Analysis Of Newton's Method to Compute Travelling Waves in Discrete Media.

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Lattice differential equations (LDE's)

We are interested in infinite dimensional systems of differential equations,

$$\dot{x}_\eta = F_\eta(\{x_\lambda\}_{\lambda\in\Lambda}), \qquad \eta \in \Lambda, \tag{1}$$

for some lattice Λ, e.g. $\mathbb{Z}$ or $\mathbb{Z}^2$.

The numerical and experimental work of Leon Chua and Martin Hasler is a strong motivation for the study of LDEs.

They are developing algorithms based on LDEs which identify various prescribed patterns, for example edges, or corners, in a digitized image.

Cellular Neural Networks

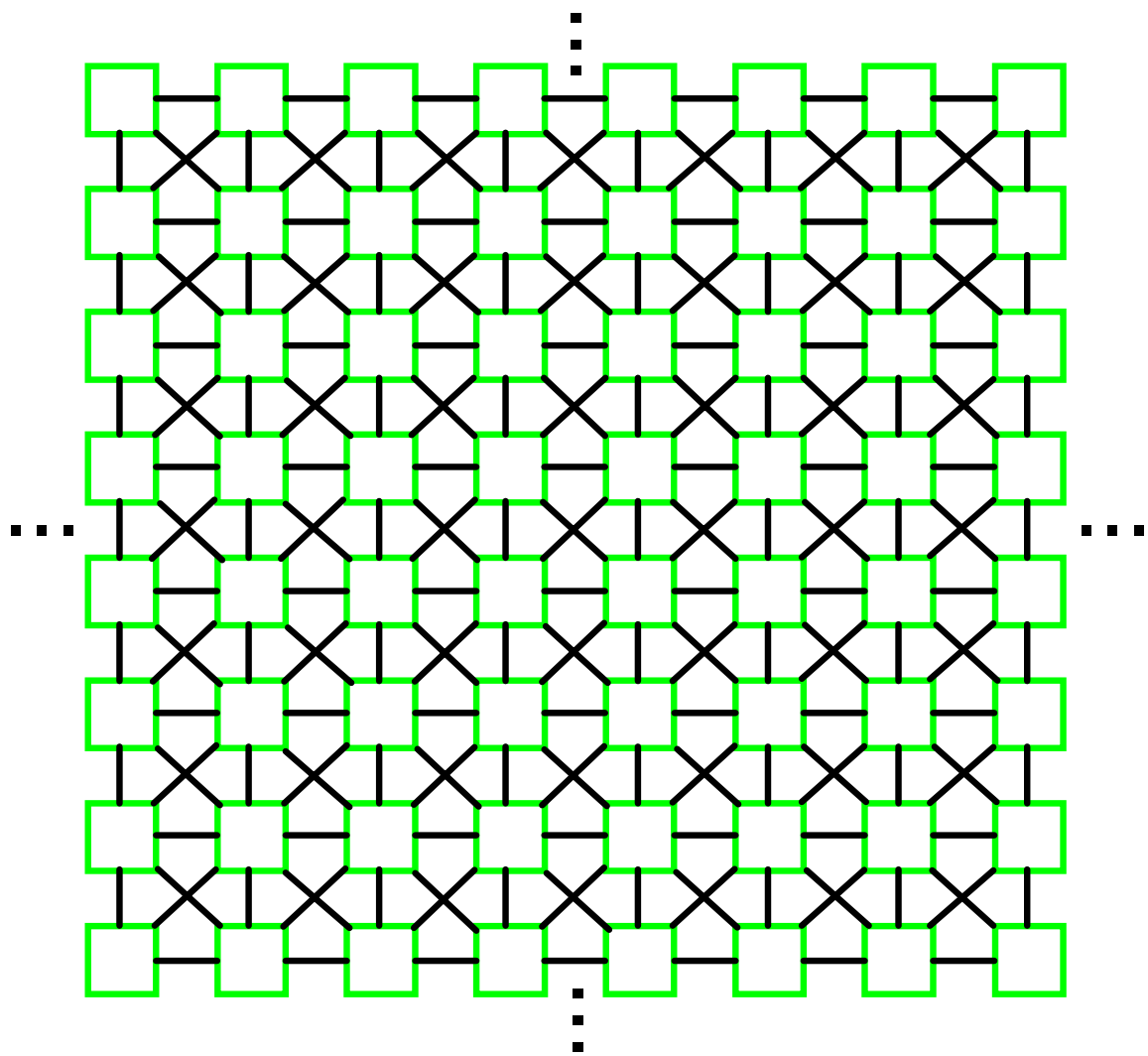


Figure 1: Already in 1988 Leon O. Chua and Lin Yang developed the concept of Cellular Neural Networks: large neural nets with local interactions.

CNN Automata

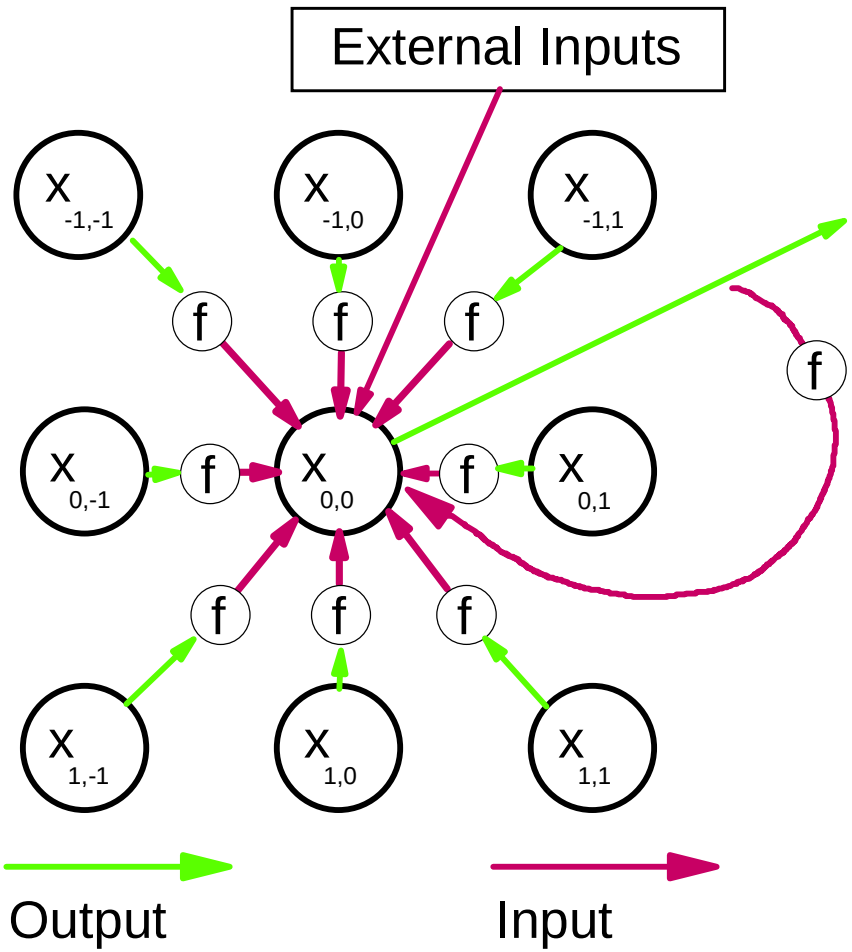
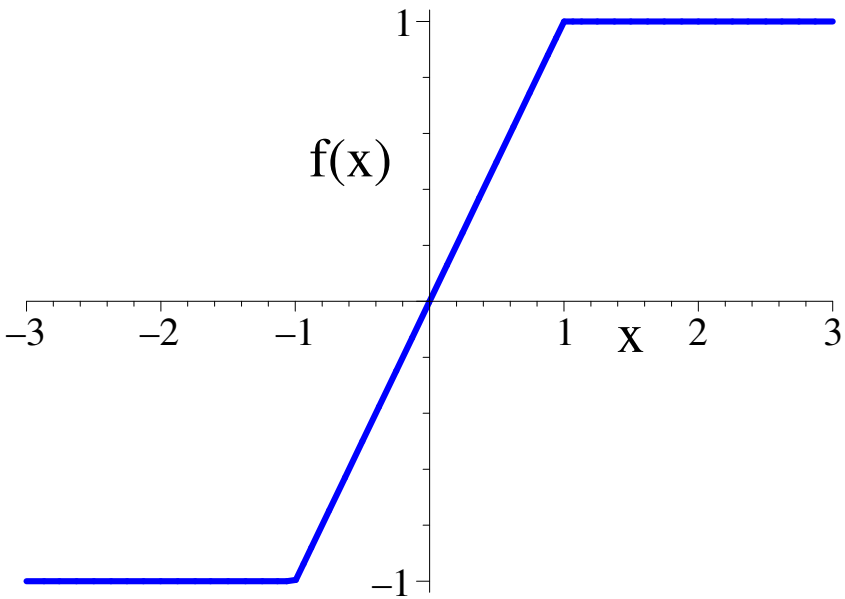


