cdot}\sigma(\epsilon\cdot)\|_{L^\infty(\mathbb{R}_+)} = \mathcal{O}(1)$. For the other term in (D.2.1), we rewrite

$$\begin{aligned}\|e^{q\cdot}(\varsigma_{c_\epsilon} - \epsilon^2\sigma(\epsilon\cdot))\|_{L^\infty(\mathbb{R}_+)} &\le \|e^{a|\cdot|}(\varsigma_{c_\epsilon} - \epsilon^2\sigma(\epsilon\cdot))\|_{L^\infty(\mathbb{R}_+)} \le \|e^{a|\cdot|}(\varsigma_{c_\epsilon} - \epsilon^2\sigma(\epsilon\cdot))\|_{L^\infty} \\ &= \epsilon^2\left\|e^{(a/\epsilon)|\cdot|}\left(\frac{1}{\epsilon^2}\varsigma_{c_\epsilon}\left(\frac{\cdot}{\epsilon}\right) - \sigma\right)\right\|_{L^\infty} \le C\epsilon^4\end{aligned}$$

by Lemma D.1. □

Appendix E. Some Proofs for the Verification of Hypotheses 3 and 4 in the case $|c| \gtrsim 1$

E.1. **Background from the theory of modified functional differential equations.** We extract the content of the following theorem from Theorem 3.2 and Proposition 3.4 in [HL07].

Theorem E.1 (Hupkes & Verduyn Lunel)**.** *Let $A_0, A_1, \ldots, A_n \in \mathbb{C}^{m\times m}$ and let $d_0 < d_1 < \cdots < d_n$ be real numbers. For $\mathbf{f} \in W^{1,\infty}$, define*

$$\Lambda\mathbf{f} := \mathbf{f}' - \sum_{j=1}^{n} A_j S^{d_j}\mathbf{f}. \tag{E.1.1}$$

Let

$$\Delta(z) := z\mathbb{1} - \sum_{j=1}^{n} e^{d_j z}A_j \in \mathbb{C}^{m\times m}, \tag{E.1.2}$$

where $\mathbb{1}$ *is the* $m \times m$ *identity matrix. We call* Δ *the characteristic matrix corresponding to the system* (E.1.1)*. Suppose that* $q \in \mathbb{R}$ *with* $\det[\Delta(z)] \neq 0$ *for* $\mathrm{Re}(z) = q$*. Then* Λ *is an isomorphism between* $W_q^{1,\infty}(\mathbb{R}, \mathbb{C}^m)$ *and* $L_q^\infty(\mathbb{R}, \mathbb{C}^m)$*, where these spaces were defined in Definition 2.2. We denote its inverse by* Λ_q^{-1}*. Moreover, we have two formulas for* Λ_q^{-1}*.*

(i) *There is a function* $\mathscr{G}_q \in \cap_{p=1}^\infty L^p(\mathbb{R}, \mathbb{C}^{m \times m})$ *such that for* $\mathbf{g} \in L_q^\infty(\mathbb{R}, \mathbb{C}^{m \times m})$*,*

(E.1.3)
$$(\Lambda_q^{-1})(x) = e^{qx} \int_{-\infty}^{\infty} e^{-qs} \mathscr{G}_q(x-s)\mathbf{g}(s)\ ds$$

Moreover,

(E.1.4)
$$\widehat{\mathscr{G}}_q(k) = \Delta(ik+q)^{-1}, \ k \in \mathbb{R}.$$

(ii) *Given* $\delta \in (0, |q|)$ *and* $\mathbf{g} \in L_q^\infty(\mathbb{R}, \mathbb{C}^m)$*, we have*

(E.1.5)
$$(\Lambda_q^{-1}\mathbf{g})(x) = \frac{1}{2\pi i} \int_{q+\delta-i\infty}^{q+\delta+i\infty} e^{xz} \Delta(z)^{-1} \mathscr{L}_+[\mathbf{g}](z)\ dz + \frac{1}{2\pi i} \int_{q-\delta-i\infty}^{q-\delta+i\infty} e^{xz} \Delta(z)^{-1} \mathscr{L}_-[\mathbf{g}](z)\ dz,$$

where $\mathscr{L}_\pm$ *are the Laplace transforms defined in Appendix A.2.*

E.2. **Proof of Proposition 6.4.** That $\varsigma_{c_\epsilon} g \in L_{q_\epsilon}^\infty$ for $g \in L_{-q_\epsilon}^\infty$ is a straightforward consequence of part (vi) of Proposition D.2.

E.2.1. *Residue theory.* Take $R > 0$ so large that $0 < \omega_{c_\epsilon} < R$ for all $0 < \epsilon < \epsilon_{\mathcal{B}}$ and consider the contour $\varrho_{\epsilon,1}^\pm(R) + \varrho_{\epsilon,2}^\pm(R) + \varrho_{\epsilon,3}^\pm(R) + \varrho_{\epsilon,4}^\pm(R)$ sketched in Figure 5. The residue theorem tells us

(E.2.1)
$$\frac{1}{2\pi i} \int_{\varrho_{\epsilon,1}^\pm(R)+\varrho_{\epsilon,2}^\pm(R)+\varrho_{\epsilon,3}^\pm(R)+\varrho_{\epsilon,4}^\pm(R)} \mathcal{K}_\epsilon(x,z)\mathscr{L}_\pm[\varsigma_{c_\epsilon} g](z)\ dz = \mathrm{Res}\left(\mathcal{K}_\epsilon(x,z)\mathscr{L}_\pm[\varsigma_{c_\epsilon} g](z); z = i\omega\right)$$
$$+ \mathrm{Res}\left(\mathcal{K}_\epsilon(x,z)\mathscr{L}_\pm[\varsigma_{c_\epsilon} g](z); z = -i\omega\right).$$

By the inverse formulas in (6.3.11), we can write

(E.2.2)
$$[\mathcal{B}_{c_\epsilon}^-]^{-1}[\varsigma_{c_\epsilon} g](x) = -\frac{1}{2\pi i} \lim_{R \to \infty} \left(\int_{\varrho_{\epsilon,3}^+(R)} \mathcal{K}_\epsilon(x,z)\mathscr{L}_+[\varsigma_{c_\epsilon} g](z)\ dz + \int_{\varrho_{\epsilon,3}^-(R)} \mathcal{K}_\epsilon(x,z)\mathscr{L}_+[\varsigma_{c_\epsilon} g](z)\ dz \right)$$

and

(E.2.3)
$$[\mathcal{B}_{c_\epsilon}^+]^{-1}[\varsigma_{c_\epsilon} g](x) = \frac{1}{2\pi i} \lim_{R \to \infty} \left(\int_{\varrho_{\epsilon,1}^+(R)} \mathcal{K}_\epsilon(x,z)\mathscr{L}_+[\varsigma_{c_\epsilon} g](z)\ dz + \int_{\varrho_{\epsilon,1}^-(R)} \mathcal{K}_\epsilon(x,z)\mathscr{L}_-[\varsigma_{c_\epsilon} g](z)\ dz \right).$$

If we can show that

(E.2.4)
$$\lim_{R \to \infty} \int_{\varrho_{\epsilon,2}^\pm(R)} \mathcal{K}_\epsilon(x,z)\mathscr{L}_\pm[\varsigma_{c_\epsilon} g](z)\ dz = \lim_{R \to \infty} \int_{\varrho_{\epsilon,4}^\pm(R)} \mathcal{K}_\epsilon(x,z)\mathscr{L}_\pm[\varsigma_{c_\epsilon} g](z)\ dz = 0$$

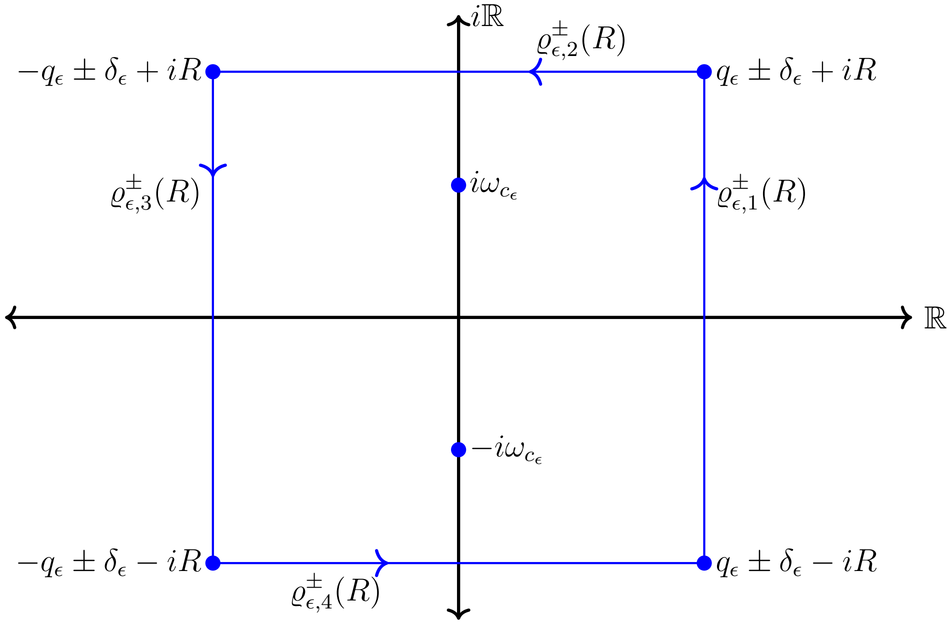

FIGURE 5. The contour $\varrho_{\epsilon,1}^{\pm}(R) + \varrho_{\epsilon,2}^{\pm}(R) + \varrho_{\epsilon,3}^{\pm}(R) + \varrho_{\epsilon,4}^{\pm}(R)$

then we can combine (E.2.1), (E.2.2), and (E.2.3) to conclude
(E.2.5)
$$\begin{aligned}[\mathcal{B}_{c_\epsilon}^{-}]^{-1}[\varsigma_{c_\epsilon} g](x) = [\mathcal{B}_{c_\epsilon}^{+}]^{-1}[\varsigma_{c_\epsilon} g](x) &- \mathrm{Res}\left(\mathcal{K}_\epsilon(x,z)\mathscr{L}_+[\varsigma_{c_\epsilon} g](z); z = i\omega_{c_\epsilon}\right) \\ &- \mathrm{Res}\left(\mathcal{K}_\epsilon(x,z)\mathscr{L}_+[\varsigma_{c_\epsilon} g](z); z = -i\omega_{c_\epsilon}\right) - \mathrm{Res}\left(\mathcal{K}_\epsilon(x,z)\mathscr{L}_-[\varsigma_{c_\epsilon} g](z); z = i\omega_{c_\epsilon}\right) \\ &- \mathrm{Res}\left(\mathcal{K}_\epsilon(x,z)\mathscr{L}_-[\varsigma_{c_\epsilon} g](z); z = -i\omega_{c_\epsilon}\right).\end{aligned}$$

We will then be able to massage the right side of (E.2.5) into our desired formula (4.3.8). So, we commence with the residue theory.