Figure 2: Overview of inputs and outputs for the cell at $(0, 0)$.

State equation

$$C\dot{x}_{i,j}(t) = -\frac{1}{R_x}x_{i,j}(t) + \sum_{(k,l)\in N} A_{k,l} f(x_{i+k,j+l}) + I_{\text{ext}} \tag{2}$$

Here N denotes the 3×3 neighbourhood $\{(i, j) \mid -1 \leq i \leq 1,\ -1 \leq j \leq 1\}$.



$f(x) = \frac{1}{2}(|x + 1| - |x - 1|)$.

CNN Pattern Recognition

- One CNN Cell represents one pixel.
- Original state and I_{ext} correspond with input picture.
- Input picture is greyscale with values in range $[-1, 1]$.
- Neural Network should converge to equilibrium state $x(\infty)$.
- Output should be black and white, i.e. $f(x(\infty)) \in \{-1, 1\}$. This is equivalent to $|x(\infty)| \geq 1$.

Theorem 1. *Suppose that $A_{0,0} > R_x^{-1}$. Then for inputs corresponding to greyscale images, the limits*

$$\lim_{t \to \infty} x_{i,j}(t) = x_{i,j}(\infty) \tag{3}$$

exist and satisfy $|x_{i,j}(\infty)| \geq 1$.

This theorem guarantees that the final output $f(x_{i,j}(\infty))$ is a black and white image.

CNN Pattern Recognition - Line Detection

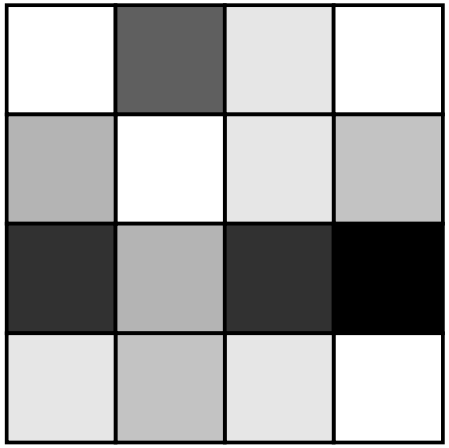
The coupling constants $A_{i,j}$ should be chosen according to the task at hand.

0.0	0.0	0.0
1.0	2.0	1.0
0.0	0.0	0.0

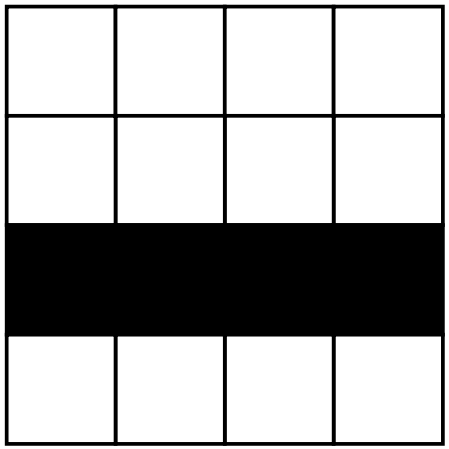
Horizontal line detector template.

0.0	1.0	0.0
0.0	2.0	0.0
0.0	1.0	0.0

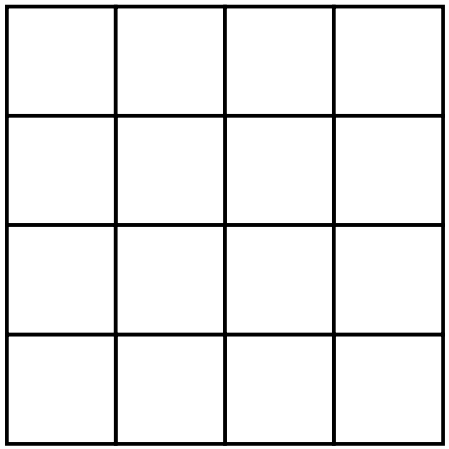
Vertical line detector template.



Original greyscale image.



Horizontal line after.



Vertical line after.

CNN Noise Reduction

Goal is to eliminate random noise applied to image.

0.0	1.0	0.0
1.0	2.0	1.0
0.0	1.0	0.0

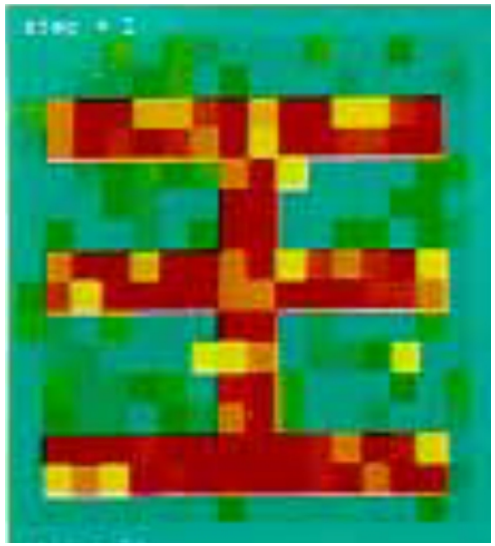
Noise reduction template A.

0.0	1.0	0.0
1.0	4.0	1.0
0.0	1.0	0.0

Noise reduction template B.

0.5	1.0	0.5
1.0	4.0	1.0
0.5	1.0	0.5

Noise reduction template C.



Original Image.

CNN Edge Recognition

Goal is to extract edges from an image.

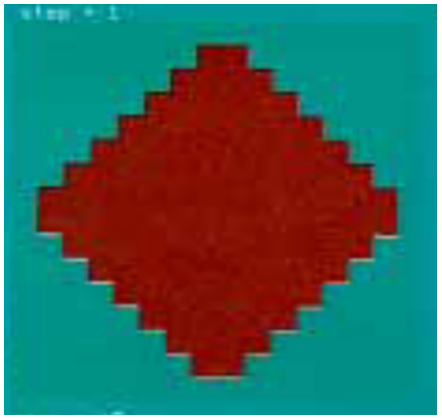
0.0	-1.0	0.0
-1.0	4.0	-1.0
0.0	-1.0	0.0

Edge recognition template A.

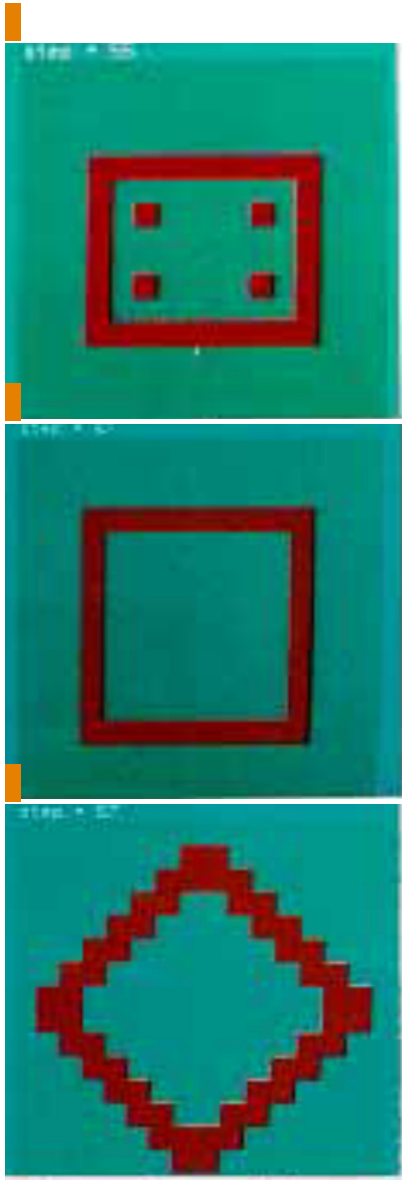
-0.25	-0.25	-0.25
-0.25	2.0	-0.25
-0.25	-0.25	-0.25

Edge recognition template B.

Original Image Square.



Original Image Diamond.



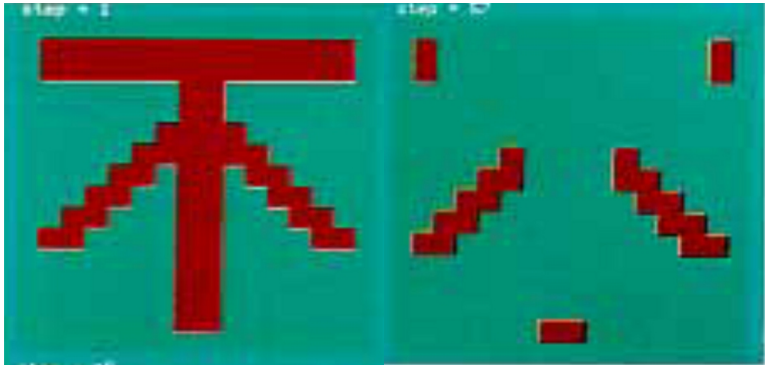
A.

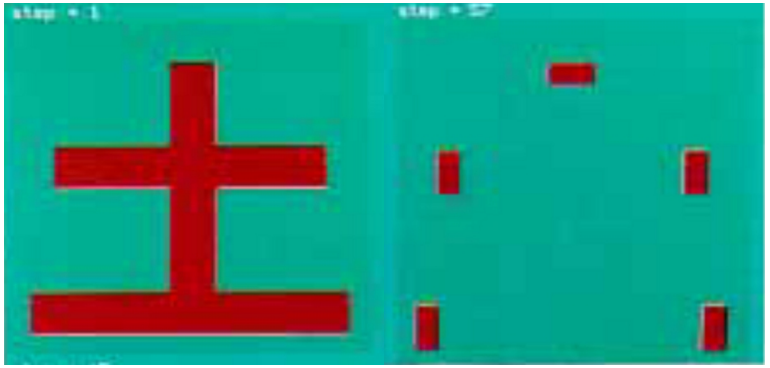
B.

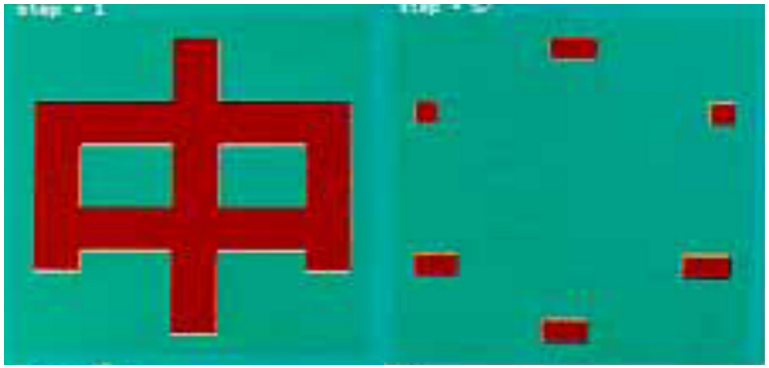
B.

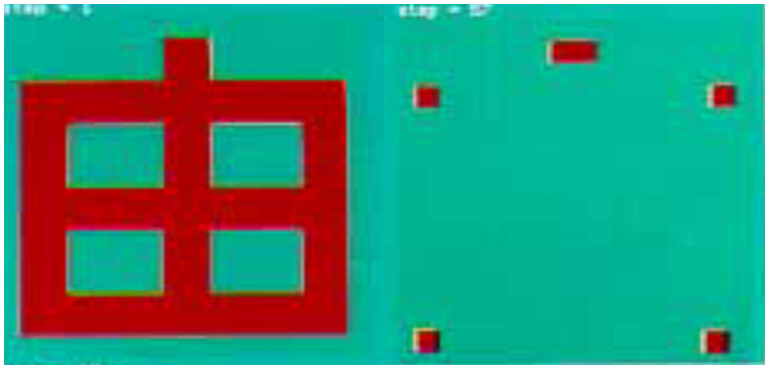
CNN Corner Recognition

Template the same as for edge recognition; Inputs I_{ext} get extra biasterm.









CNN Circuits

- Cellular Neural Networks can be implemented as electronic circuits.
- Couplings $A_{k,l}$ can be set by changing impedances of circuit elements.
- Very fast parallel processing possible.