E.2.2. *Calculation of the residues* (E.2.1). Recall from part (i) of Proposition 4.2 that

$$|(\widetilde{\mathcal{B}}_{c_\epsilon})'(\pm\omega_{c_\epsilon})| \ge b_{\mathcal{B}} > 0$$

for $0 < \epsilon < \epsilon_{\mathcal{B}}$ and $|\mu| \le \mu_{\mathcal{B}}(c_\epsilon)$. This hypothesis and the definition of $\mathcal{K}_\epsilon$ in (6.3.12) ensures that $\mathcal{K}_\epsilon(x,z)\mathscr{L}_\pm[\varsigma_{c_\epsilon} g](z)$ has simple poles at $z = \pm i\omega_{c_\epsilon}$ and so

$$\mathrm{Res}(\mathcal{K}_\epsilon(x,z)\mathscr{L}_\pm[\varsigma_{c_\epsilon} g](z); z = i\omega_{c_\epsilon}) = i\left(\frac{\mathscr{L}_\pm[\varsigma_{c_\epsilon} g](i\omega_{c_\epsilon})}{(\widetilde{\mathcal{B}}_{c_\epsilon})'(\omega_{c_\epsilon})}\right)e^{i\omega_{c_\epsilon} x} \tag{E.2.6}$$

and

$$\mathrm{Res}(\mathcal{K}_\epsilon(x,z)\mathscr{L}_\pm[\varsigma_{c_\epsilon} g](z); z = -i\omega_{c_\epsilon}) = -i\left(\frac{\mathscr{L}_\pm[\varsigma_{c_\epsilon} g](-i\omega_{c_\epsilon})}{(\widetilde{\mathcal{B}}_{c_\epsilon})'(-\omega_{c_\epsilon})}\right)e^{-i\omega_{c_\epsilon} x}. \tag{E.2.7}$$

E.2.3. *Estimates on the Laplace transforms in* (E.2.6) *and* (E.2.7). Throughout this section, we will need the estimates on ς_{c_ϵ} from Proposition D.2.

We calculate

$$|\mathscr{L}_+[\varsigma_{c_\epsilon} g](\pm i\omega_{c_\epsilon})| = \left|\int_0^\infty e^{\mp i\omega_{c_\epsilon} s}\varsigma_{c_\epsilon}(s)g(s)\ ds\right| \le \int_0^\infty \varsigma_{c_\epsilon}(s)|g(s)|\ ds.$$

Now we multiply by $1 = e^{q_\epsilon s}e^{-q_\epsilon s}$ to find

$$\int_0^\infty \varsigma_{c_\epsilon}(s)|g(s)|\ ds = \int_0^\infty e^{-q_\epsilon s}\varsigma_{c_\epsilon}(s)e^{q_\epsilon s}|g(s)|\ ds \le \|g\|_{L^\infty_{-q_\epsilon}} \int_0^\infty e^{-q_\epsilon s}\varsigma_{c_\epsilon}(s)\ ds$$

$$\le \|g\|_{L^\infty_{-q_\epsilon}} \int_0^\infty \varsigma_{c_\epsilon}(s)\ ds \le \|g\|_{L^\infty_{-q_\epsilon}}\|\varsigma_{c_\epsilon}\|_{L^1} \le C\epsilon\|g\|_{L^\infty_{-q_\epsilon}}.$$

Here we need to recall from Theorem 6.1 that $\varsigma_{c_\epsilon} > 0$, so $\|\varsigma_{c_\epsilon}\|_{L^1} = \int_{-\infty}^\infty \varsigma_{c_\epsilon}(x)\ dx$.

The situation for $\mathscr{L}_-$ is slightly different. With the same manipulations as above, we find

$$|\mathscr{L}_-[\varsigma_{c_\epsilon} g](\pm i\omega_{c_\epsilon})| \le \|g\|_{L^\infty_{-q_\epsilon}} \int_0^\infty e^{q_\epsilon s}\varsigma_{c_\epsilon}(s)\ ds.$$

Now the exponential contains the positive factor q_ϵ, so we cannot blithely ignore it. Instead, we bound this integral as

$$\int_0^\infty e^{q_\epsilon s}\varsigma_{c_\epsilon}(s)\ ds \le \underbrace{\epsilon^2\int_0^\infty e^{q_\epsilon s}\sigma(\epsilon s)\ ds}_{\mathcal{I}_{\epsilon,1}} + \underbrace{\int_0^\infty e^{q_\epsilon s}|\varsigma_{c_\epsilon}(s) - \epsilon^2\sigma(\epsilon s)|\ ds}_{\mathcal{I}_{\epsilon,2}}.$$

Since $\sigma(X) = \operatorname{sech}^2(X/2)/4$, we use the estimates on q_ϵ from (6.3.4) to bound

$$\mathcal{I}_{\epsilon,1} \le C\epsilon^2\int_0^\infty e^{q_\epsilon s}e^{-\epsilon s}\ ds \le C\epsilon^2\int_0^\infty e^{-\epsilon s/2}\ ds = \mathcal{O}(\epsilon).$$

Next, we rewrite

$$\mathcal{I}_{\epsilon,2} = \int_0^\infty (e^{q_\epsilon s}e^{-a\epsilon s})\big(e^{a\epsilon s}|\varsigma_{c_\epsilon}(s) - \epsilon^2\sigma(\epsilon s)|\big)\ ds$$

$$\le \underbrace{\left(\int_0^\infty e^{2(q_\epsilon - a\epsilon)s}\ ds\right)^{1/2}}_{\mathcal{I}_{\epsilon,3}} \underbrace{\left(\int_0^\infty e^{2a\epsilon s}|\varsigma_{c_\epsilon}(s) - \epsilon^2\sigma(\epsilon s)|^2\ ds\right)^{1/2}}_{\mathcal{I}_{\epsilon,4}}.$$

We appeal again to (6.3.4) to find

$$\mathcal{I}_{\epsilon,3} \le \left(\int_0^\infty e^{-2a\epsilon s}\ ds\right)^{1/2} = \mathcal{O}(\epsilon^{-1/2}).$$

Last, since $0 \le e^{a\epsilon s} \le e^{as}$ for $0 \le \epsilon \le 1$ and $s \ge 0$, we use part (i) of Proposition D.2 to estimate

$$\mathcal{I}_{\epsilon,4} \le \left(\int_0^\infty e^{2as}|\varsigma_{c_\epsilon}(s) - \epsilon^2\sigma(\epsilon s)|^2\ ds\right)^{1/2} \le \|e^{a|\cdot|}(\varsigma_{c_\epsilon} - \epsilon^2\sigma(\epsilon\cdot))\|_{L^2} \le C\epsilon^2.$$

Combining all these estimates, we conclude

$$|\mathscr{L}_\pm[\varsigma_{c_\epsilon} g](\pm i\omega_{c_\epsilon})| \le C\epsilon\|g\|_{L^\infty_{-q_\epsilon}}. \tag{E.2.8}$$

E.2.4. *The limits* (E.2.4). We prove only that

$$\lim_{R\to\infty}\int_{\varrho_{\epsilon,2}^{+}(R)}\mathcal{K}_{\epsilon}(x,z)\mathscr{L}_{+}[\varsigma_{c_\epsilon}g](z)\ dz=0, \tag{E.2.9}$$

the other cases being similar. We parametrize the line segment $\varrho_{\epsilon,2}^{+}(R)$ by

$$z_{\epsilon}(t,R):=(1-t)(q_{\epsilon}+\delta_{\epsilon}+iR)+t(-q_{\epsilon}+\delta_{\epsilon}+iR)=-2q_{\epsilon}t+\delta_{\epsilon}+q_{\epsilon}+iR,$$

and so the line integral in (E.2.9) over $\varrho_{\epsilon,2}^{+}(R)$ is

$$-2q_{\epsilon}\int_0^1\mathcal{K}_{\epsilon}(x,z_{\epsilon}(t,R))\mathscr{L}_{+}[\varsigma_{c_\epsilon}g](z_{\epsilon}(t,R))\ dt.$$

To establish (E.2.9), it suffices, of course, to show that

$$\lim_{R\to\infty}\max_{0\le t\le 1}|\mathcal{K}_{\epsilon}(x,z_{\epsilon}(t,R))\mathscr{L}_{+}[\varsigma_{c_\epsilon}g](z_{\epsilon}(t,R))|=0. \tag{E.2.10}$$

The methods of Section E.2.3 can be adapted to show that $|\mathscr{L}_{+}[\varsigma_{c_\epsilon}g](z_{\epsilon}(t,R))|$ is bounded above by a constant independent of t or R (but dependent on g and ϵ, although that does not matter here). Next, we refer to the definition of $\mathcal{K}_{\epsilon}$ in (6.3.12) to estimate

$$|\mathcal{K}_{\epsilon}(x,-2q_{\epsilon}t+\delta_{\epsilon}+q_{\epsilon}+iR)|\le\frac{e^{x(-2q_{\epsilon}t+\delta_{\epsilon})}}{\widetilde{\mathcal{B}}_{c_\epsilon}(iz_{\epsilon}(t,R))}$$

The numerator on the right side above is independent of R and bounded in t. We can infer from the quadratic estimates on $\widetilde{\mathcal{B}}_{c_\epsilon}$ in (4.0.6) that the denominator is $\mathcal{O}(R^2)$ uniformly in t, and so the whole expression above vanishes as $R\to\infty$.

E.2.5. *Conclusion of the proof of Proposition 6.4.* Since we have established the vanishing of the integrals in (E.2.4), we may use the residues computed in (E.2.6) and (E.2.7) and the strategy outlined in Section E.2.1 to conclude from (E.2.5) that

$$[\mathcal{B}_{c_\epsilon}^{-}]^{-1}[\varsigma_{c_\epsilon}g](x)=[\mathcal{B}_{c_\epsilon}^{+}]^{-1}[\varsigma_{c_\epsilon}g](x)+-\left(\frac{\mathscr{L}_{+}[\varsigma_{c_\epsilon}g](i\omega_{c_\epsilon})+\mathscr{L}_{-}[\varsigma_{c_\epsilon}g](i\omega_{c_\epsilon})}{(\widetilde{\mathcal{B}}_{c_\epsilon})'(\omega_{c_\epsilon})}\right)e^{i\omega_{c_\epsilon}x}$$

$$+i\left(\frac{\mathscr{L}_{+}[\varsigma_{c_\epsilon}g](-i\omega_{c_\epsilon})+\mathscr{L}_{-}[\varsigma_{c_\epsilon}g](-i\omega_{c_\epsilon})}{(\widetilde{\mathcal{B}}_{c_\epsilon})'(-\omega_{c_\epsilon})}\right)e^{-i\omega_{c_\epsilon}x}.$$

The estimates on the functionals α_{ϵ} and β_{ϵ} in (6.4.7) then follow from the estimates on the Laplace transforms in Section E.2.3, particularly (E.2.8).

E.3. **Proof of Proposition 6.5.**

E.3.1. *Refined estimates on α_{ϵ}.* The functional α_{ϵ} was defined in (6.4.5). Using the expression for g_{ϵ} from (6.4.8), we have

$$\alpha_{\epsilon}[\mathcal{M}g_{\epsilon}]=\alpha_{\epsilon}[\mathcal{M}e^{i\omega_{c_\epsilon}\cdot}]+\alpha_{\epsilon}\left[\mathcal{M}\sum_{k=1}^{\infty}\left([\mathcal{B}_{c_\epsilon}^{-}]^{-1}\Sigma_{c_\epsilon}^{*}\right)^{k}e^{i\omega_{c_\epsilon}\cdot}\right].$$

Since $\mathcal{M}e^{i\omega_{c_\epsilon}\cdot} = \widetilde{\mathcal{M}}(\omega_{c_\epsilon})e^{i\omega_{c_\epsilon}\cdot}$, per (6.3.14), we can calculate the first term just by computing

$$\alpha_\epsilon[e^{i\omega_{c_\epsilon}\cdot}] = -i\widetilde{\mathcal{M}}(\omega_{c_\epsilon})\left(\frac{\mathscr{L}_+[\varsigma_{c_\epsilon}e^{i\omega_{c_\epsilon}\cdot}](i\omega_{c_\epsilon}) + \mathscr{L}_-[\varsigma_{c_\epsilon}e^{i\omega_{c_\epsilon}\cdot}](i\omega_{c_\epsilon})}{(\widetilde{\mathcal{B}}_{c_\epsilon})'(\omega_{c_\epsilon})}\right).$$

The Laplace transforms are

$$\mathscr{L}_\pm[\varsigma_{c_\epsilon}e^{i\omega_{c_\epsilon}\cdot}](i\omega_{c_\epsilon}) = \int_0^\infty e^{\mp i\omega s}\varsigma_{c_\epsilon}(s)e^{\pm i\omega_{c_\epsilon}s}\,ds = \int_0^\infty \varsigma_{c_\epsilon}(s)\,ds = \frac{\|\varsigma_{c_\epsilon}\|_{L^1}}{2}.$$

Here we needed the result from Theorem 6.1 that ς_{c_ϵ} is positive. Thus

$$\alpha_\epsilon[\mathcal{M}e^{i\omega_{c_\epsilon}\cdot}] = -i\epsilon\left(\frac{\widetilde{\mathcal{M}}(\omega_{c_\epsilon})}{(\widetilde{\mathcal{B}}_{c_\epsilon})'(\omega_{c_\epsilon})}\right)\left(\frac{\|\varsigma_{c_\epsilon}\|_{L^1}}{\epsilon}\right) =: i\epsilon\theta^+_{\epsilon,0}, \tag{E.3.1}$$

and we note that $\theta^+_{\epsilon,0}$ is real. The uniform boundedness of ω_{c_ϵ}, the estimate (4.0.5), the definition of $\widetilde{\mathcal{M}}$ in (6.3.14), and part (iii) of Proposition D.2 tell us there are C_1, $C_2 > 0$ such that

$$0 < C_1 \le |\theta^+_{\epsilon,0}| \le C_2 < \infty$$

for all $0 < \epsilon < \epsilon_{\mathcal{B}}$.

Now we show

$$\alpha_\epsilon\left[\mathcal{M}\sum_{k=1}^\infty \left([\mathcal{B}^-_{c_\epsilon}]^{-1}\Sigma^*_{c_\epsilon}\right)^k e^{i\omega_{c_\epsilon}\cdot}\right] = \mathcal{O}(\epsilon^2). \tag{E.3.2}$$

This holds if there exists $C = \mathcal{O}(1)$ such that

$$\left|\alpha_\epsilon[\mathcal{M}\left([\mathcal{B}^-_{c_\epsilon}]^{-1}\Sigma^*_{c_\epsilon}\right)^k e^{i\omega_{c_\epsilon}\cdot}]\right| \le C^k\epsilon^{k+1}. \tag{E.3.3}$$

In that case, we have

$$\alpha_\epsilon\left[\mathcal{M}\sum_{k=1}^\infty \left([\mathcal{B}^-_{c_\epsilon}]^{-1}\Sigma^*_{c_\epsilon}\right)^k e^{i\omega_{c_\epsilon}\cdot}\right] \le \sum_{k=1}^\infty C^k\epsilon^{k+1} = C\epsilon^2\sum_{k=0}^\infty (C\epsilon)^k. \tag{E.3.4}$$

Then we set

$$\epsilon_{\mathsf{res},1} := \min\left\{\epsilon_{\mathcal{B}}, \frac{1}{2C}\right\}, \tag{E.3.5}$$

in which case the geometric series in (E.3.4) converges for[7] $0 < \epsilon < \epsilon_{\mathsf{res},1}$. We conclude

$$\alpha_\epsilon\left[\mathcal{M}g_\epsilon\right] = i\epsilon\theta^+_{\epsilon,0} + \epsilon^2\theta^+_\epsilon, \tag{E.3.6}$$

where $\theta^+_{\epsilon,0} \in \mathbb{R}\setminus\{0\}$ and $\theta^+_\epsilon = \mathcal{O}(1)$.

So, we only need to prove (E.3.3). Fix some $h \in L^\infty \cup L^\infty_{-q_\epsilon}$. Then

$$\|\varsigma_{c_\epsilon}h\|_{L^\infty_{-q_\epsilon}} \le C\epsilon^2\min\left\{\|h\|_{L^\infty}, \|h\|_{L^\infty_{-q_\epsilon}}\right\}, \tag{E.3.7}$$

for if $h \in L^\infty_{-q_\epsilon}$, then

$$\|e^{q_\epsilon\cdot}\varsigma_{c_\epsilon}h\|_{L^\infty} \le \|\varsigma_{c_\epsilon}\|_{L^\infty}\|e^{q_\epsilon\cdot}h\|_{L^\infty} \le C\epsilon^2\|h\|_{L^\infty_{-q_\epsilon}}$$

[7] We are, admittedly, being rather cavalier about what C is. In Section 1.8, we agreed that C would denote any constant that is $\mathcal{O}(1)$ in ϵ; now we are restricting our range of ϵ based on one of these C. To be more precise, we could trace the lineage of the C defining $\epsilon_{\mathsf{res},1}$ in (E.3.5) back to three sources: the estimates in Proposition 6.3 on $[\mathcal{B}^-_{c_\epsilon}]^{-1}$, in (6.4.7) on α_ϵ, and in Proposition D.2 on ς_{c_ϵ}.

by part (v) of Proposition D.2. Otherwise, if $h \in L^\infty$, we can use the estimate $\|e^{q_\epsilon \cdot}\varsigma_{c_\epsilon}\|_{L^\infty} \le C\epsilon^2$ from part (vi) of that proposition to bound $\|\varsigma_{c_\epsilon} h\|_{L^\infty_{-q_\epsilon}} \le C\epsilon^2 \|h\|_{L^\infty}$.

Next, we estimate

(E.3.8)
$$\|[\mathcal{B}_{c_\epsilon}^-]^{-1}\varsigma_{c_\epsilon} h\|_{L^\infty_{-q_\epsilon}} \le \|[\mathcal{B}_{c_\epsilon}^-]^{-1}\varsigma_{c_\epsilon} h\|_{W^{2,\infty}_{-q_\epsilon}} \le C\epsilon^{-1}\|\varsigma_{c_\epsilon} h\|_{L^\infty_{-q_\epsilon}} \le C\epsilon \min\left\{\|h\|_{L^\infty}, \|h\|_{L^\infty_{-q_\epsilon}}\right\}$$

by Proposition 6.3 and (E.3.7). Last, we use the estimate (6.4.7) on α_ϵ to bound

$$\left|\alpha_\epsilon[\mathcal{M}[\mathcal{B}_{c_\epsilon}^-]^{-1}\varsigma_{c_\epsilon} h]\right| \le C\epsilon \|[\mathcal{B}_{c_\epsilon}^-]^{-1}\varsigma_{c_\epsilon} h\|_{L^\infty_{-q_\epsilon}} \le C\epsilon^2 \min\left\{\|h\|_{L^\infty}, \|h\|_{L^\infty_{-q_\epsilon}}\right\} \tag{E.3.9}$$

by (E.3.8).

If we rewrite

$$\mathcal{M}\big([\mathcal{B}_{c_\epsilon}^-]^{-1}\Sigma_{c_\epsilon}^*\big)^k e^{i\omega_{c_\epsilon}\cdot} = \mathcal{M}[\mathcal{B}_{c_\epsilon}^-]^{-1}\varsigma_{c_\epsilon}\mathcal{M}\big([\mathcal{B}_{c_\epsilon}^-]^{-1}\Sigma_{c_\epsilon}^*\big)^{k-1} e^{i\omega_{c_\epsilon}\cdot}$$

and take

$$h = \mathcal{M}\big([\mathcal{B}_{c_\epsilon}^-]^{-1}\Sigma_{c_\epsilon}^*\big)^{k-1} e^{i\omega_{c_\epsilon}\cdot},$$

then we can induct on k and use (E.3.9) to obtain our desired estimate (E.3.3).

E.3.2. *Refined estimates on β_ϵ.* The functional β_ϵ was defined in (6.4.6). We have

$$\beta_\epsilon[\mathcal{M}g_\epsilon] = \widetilde{\mathcal{M}}_\epsilon(\omega_{c_\epsilon})\beta_\epsilon[e^{i\omega_{c_\epsilon}\cdot}] + \beta_\epsilon\left[\mathcal{M}\sum_{k=1}^\infty \big([\mathcal{B}_{c_\epsilon}^-]^{-1}\Sigma_{c_\epsilon}^*\big)^k e^{i\omega_{c_\epsilon}\cdot}\right].$$

The methods of Appendix E.3.1 show

$$\beta_\epsilon\left[\mathcal{M}\sum_{k=1}^\infty \big([\mathcal{B}_{c_\epsilon}^-]^{-1}\Sigma_{c_\epsilon}^*\big)^k e^{i\omega_{c_\epsilon}\cdot}\right] = \mathcal{O}(\epsilon^2);$$

all we have to do is replace α_ϵ with β_ϵ. However, now we will show that $\beta_\epsilon[e^{i\omega_{c_\epsilon}\cdot}] = \mathcal{O}(\epsilon^2)$ as well.