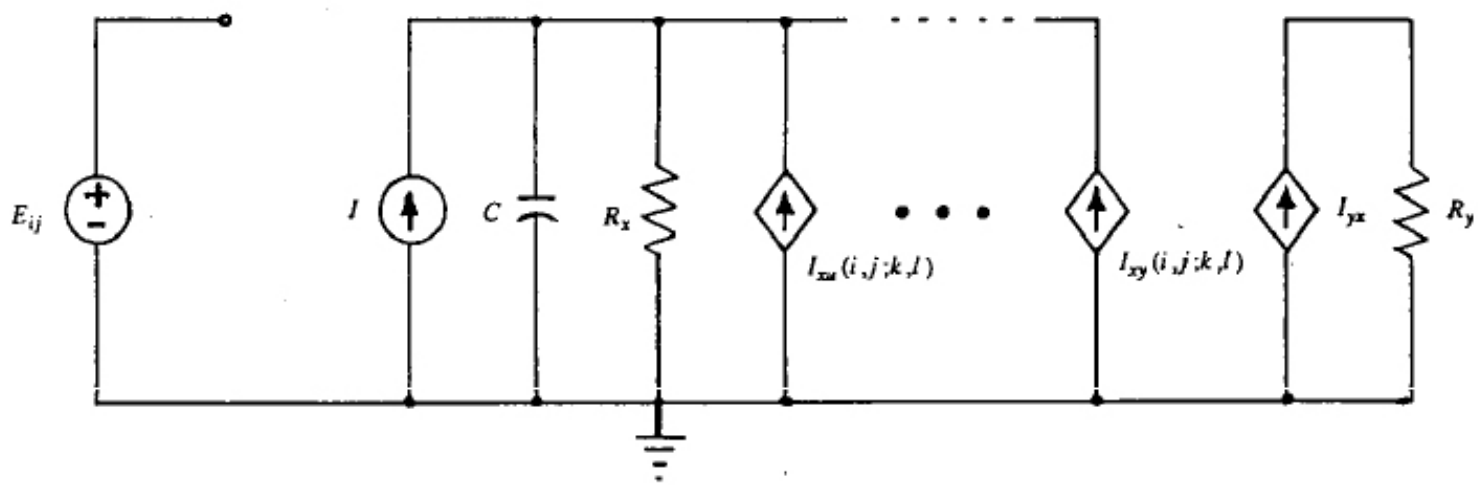


Figure 3: Circuit

CNN Final Example

Large scale edge recognition using CNN's is possible.

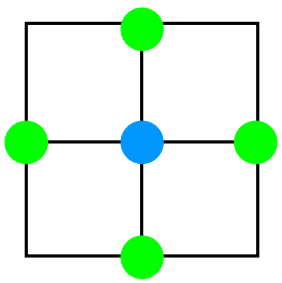
LDE vs PDE

Typical example of LDE on the integer lattice $\Lambda = \mathbb{Z}^2$,

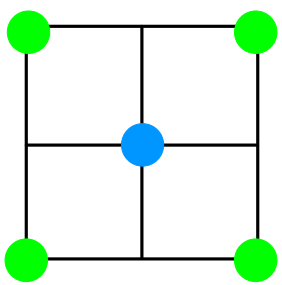
$$\dot{u}_{i,j} \quad = \quad \alpha L_D u_{i,j} - f(u_{i,j}), \quad (i,j) \in \mathbb{Z}^2, \tag{4}$$

L_D is a discrete Laplacian, which could be given by

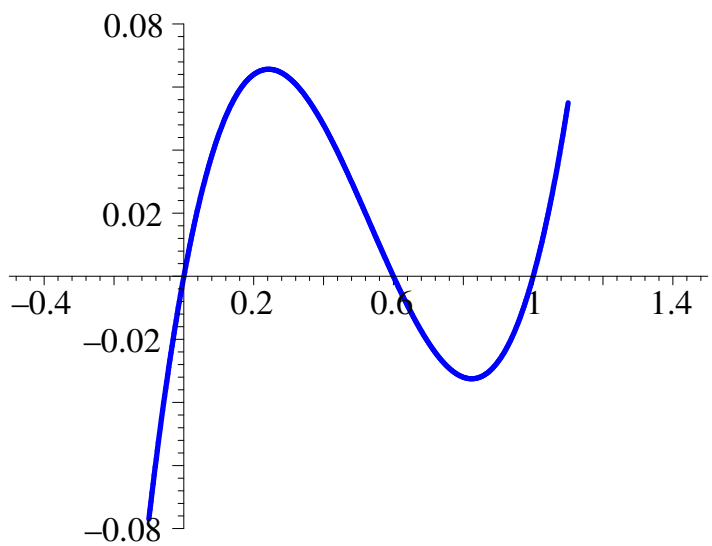
$$\begin{aligned} L_D u_{i,j} &= (\Delta^+ u)_{i,j} \equiv u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j}, \qquad \textit{or} \\ L_D u_{i,j} &= (\Delta^\times u)_{i,j} \equiv u_{i+1,j+1} + u_{i+1,j-1} + u_{i-1,j+1} + u_{i-1,j-1} - 4u_{i,j}. \end{aligned} \tag{5}$$



Discrete Laplacian Δ^+.



Discrete Laplacian $\Delta^\times$.



Bistable nonlinearity, typically

$$f_{\text{cub}}(u) = u(u-a)(u-1). \tag{6}$$

LDE vs PDE - II

The system (4), i.e.

$$\dot{u}_{i,j} \quad = \quad \alpha(\Delta^+ u)_{i,j} - f(u_{i,j}), \quad (i,j) \in \mathbb{Z}^2, \tag{7}$$

with $\alpha = h^{-2}$, arises from discretization of the reaction diffusion equation on $\mathbb{R}^2$,

$$\dot{u} = \Delta u - f(u), \tag{8}$$

to a rectangular lattice with spacing h.

- Large values of α correspond with the continuous limit $h \to 0$.
- One can also study (7) with small α and even $\alpha < 0$.
- Away from the continuous limit, (7) has a much richer structure that (8), as we shall see.

LDE vs PDE - III - Anisotropy

We consider travelling wave solutions which propagate at an angle θ.

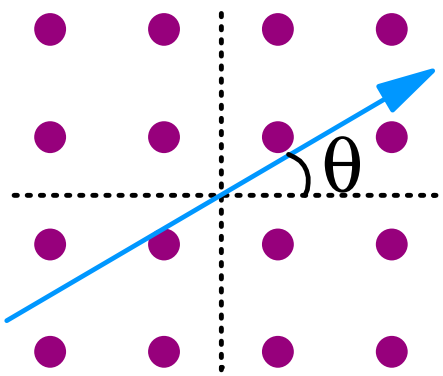
$$u_{i,j}(t) = \phi(i\cos\theta + j\sin\theta - ct). \tag{9}$$