We begin with the Laplace transforms in the definition of $\beta_\epsilon[e^{i\omega_{c_\epsilon}\cdot}]$:

$$\begin{aligned}\mathscr{L}_+[\varsigma_{c_\epsilon}e^{i\omega\cdot}](-i\omega_{c_\epsilon}) &= \int_0^\infty e^{-(-i\omega_{c_\epsilon})s}\varsigma_{c_\epsilon}(s)e^{i\omega_{c_\epsilon}s}\,ds = \int_0^\infty e^{2i\omega_{c_\epsilon}s}\varsigma_{c_\epsilon}(s)\,ds \\ &= \underbrace{\int_0^\infty \cos(2\omega_{c_\epsilon}s)\varsigma_{c_\epsilon}(s)\,ds}_{\mathcal{I}_{\epsilon,1}} + \underbrace{i\int_0^\infty \sin(2\omega_{c_\epsilon}s)\varsigma_{c_\epsilon}(s)\,ds}_{i\mathcal{I}_{\epsilon,2}}.\end{aligned} \tag{E.3.10}$$

We decompose

$$\mathcal{I}_{\epsilon,1} = \underbrace{\frac{\epsilon^2}{2}\int_{-\infty}^\infty \cos(2\omega_{c_\epsilon}s)\sigma(\epsilon s)\,ds}_{\mathcal{I}_{\epsilon,3}} + \underbrace{\int_0^\infty \cos(2\omega_{c_\epsilon}s)(\varsigma_{c_\epsilon}(s) - \epsilon^2\sigma(\epsilon s))\,ds}_{\mathcal{I}_{\epsilon,4}}.$$

Note that the evenness of the integrand allows us to integrate over all of $\mathbb{R}$ in $\mathcal{I}_{\epsilon,3}$. Moreover, after changing variables with $u = \epsilon s$ in $\mathcal{I}_{\epsilon,3}$, evenness also implies

$$\mathcal{I}_{\epsilon,3} = \frac{\epsilon}{2}\int_{-\infty}^\infty \cos\left(\frac{2\omega_{c_\epsilon}u}{\epsilon}\right)\sigma(u)\,du = \frac{\epsilon}{2}\widehat{\sigma}\left(\frac{2\omega_{c_\epsilon}}{\epsilon}\right).$$

Since $\sigma \in \cap_{r=1}^{\infty} H_1^r$ and ω_{c_ϵ} is bounded uniformly in ϵ, a variant on the Riemann-Lebesgue lemma (specifically, Lemma A.5 in [FW18]) shows $\mathcal{I}_{\epsilon,3} = \mathcal{O}(\epsilon^2)$. Next, we estimate directly

$$|\mathcal{I}_{\epsilon,4}| \le \|\varsigma_{c_\epsilon} - \epsilon^2\sigma(\epsilon\cdot)\|_{L^1} \le C\epsilon^2 \tag{E.3.11}$$

by part (ii) of Proposition D.2.

We make the same decomposition on $\mathcal{I}_{\epsilon,2}$ from (E.3.10) and use the L^1-estimate as in (E.3.11) to find

$$|\mathcal{I}_{\epsilon,2}| \le \left|\epsilon^2 \int_0^\infty \sin(2\omega_{c_\epsilon} s)\sigma(\epsilon s)\ ds\right| + C\epsilon^2.$$

An L^1-estimate on $\int_0^\infty \sin(2\omega_{c_\epsilon} s)\sigma(\epsilon s)\ ds$ will cost us a power of ϵ and reduce the estimate on $\mathcal{I}_{\epsilon,2}$ to only $\mathcal{O}(\epsilon)$. We can do better by changing variables with $u = \epsilon s$ and integrating by parts to find

$$\begin{aligned}
\epsilon^2 \int_0^\infty \sin(2\omega_{c_\epsilon} s)\sigma(\epsilon s)\ ds &= \epsilon \int_0^\infty \sin\left(\frac{2\omega_{c_\epsilon} u}{\epsilon}\right)\sigma(u)\ du \\
&= \epsilon\left(-\frac{\epsilon}{2\omega_{c_\epsilon}}\cos\left(\frac{2\omega_{c_\epsilon} u}{\epsilon}\right)\sigma(u)\Big|_{u=0}^{u=\infty} + \frac{\epsilon}{2\omega_{c_\epsilon}}\int_0^\infty \cos\left(\frac{2\omega_{c_\epsilon} u}{\epsilon}\right)\sigma(u)\ du\right) \\
&= \frac{\epsilon^2\sigma(0)}{2\omega_{c_\epsilon}} + \epsilon^2\int_0^\infty \cos\left(\frac{2\omega_{c_\epsilon} u}{\epsilon}\right)\sigma(u)\ du.
\end{aligned}$$

This is plainly $\mathcal{O}(\epsilon^2)$.

We have

$$\mathscr{L}_-[\varsigma_{c_\epsilon} e^{i\omega_{c_\epsilon}\cdot}](-i\omega_{c_\epsilon})\overline{\mathscr{L}_+[\varsigma_{c_\epsilon} e^{i\omega_{c_\epsilon}\cdot}](i\omega_{c_\epsilon})} = \mathcal{O}(\epsilon^2),$$

and so we conclude that for some $\epsilon_{\mathsf{res}} \le \epsilon_{\mathsf{res},1}$ and all $\epsilon \in (0,\overline{\epsilon})$, we have

$$\beta_\epsilon[\mathcal{M}g_\epsilon] = \epsilon^2\theta_\epsilon^-, \tag{E.3.12}$$

where $\theta_\epsilon^- = \mathcal{O}(1)$.

Appendix F. Proofs Involved in the Construction of the Nonlocal Solitary Wave Problem

F.1. **Proof of Proposition 2.1.** We simply rearrange (2.0.1) into

$$\varsigma_c'' = \frac{(A-2)(\varsigma_c + \varsigma_c^2)}{c^2}. \tag{F.1.1}$$

The operator $(A-2)/c^2$ maps E_q^r into E_q^r for any q and r. Since $\varsigma_c \in E^2_{q_\varsigma(c)}$, we also have $\varsigma_c^2 \in E^2_{q_\varsigma(c)}$, and so (F.1.1) implies $\varsigma_c'' \in E^2_{q_\varsigma(c)}$, i.e., $\varsigma_c \in E^4_{q_\varsigma(c)}$. We continue to bootstrap with (F.1.1) to obtain $\varsigma_c \in \cap_{r=1}^\infty E^r_{q_\varsigma(c)}$.

F.2. **Proof of Proposition 5.1.** First we study an auxiliary Fourier multiplier. Let

$$\widetilde{\mathcal{M}}_c(z) := -c^2z^2 + 2 - 2\cos(z).$$

Clearly $\widetilde{\mathcal{M}}_c$ is entire and one can check that $\widetilde{\mathcal{M}}_c$ has a double zero at $z = 0$ and no other zeros on $\mathbb{R}$. Next, as in the proof of part (iii) of Proposition 4.2, one can show the existence of $C(c)$, $z_0(c) > 0$ such that if $q_{\mathcal{H}}(c) \le |\operatorname{Im}(z)| \le \min\{q_\varsigma(c), 1\}$ and $|z| \ge z_0(c)$, then

$$C(c)|\operatorname{Re}(z)|^2 \le |\widetilde{\mathcal{M}}_c(z)|.$$

Lemma A.3 then provides $q_{\mathcal{H}}^\star(c)$, $q_{\mathcal{H}}^{\star\star}(c)$ such that $q_{\mathcal{H}}(c) < q_{\mathcal{H}}^\star(c) < q_{\mathcal{H}}^{\star\star}(c) < \min\{q_\varsigma(c), 1\}$ such that for $q \in [q_{\mathcal{H}}^\star(c), q_{\mathcal{H}}^{\star\star}(c)]$ and $r \ge 0$, the Fourier multiplier $\mathcal{M}_c$ with symbol $\widetilde{\mathcal{M}}_c$ is invertible from E_q^{r+2} to $E_{q,0}^r$.

Now we are ready to prove the proposition. Fix $q \in [q_{\mathcal{H}}^\star(c), q_{\mathcal{H}}^{\star\star}(c)]$ and $r \ge 0$. Since $q \ge q_{\mathcal{H}}^\star(c) > q_{\mathcal{H}}(c)$, we have $E_q^{r+2} \subseteq E_{q_{\mathcal{H}}(c)}^2$, and so $\mathcal{H}_c$ is injective from E_q^{r+2} to $E_{q,0}^r$. For surjectivity, take $g \in E_{q,0}^r$. Then $g \in E_{q_{\mathcal{H}}(c),0}^0$, so there exists $f \in E_{q_{\mathcal{H}}(c)}^2$ such that $\mathcal{H}_c f = g$.

Next, we need to show that f that $f \in E_q^{r+2}$. We rearrange the equality $\mathcal{H}_c f = g$ into

$$\mathcal{M}_c f = 2(A-2)\varsigma_c f + g.$$

Since $\varsigma_c \in E_{q_\varsigma(c)}^2$ and $f \in E_{q_{\mathcal{H}}(c)}^2$, and since $q \le q_{\mathcal{H}}^{\star\star} < q_\varsigma(c)$, we have $\varsigma_c f \in E_{q_\varsigma(c)+q_{\mathcal{H}}(c)}^2 \subseteq E_q^2$. Then $2(A-2)\varsigma_c f + g \in E_{q,0}^0$, and so

$$f = \mathcal{M}_c^{-1}[2(A-2)\varsigma_c f + g] \in E_q^2.$$

If $r = 0$, then we are done; otherwise, we rearrange the equality $\mathcal{H}_c f = g$ into the different form

$$f'' = \frac{1}{c^2}(A-2)(1+2\varsigma_c)f + g,$$

and we may bootstrap from this equality until we achieve $f \in E_q^{r+2}$.

F.3. **Proof of the formula 5.3.7.** We use the definition of χ_c in (5.3.3), the definitions of $\mathcal{B}_c$ and Σ_c in (1.5.2), the definition of Σ_c^* in (4.1.1), the fundamental property $(\mathcal{B}_c - \Sigma_c^*)\gamma_c = 0$, and the evenness of $\mathcal{B}_c\gamma_c\sin(\omega_c\cdot)$ to calculate

$$\iota_c[\chi_c] = 2\lim_{R\to\infty}\int_0^R \big(c^2\gamma_c''(x) + (2+A)\gamma_c(x)\big)\sin(\omega_c x)\ dx.$$

Now we integrate by parts. First, we use the formula (as stated in Appendix D of [HW17])

$$\int_0^R f''(x)g(x)\ dx = f'(R)g(R) - f(R)g'(R) + \int_0^R f(x)g''(x)\ dx,$$

valid for f and g odd, to rewrite

$$2c^2\int_0^R \gamma_c''(x)\sin(\omega_c x)\ dx = 2c^2\big(\gamma_c'(R)\sin(\omega_c R) - \omega_c\gamma_c(R)\cos(\omega_c R)\big) - 2c^2\omega_c^2\int_0^R \gamma_c(x)\sin(\omega_c x)\ dx. \tag{F.3.1}$$

Observe

$$-\omega_c^2\int_0^R \gamma_c(x)\sin(\omega_c x)\ dx = \int_0^R \gamma_c(x)[\partial_x^2\sin(\omega_c\cdot)](x)\ dx. \tag{F.3.2}$$

We have another formula, easily established through direct calculation and u-substitution:

(F.3.3)
$$\int_0^R (Af)(x)g(x)\,dx = \int_0^R f(x)(Ag)(x)\,dx + \frac{1}{2}\int_R^{R+1} f(x)(S^{-1}g)(x)\,dx - \frac{1}{2}\int_{R-1}^R f(x)(S^1 g)(x)\,dx + \frac{1}{2}\int_{-1}^0 f(x)(S^1 g)(x)\,dx - \frac{1}{2}\int_0^1 f(x)(S^{-1})g(x)\,dx.$$

We combine (F.3.1) and (F.3.2), apply (F.3.3), and use $\mathcal{B}_c^* = \mathcal{B}_c$ to find

$$\begin{aligned}
\iota_c[\chi_c] &= 2c^2 \lim_{R\to\infty}\big(\gamma_c'(R)\sin(\omega_c R) - \omega_c\gamma_c(R)\cos(\omega_c R)\big) \\
&\quad + 2\lim_{R\to\infty}\int_0^R \gamma_c(x)(\mathcal{B}_c \sin(\omega_c\cdot))(x)\,dx \\
&\quad + \left(\int_{-1}^0 \gamma_c(x)\sin(\omega_c(x+1))\,dx - \int_0^1 \gamma_c(x)\sin(\omega_c(x-1))\,dx\right) \\
&\quad + \lim_{R\to\infty}\left(\int_R^{R+1} \gamma_c(x)\sin(\omega_c(x-1))\,dx - \int_{R-1}^R \gamma_c(x)\sin(\omega_c(x+1))\,dx\right).
\end{aligned} \tag{F.3.4}$$

We can evaluate exactly each of the four terms above. Trigonometric identities and the asymptotics of γ_c from (4.0.2) give

$$\lim_{R\to\infty}\big(\gamma_c'(R)\sin(\omega_c R) - \omega_c\gamma_c(R)\cos(\omega_c R)\big) = \omega_c\sin(\omega_c\vartheta_c).$$

Next, since $\mathcal{B}_c e^{\pm i\omega_c\cdot} = 0$, we have

$$\lim_{R\to\infty}\int_0^R \gamma_c(x)(\mathcal{B}_c\sin(\omega_c\cdot))(x)\,dx = 0,$$

and so the second term is zero. For the third term, we rewrite the integrals using the addition formula for sine and then use the evenness of $\gamma_c\sin(\omega_c\cdot)$ and the oddness of $\gamma_c\cos(\omega_c\cdot)$ to show that the resulting integrals all add up to zero.

Last, we rewrite the first integral in the fourth term as

$$\int_R^{R+1} \gamma_c(x)\sin(\omega_c(x-1))\,dx = \int_R^{R+1}\big(\gamma_c(x) - \sin(\omega_c(x+\vartheta_c))\big)\sin(\omega_c(x-1))\,dx + \int_R^{R+1}\sin(\omega_c(x+\vartheta_c))\sin(\omega_c(x-1))\,dx,$$

where

$$\left|\int_R^{R+1}\big(\gamma_c(x) - \sin(\omega_c(x+\vartheta_c))\big)\sin(\omega_c(x-1))\,dx\right| \le \max_{R\le x\le R+1}|\gamma_c(x) - \sin(\omega_c(x+\vartheta_c))| \to 0$$

as $R \to \infty$ by the asymptotics of γ_c in (4.0.2). Similar manipulations in the other integral in the fourth term and a host of trig identities then yield

$$\lim_{R\to\infty}\left(\int_R^{R+1}\gamma_c(x)\sin(\omega_c(x-1))\,dx - \int_{R-1}^{R}\gamma_c(x)\sin(\omega_c(x+1))\,dx\right)$$

$$= \lim_{R\to\infty}\left(\int_R^{R+1}\sin(\omega_c(x+\vartheta_c))\sin(\omega_c(x-1))\,dx - \int_{R-1}^{R}\sin(\omega_c(x+\vartheta_c))\sin(\omega_c(x+1))\,dx\right)$$

$$= -\sin(\omega_c\vartheta_c)\sin(\omega_c).$$

All together, we have

$$\iota_c[\chi_c] = \big(2c^2\omega_c - \sin(\omega_c)\big)\sin(\omega_c\vartheta_c),$$

and this is exactly the formula (5.3.7). The preceding calculation of $\iota_c[\chi_c]$ is similar to the work in Appendix D of [HW17], except there the small parameter μ appeared throughout the calculations as well, which allowed Hoffman and Wright to ignore some terms analogous to those in (F.3.4).

Appendix G. Proof of Proposition 5.3

G.1. **General estimates in H_q^r.** We begin with a collection of very useful estimates that we will invoke throughout the proof of Proposition 5.3. Most of these estimates follow from straightforward calculus and occasional recourses to the norms

$$f \mapsto \|f\|_{L^2} + \|\cosh(q\cdot)\partial_x^r[f]\|_{L^2}, \qquad \text{and} \qquad f \mapsto \|f\|_{L^2} + \|\cosh^q(\cdot)\partial_x^r[f]\|_{L^2},$$

which are equivalent on H_q^r to the norm defined in (1.4.1). Further details of the proof are given in Appendix C.3.3 of [Fav18].

Proposition G.1. **(i)** *If $f \in H_q^r$ and $g \in W^{r,\infty}$, then*

$$\|fg\|_{r,q} \le \|f\|_{r,q}\|g\|_{W^{r,\infty}}. \tag{G.1.1}$$

(ii) *If f, $g \in H_q^r$, then*

$$\|fg\|_{r,q} \le \|\operatorname{sech}(q\cdot)\|_{W^{r,\infty}}\|f\|_{r,q}\|g\|_{r,q}. \tag{G.1.2}$$

(iii) *If $f \in W^{r,\infty}$ and $\omega \in \mathbb{R}$, then*

$$\|f(\omega\cdot)\|_{W^{r,\infty}} \le \left(\max_{0\le k\le r}|\omega|^k\right)\|f\|_{W^{r,\infty}}. \tag{G.1.3}$$

(iv) *If $f \in H_{q_2}^r$; $g \in W^{r,\infty}$; ω, $\grave{\omega} \in \mathbb{R}$; and $0 < q_1 < q_2$, then*

$$\|f\cdot(g(\omega\cdot) - g(\grave{\omega}\cdot))\|_{r,q_1} \le C(r,q_2-q_1)\left(\max_{0\le k\le r}\operatorname{Lip}(\partial_x^k[g])\right)\|f\|_{r,q_2}\|g\|_{W^{r,\infty}}|\omega-\grave{\omega}|, \tag{G.1.4}$$

where the Lipschitz constant $\operatorname{Lip}(\cdot)$ *was defined in* (A.1.2)*.*

(v) *If $f \in H_{q_2}^r$; g, $\grave{g} \in W^{r,\infty}$; ω, $\grave{\omega} \in \mathbb{R}$; and $0 < q_1 < q_2$, then*

(G.1.5) $\|f\cdot(g(\omega\cdot) - \grave{g}(\grave{\omega}\cdot))\|_{r,q_1}$

$$\leq C(r, q_2 - q_1)\|f\|_{r,q_2}\left[\left(\max_{0\leq k\leq r}\mathrm{Lip}(\partial_x^k[g])\right)\|g\|_{W^{r,\infty}}|\omega - \grave{\omega}| + \left(\max_{0\leq k\leq r}|\grave{\omega}|^k\right)\|g - \grave{g}\|_{W^{r,\infty}}\right].$$

We refer to parts (iv) and (v) as "decay borrowing" estimates, as they permit us to achieve a Lipschitz estimate on a product starting in a space of lower decay by "borrowing" from the decay rates of one of the faster-decaying functions in the product. A version of part (iv) in particular was stated and proved as Lemma A.2 in [FW18]. From this proposition we immediately deduce two estimates for our quadratic nonlinearity $\mathcal{Q}$ from (1.3.3).

Lemma G.2. **(i)** *If $\boldsymbol{\rho}$, $\grave{\boldsymbol{\rho}} \in H_q^r \times H_q^r$, then*

$$\|\mathcal{Q}(\boldsymbol{\rho}, \grave{\boldsymbol{\rho}})\|_{r,q} \leq C(r,q)\|\boldsymbol{\rho}\|_{r,q}\|\grave{\boldsymbol{\rho}}\|_{r,q} \tag{G.1.6}$$

(ii) *If $\boldsymbol{\rho} \in H_q^r \times H_q^r$ and $\boldsymbol{\phi} \in W^{r,\infty} \times W^{r,\infty}$, then*

$$\|\mathcal{Q}(\boldsymbol{\rho}, \boldsymbol{\phi})\|_{r,q} \leq C(r,q)\|\boldsymbol{\rho}\|_{r,q}\|\boldsymbol{\phi}\|_{W^{r,\infty}\times W^{r,\infty}}. \tag{G.1.7}$$

Proposition G.1 also allows us to relate χ_c^μ from (5.3.2) and χ_c from (5.3.3).

Lemma G.3. *For any $q \in (0, q_\varsigma(c))$ and $r \geq 0$, there is a constant $C(c,q,r) > 0$ such that*

$$\|\chi_c^\mu - \chi_c\|_{r,q} \leq C(c,q,r)|\mu|$$

for all $|\mu| \leq \mu_{\mathrm{per}}(c)$.

Proof. Recall that

$$\chi_c^\mu = 2\mathcal{D}_\mu\mathcal{Q}(\boldsymbol{\varsigma}_c, \boldsymbol{\phi}_c^\mu[0]) \cdot \mathbf{e}_2 \qquad \text{and} \qquad \chi_c = 2\mathcal{D}_0\mathcal{Q}(\boldsymbol{\varsigma}_c, \boldsymbol{\phi}_c^0[0]) \cdot \mathbf{e}_2.$$

Then

$$\chi_c^\mu - \chi_c = \underbrace{(\mathcal{D}_\mu - \mathcal{D}_0)\mathcal{Q}(\boldsymbol{\varsigma}_c, \boldsymbol{\phi}_c^\mu[0]) \cdot \mathbf{e}_2}_{I} + \underbrace{\mathcal{D}_0\mathcal{Q}(\boldsymbol{\varsigma}_c, \boldsymbol{\phi}_c^\mu[0] - \boldsymbol{\phi}_c^0[0]) \cdot \mathbf{e}_2}_{II}.$$

From the definition of $\mathcal{D}_\mu$ in (1.3.2), the uniform bounds on the periodic solutions from (3.0.10), and the product estimate (G.1.7), we estimate that if $0 < q < q_\varsigma(c)$, then

$$\|I\|_{r,q} \leq C(c,r)|\mu|.$$

Next, we have

$$\boldsymbol{\phi}_c^\mu[0] = \begin{pmatrix} \upsilon_c^\mu \cos(\omega_c^\mu \cdot) \\ \sin(\omega_c^\mu \cdot) \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{\phi}_c^0[0] = \begin{pmatrix} 0 \\ \sin(\omega_c \cdot) \end{pmatrix},$$

per (3.0.8) and the estimate $\upsilon_c^\mu = \mathcal{O}_c(\mu)$. We use this to calculate

$$II = (2 + A)(\varsigma_c \cdot (\sin(\omega_c^\mu \cdot) - \sin(\omega_c \cdot))).$$

Then we use the decay-borrowing estimate (G.1.4) with the condition $0 < q < q_\varsigma(c)$ as well as part (iv) of Proposition 3.1 to find

$$\|II\|_{r,q} \leq C(c,q,r)|\omega_c^\mu - \omega_c| \leq C(c,q,r)|\mu|. \qquad \square$$

G.2. **Proof of Proposition 5.3.** We rely on the following lemma, which we prove in the next three sections. We inherit the general techniques from the myriad nanopteron estimates in [FW18], [HW17], [Fav], and [JW].