We require $\phi(-\infty) = 0$ and $\phi(\infty) = 1$. Substitution into (7) and (8) yields

$$\begin{aligned} -c\phi'(\xi) &= \alpha L_\theta(\phi)(\xi) - f_{\mathrm{cub}}(\phi(\xi), a), && \text{(discrete)}, \\ -c\phi'(\xi) &= \phi''(\xi) - f_{\mathrm{cub}}(\phi(\xi), a), && \text{(continuous)}, \end{aligned} \tag{10}$$

where

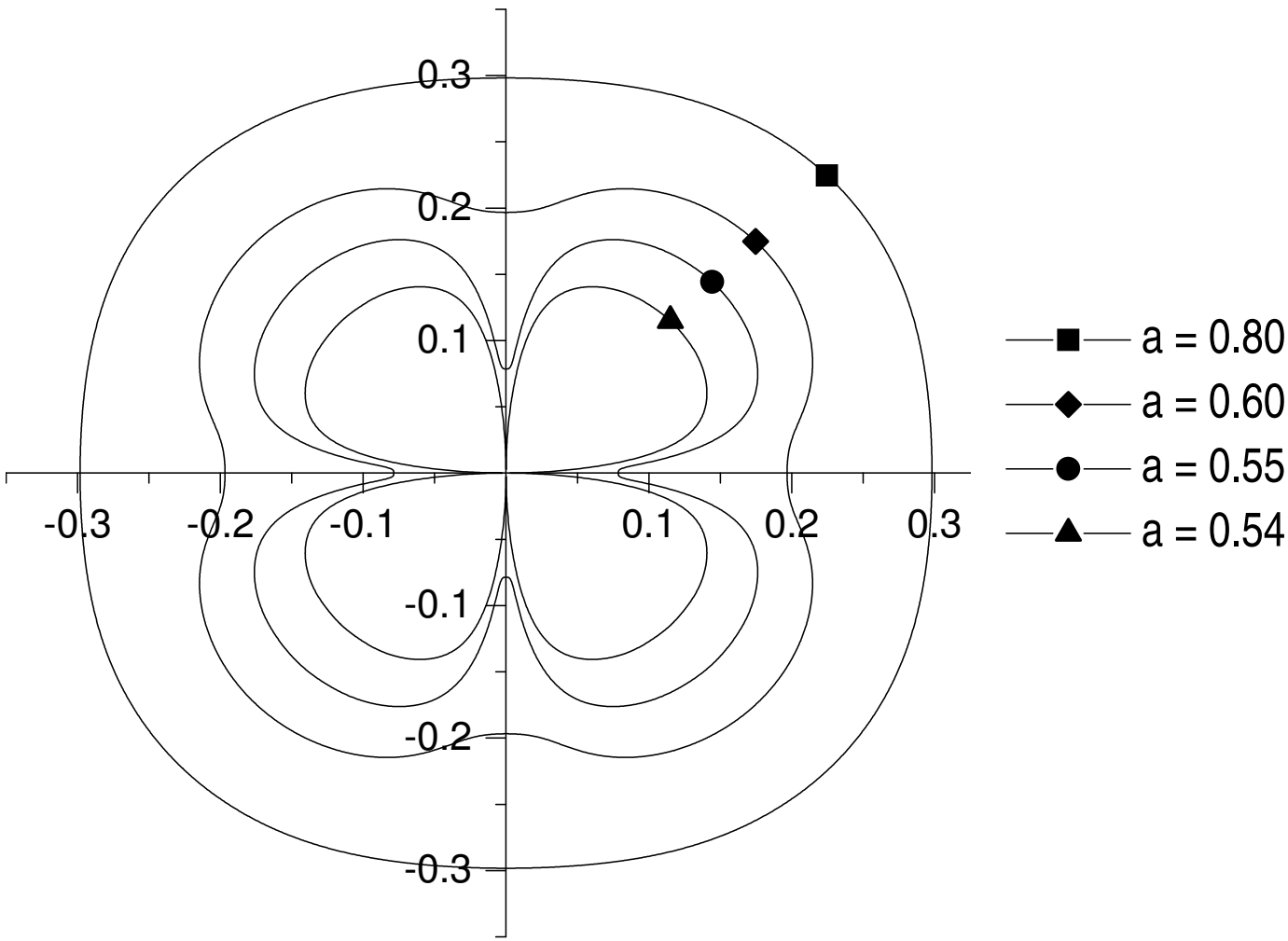
$$L_\theta(\phi) = \phi(\xi + \cos\theta) + \phi(\xi - \cos\theta) + \phi(\xi + \sin\theta) + \phi(\xi - \sin\theta) - 4\phi(\xi). \tag{11}$$



Notice the θ - dependence in the discrete case, which is absent in continuous case.
In continuous case the medium looks the same from each direction. No longer so in discrete case.

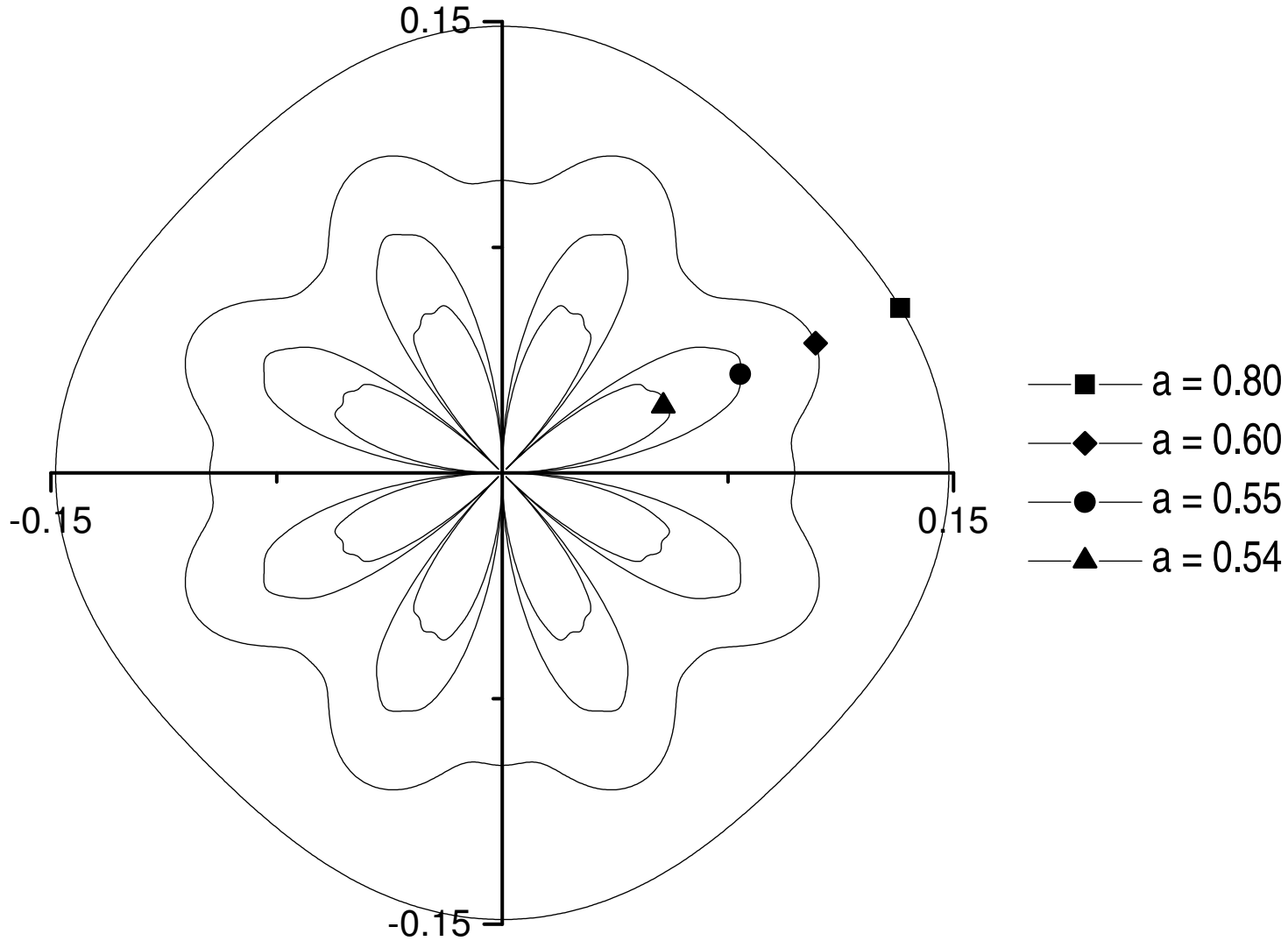
Spatial anisotropy

The lattice anisotropy can be illustrated by studing the $c(\theta)$ relation. Example LDE: $\dot{u}_{i,j} = (\Delta^+ u)_{i,j} - 10 f_{\mathrm{cub}}(u_{i,j}, a)$.



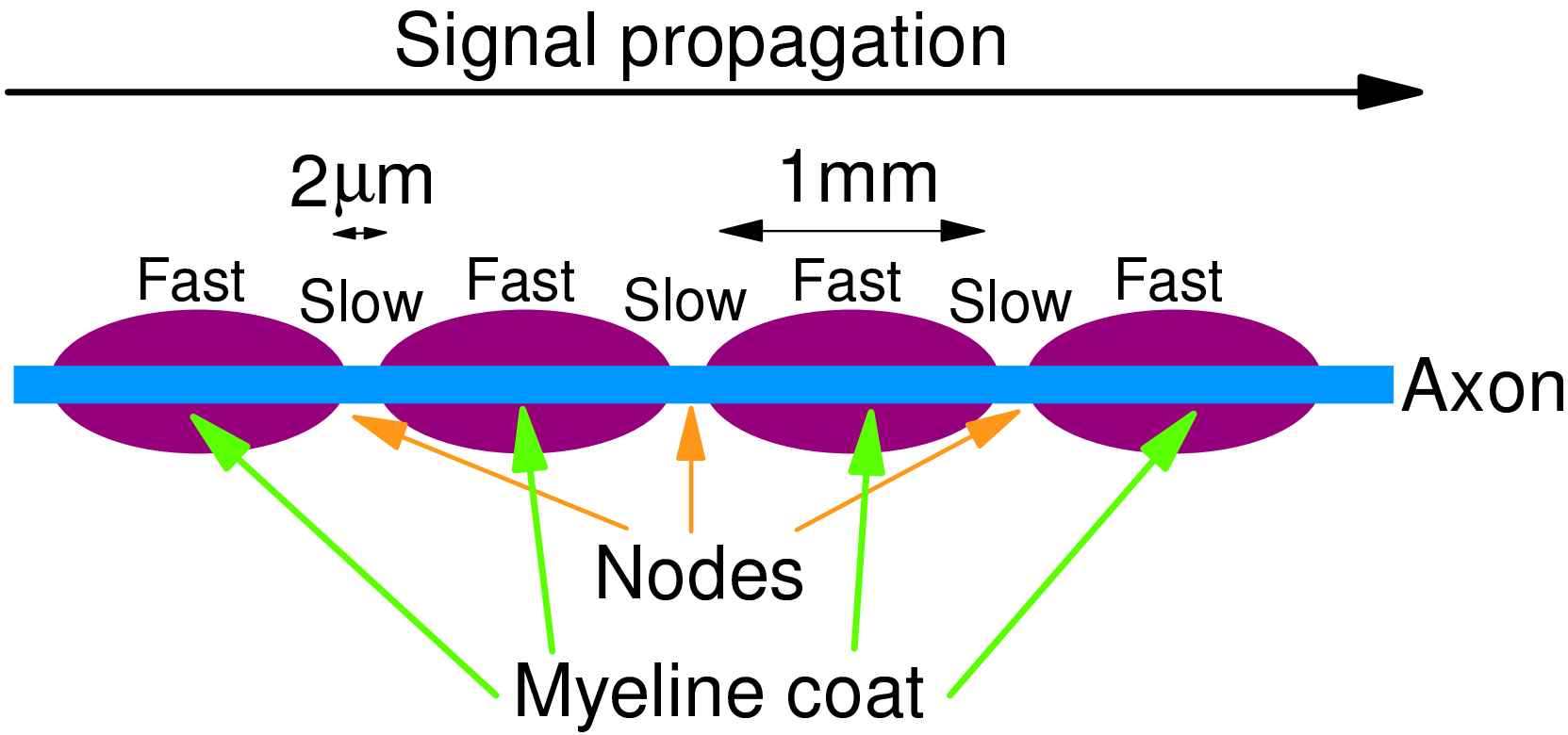
Spatial anisotropy Continued

Another $c(\theta)$ plot for $\dot{u}_{i,j} = \frac{1}{4}\big((\Delta^{+}u)_{i,j} + (\Delta^{\times}u)_{i,j}\big) - 10 f_{\mathrm{cub}}(u_{i,j}, a)$.



Nerve Conduction Theory

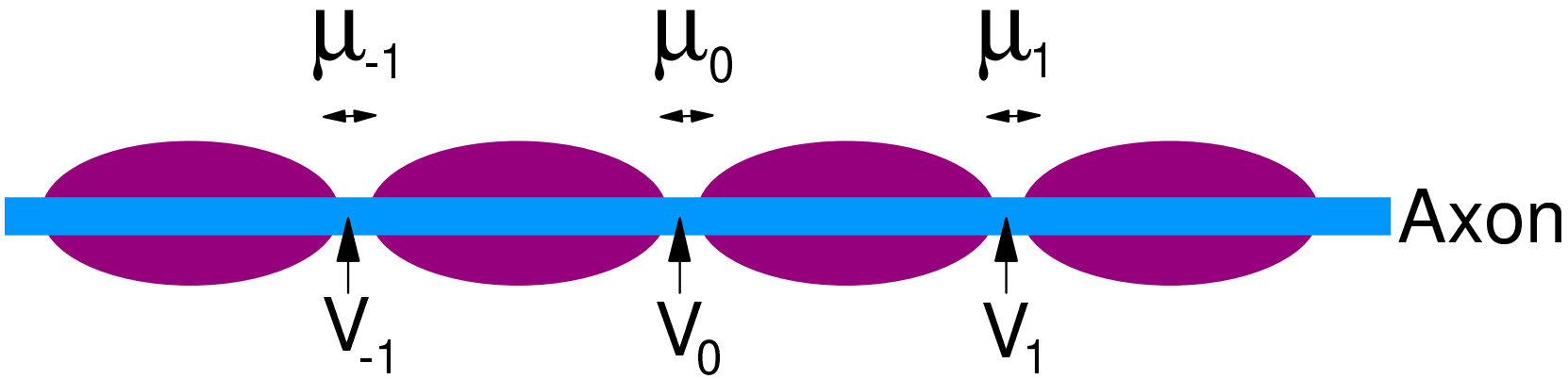
Nerve fibres carry signals over large distances (meter range).



- Fiber has myeline coating with periodic gaps called *nodes* .
- Fast propagation in coated regions, but signal loses strength rapidly (mm-range)
- Slow propagation in gaps, but signal chemically reinforced.
- Nature combines best of both!

Nerve Conduction Theory: The Model

One is interested in the potential at the node sites.