Lemma G.4. *For all $r \ge 1$, there are constants $C(c,r)$, $C(c) > 0$ such that the following estimates hold for any $|\mu| \le \mu_{\mathrm{per}}(c)$.*

(i) *If $\boldsymbol{\eta} \in E^r_{q_\star(c)} \times O^r_{q_\star(c)}$ and $|a| \le a_{\mathrm{per}}(c)$, then*

(G.2.1) $$\|\boldsymbol{\mathcal{N}}^\mu_c(\boldsymbol{\eta},a)\|_{\mathcal{X}^r} \le C(c,r)\big(|\mu| + |\mu|\|\boldsymbol{\eta}\|_{r,q_\star(c)} + |\mu||a| + |a|\|\boldsymbol{\eta}\|_{r,q_\star(c)} + \|\boldsymbol{\eta}\|^2_{r,q_\star(c)} + a^2\big).$$

(ii) *If $\boldsymbol{\eta}$, $\grave{\boldsymbol{\eta}} \in E^1_{q_\star(c)} \times O^1_{q_\star(c)}$ and $|a|$, $|\grave{a}| \le a_{\mathrm{per}}(c)$, then*

(G.2.2)
$$\|\boldsymbol{\mathcal{N}}^\mu_c(\boldsymbol{\eta},a) - \boldsymbol{\mathcal{N}}^\mu_c(\grave{\boldsymbol{\eta}},\grave{a})\|_{\mathcal{X}^0} \le C(c)\big(|\mu| + |a| + |\grave{a}| + \|\boldsymbol{\eta}\|_{1,q_\star(c)} + \|\grave{\boldsymbol{\eta}}\|_{1,q_\star(c)}\big)\big(\|\boldsymbol{\eta} - \grave{\boldsymbol{\eta}}\|_{1,\overline{q}_\star(c)} + |a - \grave{a}|\big).$$

(iii) *If $\boldsymbol{\eta} \in E^r_{q_\star(c)} \times O^r_{q_\star(c)}$ and $|a| \le a_{\mathrm{per}}(c)$, then*

(G.2.3) $$\|\boldsymbol{\mathcal{N}}^\mu_c(\boldsymbol{\eta},a)\|_{\mathcal{X}^{r+1}} \le C(c,r)\big(|\mu| + |\mu|\|\boldsymbol{\eta}\|_{r,q_\star(c)} + |\mu||a| + |a|\|\boldsymbol{\eta}\|_{r,q_\star(c)} + \|\boldsymbol{\eta}\|^2_{r,q_\star(c)} + a^2\big).$$

Now we prove Proposition 5.3. Set

$$\tau_c = 6C(c,1)$$

with $C(c,1)$ from part (i) of Lemma G.4, and let

$$\mu_\star(c) := \min\left\{1, \frac{1}{\tau_c}, \frac{1}{\tau_c^2}, \mu_{\mathrm{per}}(c), \frac{a_{\mathrm{per}}(c)}{\tau_c}, \frac{1}{2C(c)(1+4\tau_c)}\right\},$$

with $C(c)$ from part (ii) of the same lemma. Note in particular that if $|\mu| \le \mu_\star(c)$ and $(\boldsymbol{\eta},a) \in \mathcal{U}^r_{\tau,\mu}$, then

$$|a| \le \|\boldsymbol{\eta}\|_{r,q_\star(c)} + |a| \le \tau_c|\mu| \le a_{\mathrm{per}}(c),$$

and so $\boldsymbol{\mathcal{N}}^\mu_c(\boldsymbol{\eta},a)$ is well-defined.

Proof of (i). Assume $|\mu| \le \mu_\star(c)$ and $(\boldsymbol{\eta},a) \in \mathcal{U}^1_{\tau_c,\mu}$. We estimate from (G.2.1) with $r = 1$ that

$$\|\boldsymbol{\mathcal{N}}^\mu_c(\boldsymbol{\eta},a)\|_{1,q_\star(c)} \le C(c,1)(1 + 3\tau_c|\mu| + 2\tau_c^2|\mu|)|\mu| \le 6C(c,1)|\mu| = \tau_c|\mu|.$$

That is, $\boldsymbol{\mathcal{N}}^\mu_c(\boldsymbol{\eta},a) \in \mathcal{U}^1_{\tau_c,\mu}$.

Proof of (ii). Next, let $(\boldsymbol{\eta},a)$, $(\grave{\boldsymbol{\eta}},\grave{a}) \in \mathcal{U}^1_{\tau_c,\mu}$. We estimate from (G.2.2) that

$$\|\boldsymbol{\mathcal{N}}^\mu_c(\boldsymbol{\eta},a) - \boldsymbol{\mathcal{N}}^\mu_c(\grave{\boldsymbol{\eta}},\grave{a})\|_{1,\overline{q}_\star(c)} \le C(c)|\mu|(1+4\tau_c)\big(\|\boldsymbol{\eta}-\grave{\boldsymbol{\eta}}\|_{1,\overline{q}_\star(c)} + |a-\grave{a}|\big) < \frac{1}{2}\big(\|\boldsymbol{\eta}-\grave{\boldsymbol{\eta}}\|_{1,\overline{q}_\star(c)} + |a-\grave{a}|\big).$$

Proof of (iii). Finally, let $(\boldsymbol{\eta},a) \in \mathcal{U}^1_{\tau_c,\mu} \cap \mathcal{U}^r_{\tau,\mu}$, where $\tau > 0$ is arbitrary. Then (G.2.3) implies

$$\|\boldsymbol{\mathcal{N}}^\mu_c(\boldsymbol{\eta},a)\|_{r,q_\star(c)} \le C(c,r)(1 + 3\tau|\mu| + 2\tau^2|\mu|)|\mu| \le C(c,r)(1 + 3\tau + 2\tau^2)|\mu|.$$

With

$$\widetilde{\tau} := C(c,r)(1 + 3\tau + 2\tau^2),$$

we see that $\boldsymbol{\mathcal{N}}^\mu_c(\boldsymbol{\eta},a) \in \mathcal{U}^{r+1}_{\widetilde{\tau},\mu}$.

G.3. **Mapping estimates.** In this appendix we prove part (i) of Lemma G.4. The definitions of $\mathcal{N}_{c,1}^{\mu}$ in (5.2.1), $\mathcal{N}_{c,2}^{\mu}$ in (5.4.4), and $\mathcal{N}_{c,3}^{\mu}$ in (5.3.9) imply

$$\|\mathcal{N}_{c,1}^{\mu}(\boldsymbol{\eta},a)\|_{r,q_\star(c)} \le \|\mathcal{H}_c^{-1}\|_{\mathbf{B}(E_{q_\star(c),0}^r,E_{q_\star(c)}^{r+2})} \sum_{k=1}^{5} \|h_{c,k}^{\mu}(\boldsymbol{\eta},a)\|_{r,q_\star(c)},$$

$$\|\mathcal{N}_{c,2}^{\mu}(\boldsymbol{\eta},a)\|_{r,q_\star(c)} \le \|\mathcal{L}_c^{-1}\mathcal{P}_c\|_{\mathbf{B}(O_{q_\star(c)}^r,O_{q_\star(c)}^{r+2})} \sum_{k=1}^{5} \|\widetilde{\ell}_{c,k}^{\mu}(\boldsymbol{\eta},a)\|_{r,q_\star(c)},$$

and

$$|\mathcal{N}_{c,3}^{\mu}(\boldsymbol{\eta},a)| \le \frac{1}{|\iota_c[\chi_c]|}\|\iota_c\|_{(O_q^r)^*} \sum_{k=1}^{5} \|\widetilde{\ell}_{c,k}^{\mu}(\boldsymbol{\eta},a)\|_{r,q_\star(c)}.$$

Then to obtain the estimate (G.2.1), it suffices to find bounds of the form

$$\|h_{c,k}^{\mu}(\boldsymbol{\eta},a)\|_{r,q_\star(c)} + \|\widetilde{\ell}_{c,k}^{\mu}(\boldsymbol{\eta},a)\|_{r,q_\star(c)} \le C(c,r)\mathcal{R}_{\mathrm{map}}(\|\boldsymbol{\eta}\|_{r,q_\star(c)},a,\mu), \ \ k=1,\ldots,6,$$

where for a, μ, $\rho \in \mathbb{R}$, we define

$$\mathcal{R}_{\mathrm{map}}(\rho,a,\mu) := |\mu| + |\mu\rho| + |\mu a| + |a\rho| + a^2 + \rho^2.$$

We do this in the following sections; throughout, we recall that these $h_{c,k}^{\mu}$ and $\widetilde{\ell}_{c,k}^{\mu}$ terms were defined in (5.1.3) and (5.3.4).

G.3.1. *Mapping estimates for $h_{c,1}^{\mu}$ and $\widetilde{\ell}_{c,1}^{\mu}$.* Since $\varsigma_c \in \cap_{r=0}^{\infty} E_{q_\varsigma(c)}^r$, we have

$$\|h_{c,1}^{\mu}(\boldsymbol{\eta},a)\|_{r,q_\star(c)} + \|\widetilde{\ell}_{c,1}^{\mu}(\boldsymbol{\eta},a)\|_{r,q_\star(c)} = |\mu|\|\mathring{\mathcal{D}}(\boldsymbol{\varsigma}_c + \mathcal{Q}(\boldsymbol{\varsigma}_c,\boldsymbol{\varsigma}_c)\|_{r,q_\star(c)} = \mathcal{O}_c(\mu).$$

G.3.2. *Mapping estimates for $h_{c,2}^{\mu}$ and $\widetilde{\ell}_{c,2}^{\mu}$.* We use (G.1.2) to estimate

$$\|h_{c,1}^{\mu}(\boldsymbol{\eta},a)\|_{r,q_\star(c)} \le C(c,r)|\mu|\|\boldsymbol{\eta}\|_{r,q_\star(c)} + C(c,r)|\mu|\|\varsigma_c\eta_1\|_{r,q_\star(c)} \le C(c,r)|\mu|\|\boldsymbol{\eta}\|_{r,q_\star(c)}.$$

G.3.3. *Mapping estimates for $h_{c,3}^{\mu}$.* The estimates for this term and its counterpart $\widetilde{\ell}_{c,3}^{\mu}$ are probably the most intricate of all the mapping estimates. First, we compute

$$h_{c,3}^{\mu}(\boldsymbol{\eta},a) = -2a\mathcal{D}_\mu\mathcal{Q}(\boldsymbol{\varsigma}_c,\boldsymbol{\phi}_c^{\mu}[a])\cdot\mathbf{e}_1 = \underbrace{-a(2+\mu)(2-A)(\varsigma_c\phi_{c,1}^{\mu}[a])}_{aI} - \underbrace{\mu a\delta(\varsigma_c\phi_{c,2}^{\mu}[a])}_{aII}. \tag{G.3.1}$$