Electro-chemical analysis leads to the 1 dimensional LDE

$$\dot{V}_j = \frac{C}{\mu_j}\left(V_{j+1} + V_{j-1} - 2V_j\right) - I_{\text{ion}}(V_j), \qquad j \in \mathbb{Z}. \tag{12}$$

- Node j has potential V_j.
- Node j has length μ_j.
- Constant C describes electrical properties of the nerve.
- Ionic current I_{ion} well described by our cubic f_{cub}.

Nerve Conduction Theory: Travelling Wave Solutions

Substituting $V_j(t) = \phi(j - ct)$ and taking the node lengths $\mu_j = \mu$ constant, one gets

$$-c\phi'(\xi) = \alpha\big(\phi(\xi+1) + \phi(\xi-1) - 2\phi(\xi)\big) - f_{\mathrm{cub}}(\phi(\xi), a). \tag{13}$$

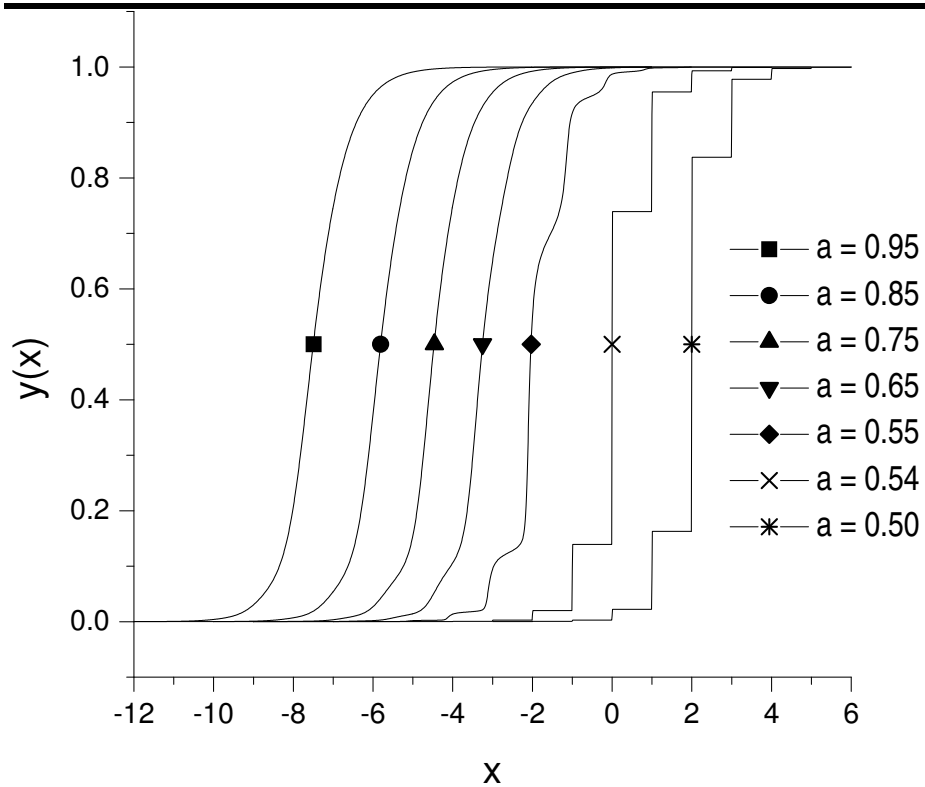
This is called a Differential Difference Equation (DDE). We require $\phi(0) = a$ and impose the limits

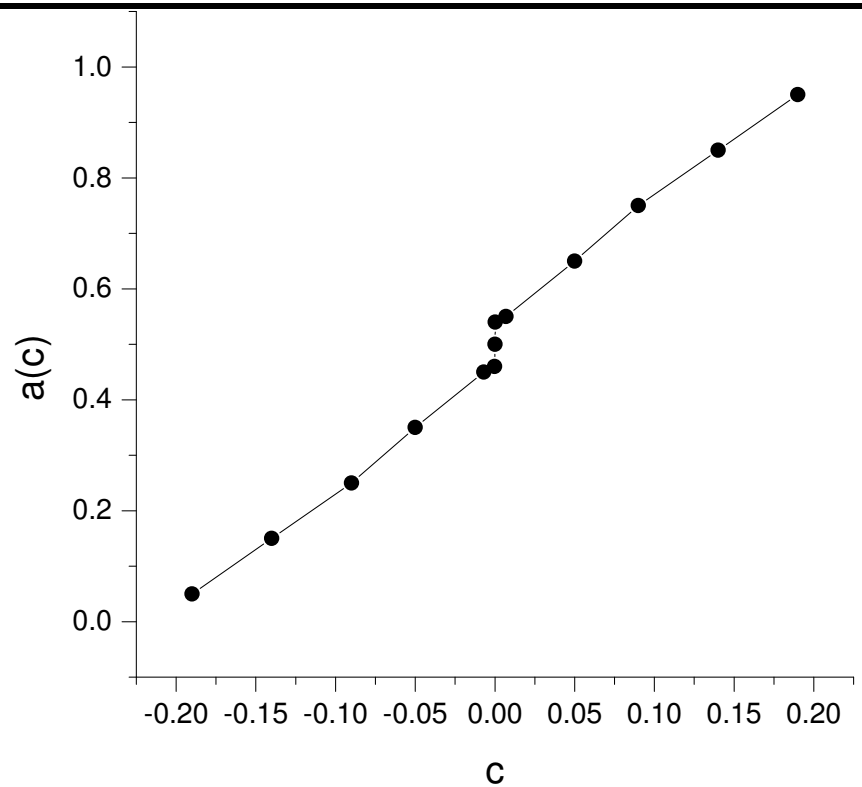
$$\lim_{\xi\to-\infty} \phi(\xi) = 0, \qquad \lim_{\xi\to\infty} \phi(\xi) = 1. \tag{14}$$

Note that finding wavespeed c is part of the problem.

- Algorithm was developed to solve (13) numerically.
- Enormous amount of literature on solving ODE's *without* time-shifted terms.
- Large amount of literature on ODE's with only time-delay terms.
- However, almost nothing known about DDE's with time-delay and time-advanced terms.
- New and exciting subject!

Nerve Conduction Theory: Propagation failure





Solutions to the problem

$$-c\phi'(\xi) = 0.1\big(\phi(\xi+1) + \phi(\xi-1) - 2\phi(\xi)\big) - f_{\mathrm{cub}}(\phi(\xi), a).$$

- Note the nontrivial interval of a in which $c = 0$!
- Note the discontinuities in the wave profiles in this region.
- Propagation failure has been established both theoretically and numerically.
- Propagation failure is DEADLY: for owner of nerve and computational method.

Dealing with propagation failure - Our contribution

Analysis of numerical method to solve class of DDEs including

$$-\gamma\phi''(\xi) - c\phi'(\xi) = \alpha \sum_{j=1}^{N} (\phi(\xi + r_j) - \phi(\xi)) - f_{\mathrm{cub}}(\phi(\xi), a) \tag{15}$$

for $\gamma > 0$ and $\alpha > 0$, under the condition $\phi(0) = a$ and the limits

$$\begin{array}{rcl} \lim_{\xi \to -\infty} \phi(\xi) & = & 0, \\ \lim_{\xi \to \infty} \phi(\xi) & = & 1, \end{array} \tag{16}$$

A connecting solution to the DDE (15) is a pair $(\phi, c) \in W^{2,\infty} \times \mathbb{R}$ which satisfies the DDE (15) and the above conditions (16).