The bound (3.0.10) on $\|\boldsymbol{\phi}_c^{\mu}[a]\|_{W^{r,\infty}}$ from Proposition 3.3 and the product estimate (G.1.1) from Proposition G.1 tell us

$$\|II\|_{r,q_\star(c)} \le C(c,r)|\mu|. \tag{G.3.2}$$

Now we use the expansion (3.0.8) in Proposition 3.3 to write

$$\phi_{c,1}^{\mu}[a] = \upsilon_c^{\mu}\cos(\omega_c^{\mu}[a]\cdot) + \psi_{c,1}^{\mu}[a](\omega_c^{\mu}[a]\cdot),$$

We then use the product estimate (G.1.1) and this expansion to bound

$$\begin{aligned}\|I\|_{r,q_\star(c)} &\le C(c,r)\|\varsigma_c\phi_{c,1}^{\mu}[a]\|_{r,q_\star(c)} \\ &\le C(c,r)\|\upsilon_c^{\mu}\cos(\omega_c^{\mu}[a]\cdot) + \psi_{c,1}^{\mu}[a](\omega_c^{\mu}[a]\cdot)\|_{W^{r,\infty}} \\ &\le C(c,r)|\upsilon_c^{\mu}|\|\cos(\omega_c^{\mu}[a]\cdot)\|_{W^{r,\infty}} + C(c,r)\|\psi_{c,1}^{\mu}[a](\omega_c^{\mu}[a]\cdot)\|_{W^{r,\infty}}.\end{aligned}$$

The estimate $\upsilon_c^\mu = \mathcal{O}_c(\mu)$ and the bounds on $|\omega_c^\mu[a]|$ from (3.0.10) give

$$|\upsilon_c^\mu|\|\cos(\omega_c^\mu[a]\cdot)\|_{W^{r,\infty}} \le C(c,r)|\mu|.$$

Last, from the uniform bound (3.0.11) on $\boldsymbol{\psi}_c^\mu$ and the scaling estimate (G.1.3), we have

$$\|\psi_{c,1}^\mu[a](\omega_c^\mu[a]\cdot)\|_{W^{r,\infty}} \le C(c,r)|a|$$

and thus

$$\|I\|_{r,q_\star(c)} \le C(c,r)(|\mu|+|a|). \tag{G.3.3}$$

We conclude

$$\|h_{c,3}^\mu(\boldsymbol{\eta},a)\|_{r,q_\star(c)} \le |a|\big(\|I\|_{r,q_\star(c)} + \|II\|_{r,q_\star(c)}\big) \le C(c,r)(|a\mu|+a^2).$$

G.3.4. *Mapping estimates for* $\widetilde{\ell}_{c,3}^\mu$. Recall the definitions of $\widetilde{\ell}_{c,3}^\mu$ in (5.3.4) and of χ_c from (5.3.3) and χ_c^μ from (5.3.2). Then

$$\begin{aligned}
\widetilde{\ell}_{c,3}^\mu(\boldsymbol{\eta},a) &= -2a\mathcal{D}_\mu\mathcal{Q}(\boldsymbol{\varsigma}_c,\boldsymbol{\phi}_c^\mu[a])\cdot\mathbf{e}_2 + a\chi_c \\
&= -2a\mathcal{D}_\mu\mathcal{Q}(\boldsymbol{\varsigma}_c,\boldsymbol{\phi}_c^\mu[a])\cdot\mathbf{e}_2 + a\chi_c^\mu + a(\chi_c-\chi_c^\mu) \\
&= -2a\mathcal{D}_\mu\mathcal{Q}(\boldsymbol{\varsigma}_c,\boldsymbol{\phi}_c^\mu[a]-\boldsymbol{\phi}_c^\mu[0])\cdot\mathbf{e}_2 + a(\chi_c-\chi_c^\mu) \\
&= \underbrace{\mu a\delta(\varsigma_c\cdot\big(\phi_{c,1}^\mu[a]-\phi_{c,2}^\mu[0]\big))}_{\mu a I} - \underbrace{a(2+\mu)(2+A)\big(\sigma_c\cdot(\phi_{c,2}^\mu[a]-\phi_{c,2}^\mu[0])\big)}_{aII} + \underbrace{a(\chi_c-\chi_c^\mu)}_{aIII}.
\end{aligned}$$

We know from Lemma G.3 that $\|III\|_{r,q_\star(c)} \le C(c,q_\star(c),r)|\mu|$, so we only need to estimate the terms I and II.

We can use the triangle inequality and the periodic bounds (3.0.10) to obtain the crude estimate

$$|\mu a|\|I\|_{r,q_\star(c)} \le C(c,r)|\mu a|,$$

but we need to do more work with II. From the periodic expansion (3.0.8) and part (iii) of Proposition 3.3, we calculate

$$\phi_{c,2}^\mu[a]-\phi_{c,2}^\mu[0] = \big(\sin(\omega_c^\mu[a]\cdot)-\sin(\omega_c^\mu[0]\cdot)\big) + \psi_{c,2}^\mu[a](\omega_c^\mu[a]\cdot).$$

Then

$$\|II\|_{r,q_\star(c)} \le C(c,r)\underbrace{\|\varsigma_c\cdot\big(\sin(\omega_c^\mu[a]\cdot)-\sin(\omega_c^\mu[0]\cdot)\big)\|_{r,q_\star(c)}}_{IV} + \underbrace{\|\varsigma_c\psi_{c,2}^\mu[a](\omega_c^\mu[a]\cdot)\|_{r,q_\star(c)}}_{V}.$$

We use (3.0.10), (3.0.11), and (G.1.3) to estimate

$$V \le C(c,r)|a|.$$

To estimate IV, we exploit the condition $\varsigma_c \in \cap_{r=0}^\infty(E_{q_\varsigma(c)}^r \cap E_{q_\star(c)}^r)$ with $q_\star(c) < q_\varsigma(c)$. Since this inequality is strict, we may call on the decay borrowing estimate (G.1.4) to find

$$\|IV\|_{r,q_\star(c)} \le C(c,r)\|\varsigma_c\|_{r,q_\varsigma(c)}|\omega_c^\mu[a]-\omega_c^\mu[0]|.$$

Then we use (3.0.9) to conclude

$$\|IV\|_{r,q_\star(c)} \le C(c,r)|a|.$$

We put all these estimates together to find

$$\|\widetilde{\ell}^{\mu}_{c,3}(\boldsymbol{\eta}, a)\|_{r,q_\star(c)} \le C(c,r)(|\mu a| + a^2).$$

G.3.5. *Mapping estimates for $h^{\mu}_{c,4}$ and $\widetilde{\ell}^{\mu}_{c,4}$.* These terms are roughly quadratic of the form $a\boldsymbol{\eta}$. We first use the estimate (G.1.7) in Lemma G.2 to find

$$\|h^{\mu}_{c,4}(\boldsymbol{\eta}, a)\|_{r,q_\star(c)} + \|\widetilde{\ell}^{\mu}_{c,4}(\boldsymbol{\eta}, a)\|_{r,q_\star(c)} \le 2|a|\|\mathcal{D}_\mu \mathcal{Q}(\boldsymbol{\phi}^{\mu}_c[a], \boldsymbol{\eta})\|_{r,q_\star(c)}$$

$$\le C(c,r)|a|\|\boldsymbol{\phi}^{\mu}_c[a]\|_{W^{r,\infty}}\|\boldsymbol{\eta}\|_{r,q_\star(c)}.$$

Next, we use the estimate (3.0.10) on $\|\boldsymbol{\phi}^{\mu}_c[a]\|_{W^{r,\infty}}$ to conclude

$$\|h^{\mu}_{c,4}(\boldsymbol{\eta}, a)\|_{r,q_\star(c)} + \|\widetilde{\ell}^{\mu}_{c,4}(\boldsymbol{\eta}, a)\|_{r,q_\star(c)} \le C(c,r)|a|\|\boldsymbol{\eta}\|_{r,q_\star(c)}.$$

G.3.6. *Mapping estimates for $h^{\mu}_{c,5}$ and $\widetilde{\ell}^{\mu}_{c,5}$.* These terms are both quadratic in $\boldsymbol{\eta}$, and so we estimate simultaneously

$$\|h^{\mu}_{c,5}(\boldsymbol{\eta}, a)\|_{r,q_\star(c)} + \|\widetilde{\ell}^{\mu}_{c,5}(\boldsymbol{\eta}, a)\|_{r,q_\star(c)} \le \|\mathcal{D}_\mu \mathcal{Q}(\boldsymbol{\eta}, \boldsymbol{\eta})\|_{r,q_\star(c)} \le C(c,r)\|\boldsymbol{\eta}\|^2_{r,q_\star(c)}$$

by (G.1.7) in Lemma G.2.

G.4. **Lipschitz estimates.** In this appendix we prove part (ii) of Lemma G.4. By the same reasoning from the start of Appendix G.3, it suffices to find bounds of the form

$$\|h^{\mu}_{c,k}(\boldsymbol{\eta}, a) - h^{\mu}_{c,k}(\grave{\boldsymbol{\eta}}, \grave{a})\|_{1,\overline{q}_\star(c)} + \|\widetilde{\ell}^{\mu}_{c,k}(\boldsymbol{\eta}, a) - \widetilde{\ell}^{\mu}_{c,k}(\grave{\boldsymbol{\eta}}, \grave{a})\|_{1,\overline{q}_\star(c)}$$

$$\le C(c)\mathcal{R}_{\mathrm{lip}}\big(\|\boldsymbol{\eta}\|_{1,q_\star(c)}, \|\grave{\boldsymbol{\eta}}\|_{1,q_\star(c)}, |a|, |\grave{a}|\big)(\|\boldsymbol{\eta} - \grave{\boldsymbol{\eta}}\|_{1,\overline{q}_\star(c)} + |a - \grave{a}|), \ k = 1, \ldots, 6,$$

where for ρ, $\grave{\rho}$, a, $\grave{a}$, $\mu \in \mathbb{R}$ we set

$$\mathcal{R}_{\mathrm{lip}}(\rho, \grave{\rho}, a, \grave{a}, \mu) := |\mu| + |\rho| + |\grave{\rho}| + |a| + |\grave{a}|.$$

G.4.1. *Lipschitz estimates for $h^{\mu}_{c,1}$ and $\widetilde{\ell}^{\mu}_{c,1}$.* This is obvious because these terms are constant in both $\boldsymbol{\eta}$ and a.

G.4.2. *Lipschitz estimates for $h^{\mu}_{c,2}$ and $\widetilde{\ell}^{\mu}_{c,2}$.* This is obvious because these terms are linear in $\boldsymbol{\eta}$ and come with a factor of μ.

G.4.3. *Lipschitz estimates for $h^{\mu}_{c,3}$.* First we write

$$h^{\mu}_{c,3}(\boldsymbol{\eta}, a) - h^{\mu}_{c,3}(\grave{\boldsymbol{\eta}}, \grave{a}) = \underbrace{(a - \grave{a})\mathcal{D}_\mu \mathcal{Q}(\boldsymbol{\varsigma}_c, \boldsymbol{\phi}^{\mu}_c[a]) \cdot \mathbf{e}_1}_{(a-\grave{a})I} + \underbrace{\grave{a}\mathcal{D}_\mu \mathcal{Q}\big(\boldsymbol{\varsigma}_c, \boldsymbol{\phi}^{\mu}_c[a] - \boldsymbol{\phi}^{\mu}_c[\grave{a}]\big) \cdot \mathbf{e}_1}_{\grave{a}II}$$

The methods of Appendix G.3.3, specifically, the decomposition (G.3.1) and the estimates (G.3.3) and (G.3.2), carry over to yield

$$\|I\|_{1,\overline{q}_\star(c)} \le C(q,c)(|\mu| + |a|). \tag{G.4.1}$$

Next, we estimate II using techniques similar to those in Appendix G.3.4. Rewrite

$$II = \underbrace{-(2+\mu)(2-A)\left(\varsigma_c \cdot \left(\phi^{\mu}_{c,1}[a] - \phi^{\mu}_{c,1}[\grave{a}]\right)\right)}_{-(2+\mu)(2-A)(III)} - \underbrace{\mu\delta\left(\varsigma_c \cdot \left(\phi^{\mu}_{c,2}[a] - \phi^{\mu}_{c,2}[\grave{a}]\right)\right)}_{\mu\delta(IV)}.$$

We will estimate only III explicitly; the estimates on IV are the same. We have

$$III = \underbrace{\varsigma_c \cdot \big(\upsilon_c^\mu \cos(\omega_c^\mu[a]\cdot) - \upsilon_c^\mu \cos(\omega_c^\mu[\grave{a}]\cdot)\big)}_{III_1} + \underbrace{\varsigma_c \cdot \big(\psi_{c,1}^\mu[a](\omega_c^\mu[a]\cdot) - \psi_{c,1}^\mu[\grave{a}](\omega_c^\mu[\grave{a}]\cdot)\big)}_{III_2}.$$

Apply the decay borrowing estimates (G.1.4) to III_1 and (G.1.5) to III_2 and use the periodic Lipschitz estimates (3.0.9) and the estimate $\upsilon_\mu = \mathcal{O}_c(\mu)$ to conclude

$$\|III_1\|_{1,\overline{q}_\star(c)} \le C(c)|\mu||a-\grave{a}|. \tag{G.4.2}$$

Combine this estimate and its omitted counterpart on IV with (G.4.1) to conclude

$$\|h_{c,3}^\mu(\boldsymbol{\eta}, a) - h_{c,3}^\mu(\grave{\boldsymbol{\eta}}, \grave{a})\|_{1,\overline{q}_\star(c)} \le C(c)(|\mu| + |a| + |\grave{a}|)|a-\grave{a}|.$$

G.4.4. *Lipschitz estimates for $\widetilde{\ell}_{c,3}^\mu$.* From the definitions of $\widetilde{\ell}_{c,3}^\mu$ in (5.3.4), we compute

$$\widetilde{\ell}_{c,3}^\mu(\boldsymbol{\eta}, a) - \widetilde{\ell}_{c,3}^\mu(\grave{\boldsymbol{\eta}}, \grave{a}) = \big(-2\mathcal{D}_\mu\mathcal{Q}(\boldsymbol{\varsigma}_c, \boldsymbol{\phi}_c^\mu[a]) \cdot \mathbf{e}_2 + a\chi_c\big) - \big(-2\mathcal{D}_\mu\mathcal{Q}(\boldsymbol{\varsigma}_c, \boldsymbol{\phi}_c^\mu[\grave{a}]) \cdot \mathbf{e}_2 + \grave{a}\chi_c\big)$$

$$= \underbrace{\big(-2\mathcal{D}_\mu\mathcal{Q}(\boldsymbol{\varsigma}_c, \boldsymbol{\phi}_c^\mu[a]) \cdot \mathbf{e}_2 + a\chi_c^\mu\big) - \big(-2\mathcal{D}_\mu\mathcal{Q}(\boldsymbol{\varsigma}_c, \boldsymbol{\phi}_c^\mu[\grave{a}]) \cdot \mathbf{e}_2 + \grave{a}\chi_c^\mu\big)}_{I} + \underbrace{(a-\grave{a})(\chi_c - \chi_c^\mu)}_{(a-\grave{a})II}$$

We know from Lemma G.3 that

$$\|II\|_{1,\overline{q}_\star(c)} \le C(c)|\mu|,$$

and so we just work on I, which, using the definition of χ_c^μ in (5.3.2), is

$$I = 2\mathcal{D}_\mu\mathcal{Q}\big(\boldsymbol{\varsigma}_c, a(\boldsymbol{\phi}_c^\mu[0] - \boldsymbol{\phi}_c^\mu[a]) - \grave{a}(\boldsymbol{\phi}_c^\mu[0] - \boldsymbol{\phi}_c^\mu[\grave{a}])\big) \cdot \mathbf{e}_2.$$

Adding zero, we have

$$a\big(\boldsymbol{\phi}_c^\mu[0] - \boldsymbol{\phi}_c^\mu[a])\big) - \grave{a}\big(\boldsymbol{\phi}_c^\mu[0] - \boldsymbol{\phi}_c^\mu[\grave{a}]\big) = a\big(\boldsymbol{\phi}_c^\mu[\grave{a}] - \boldsymbol{\phi}_c^\mu[a]\big) + (a-\grave{a})\big(\boldsymbol{\phi}_c^\mu[0] - \boldsymbol{\phi}_c^\mu[\grave{a}]\big),$$

and therefore

$$\|\widetilde{\ell}_{c,2}^\mu(\boldsymbol{\eta}, a) - \widetilde{\ell}_{c,2}^\mu(\grave{\boldsymbol{\eta}}, \grave{a})\|_{1,\overline{q}_\star(c)} \le \underbrace{C|a|\,\big\|\varsigma_c \cdot \big(\phi_{c,2}^\mu[\grave{a}] - \phi_{c,2}^\mu[a]\big)\big\|_{1,\overline{q}_\star(c)}}_{C|a|III}$$

$$+ \underbrace{C|a-\grave{a}|\,\big\|\varsigma_c \cdot \big(\phi_{c,2}^\mu[0] - \phi_{c,2}^\mu[\grave{a}]\big)\big\|_{1,\overline{q}_\star(c)}}_{C|a-\grave{a}|IV}.$$

For both III and IV, we use the decay borrowing estimate (G.1.5) and the periodic Lipschitz estimate (3.0.9) to bound

$$III \le C|a-\grave{a}| \qquad \text{and} \qquad IV \le C|\grave{a}|.$$

G.4.5. *Lipschitz estimates for $h_{c,4}^{\mu}$ and $\widetilde{\ell}_{c,4}^{\mu}$.* We compute

$$\|h_{c,4}^{\mu}(\boldsymbol{\eta}, a) - h_{c,4}^{\mu}(\grave{\boldsymbol{\eta}}, \grave{a})\|_{1,\bar{q}_\star(c)} + \|\widetilde{\ell}_{c,4}^{\mu}(\boldsymbol{\eta}, a) - \widetilde{\ell}_{c,4}^{\mu}(\grave{\boldsymbol{\eta}}, \grave{a})\|_{1,\bar{q}_\star(c)}$$

$$= 2\|a\mathcal{D}_\mu \mathcal{Q}(\boldsymbol{\phi}_c^\mu[a], \boldsymbol{\eta}) - \grave{a}\mathcal{D}_\mu \mathcal{Q}(\boldsymbol{\phi}_c^\mu[\grave{a}], \grave{\boldsymbol{\eta}})\|_{1,\bar{q}_\star(c)}$$

$$\leq \underbrace{C|a-\grave{a}|\|\mathcal{Q}(\boldsymbol{\phi}_c^\mu[a], \boldsymbol{\eta})\|_{1,\bar{q}_\star(c)}}_{C|a-\grave{a}|I} + \underbrace{C|\grave{a}|\left\|\mathcal{Q}\big(\boldsymbol{\phi}_c^\mu[a] - \boldsymbol{\phi}_c^\mu[\grave{a}], \boldsymbol{\eta}\big)\right\|_{1,\bar{q}_\star(c)}}_{C|\grave{a}|II} + \underbrace{C|\grave{a}|\|\mathcal{Q}(\boldsymbol{\phi}_c^\mu[\grave{a}], \boldsymbol{\eta} - \grave{\boldsymbol{\eta}})\|_{1,\bar{q}_\star(c)}}_{C|\grave{a}|III}.$$

We estimate

$$I \leq C(c)\|\boldsymbol{\eta}\|_{1,\bar{q}_\star(c)} \qquad \text{and} \qquad III \leq C(c)\|\boldsymbol{\eta} - \grave{\boldsymbol{\eta}}\|_{1,\bar{q}_\star(c)}$$

with the product estimate (G.1.7) on $\mathcal{Q}$ and the periodic bounds (3.0.10). For II, recall that $\boldsymbol{\eta}$, $\grave{\boldsymbol{\eta}} \in E_{\bar{q}_\star(c)}^1 \times O_{\bar{q}_\star(c)}^1 \subseteq E_{q_\star(c)}^1 \times O_{q_\star(c)}^1$. We can therefore invoke the decay borrowing estimate (G.1.5) and the periodic Lipschitz estimate (3.0.9) to conclude

$$II \leq C|a - \grave{a}|\|\boldsymbol{\eta}\|_{1,\bar{q}_\star(c)}.$$

G.4.6. *Lipschitz estimates for $h_{c,5}^{\mu}$ and $\widetilde{\ell}_{c,5}^{\mu}$.* These estimates follow immediately from the quadratic estimate (G.1.6) for $\mathcal{Q}$.

G.5. **Bootstrap estimates.** Since $\mathcal{H}_c^{-1} \in \mathbf{B}(E_{q_\star(c),0}^r, E_{q_\star(c)}^{r+2})$ and $\mathcal{L}_c^{-1}\mathcal{P}_c \in \mathbf{B}(O_{q_\star(c)}^r, O_{q_\star(c)}^{r+2})$, we have

$$\|\mathcal{N}_{c,1}^\mu(\boldsymbol{\eta}, a)\|_{r+1,q_\star(c)} + \|\mathcal{N}_{c,2}^\mu(\boldsymbol{\eta}, a)\|_{r+1,q_\star(c)} \leq C(c,r)\sum_{k=1}^5 \|h_{c,k}^\mu(\boldsymbol{\eta}, a)\|_{r,q_\star(c)}$$

$$+ C(c,r)\sum_{k=1}^5 \|h_{c,k}^\mu(\boldsymbol{\eta}, a)\|_{r,q_\star(c)}.$$

The individual mapping estimates above show that each of these eleven terms on the right is bounded by $C(c,r)\mathcal{R}_{\text{map}}^\mu(\|\boldsymbol{\eta}\|_{r,q_\star(c)}, a)$, and so we conclude that (G.2.3) also holds.

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