- The extra second order term required to deal with propagation failure discontinuities.
- Question: Do connecting solutions always exist?
- Question: Does the second order term prevent us from seeing the propagation failure?

Main result

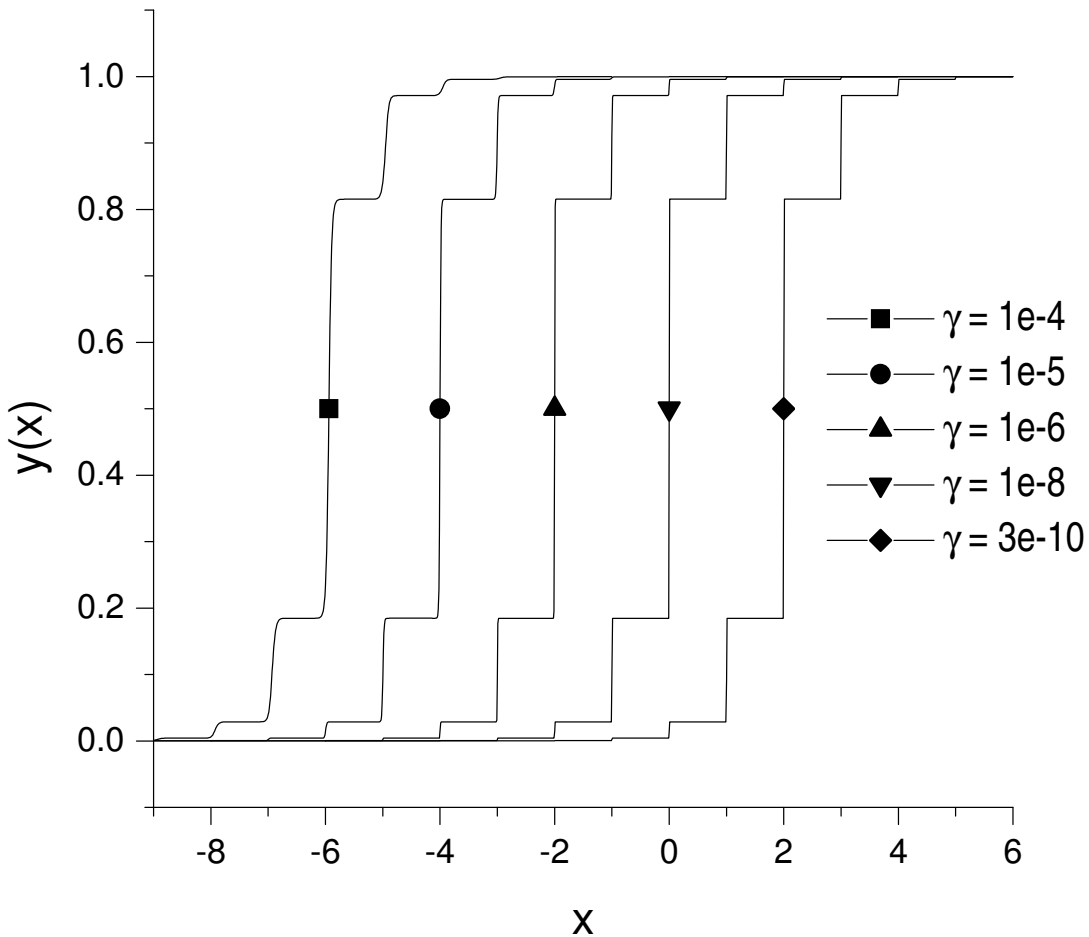
Theorem 1. *The differential difference equation*

$$-\gamma\phi''(\xi) - c\phi'(\xi) = \alpha \sum_{j=1}^{N} \big(\phi(\xi + r_j) - \phi(\xi)\big) - f_{\mathrm{cub}}(\phi(\xi), a),$$

with $\gamma > 0$ *and* $\alpha > 0$ *has a unique connecting solution* $\big(\phi(a), c(a)\big) \in W^{2,\infty} \times \mathbb{R}$ *for all* $0 < a < 1$. *Moreover, this connecting solution* $\big(\phi(a), c(a)\big)$ *depends* C^1*-smoothly on the detuning parameter* a. *Finally, our algorithm can find this connecting solution if supplied with an initial guess* (ϕ_0, c_0) *sufficiently close to this solution.*

- Solutions exist and are unique.
- Our algorithm can find them given a good enough guess.

Limit $\gamma \to 0$ in critical case $a = 0.5$



$$-\gamma\phi''(\xi) - c\phi'(\xi) = 0.1\big(\phi(\xi+1) + \phi(\xi-1) - 2\phi(\xi)\big) - f_{\mathrm{cub}}(\phi(\xi), a). \qquad (17)$$

Tentative conclusion: second order term does not mess things up!

Main Results continued

Theorem 2. *Let (ϕ_n, c_n) be a sequence of connecting solutions to the DDEs*

$$-\gamma_n \phi''(\xi) - c\phi'(\xi) = \alpha \sum_{j=1}^{N} \big(\phi(\xi + r_j) - \phi(\xi)\big) - f_{\mathrm{cub}}(\phi(\xi), a),$$

with $\gamma_n \to 0$. Then, after passing to a subsequence, the pointwise limits

$$\begin{array}{lcl} \phi_0(\xi) & = & \lim_{n\to\infty} \phi_n(\xi), \\ c_0 & = & \lim_{n\to\infty} c_n \end{array} \tag{18}$$

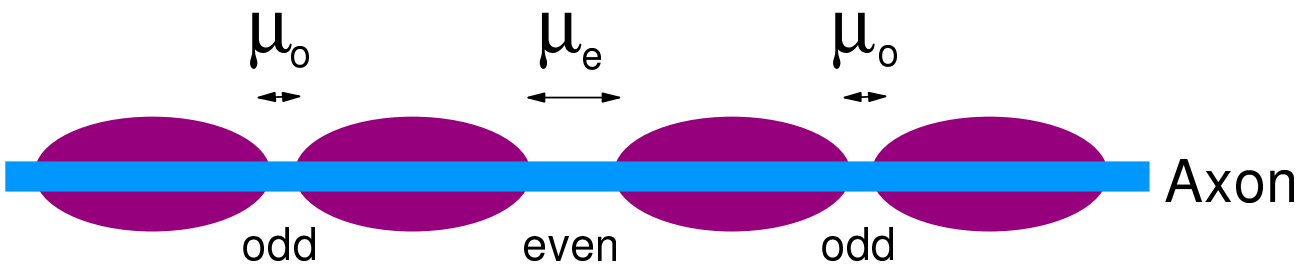
both exist and (ϕ_0, c_0) is a connecting solution to the limiting DDE

$$-c\phi'(\xi) = \alpha \sum_{j=1}^{N} \big(\phi(\xi + r_j) - \phi(\xi)\big) - f_{\mathrm{cub}}(\phi(\xi), a).$$

We thus have practical and theoretical evidence that the rich behaviour at $\gamma = 0$ can be uncovered by choosing γ small enough.

Higher Dimensional Systems

Suppose that the node lengths μ_j are periodic with period two.

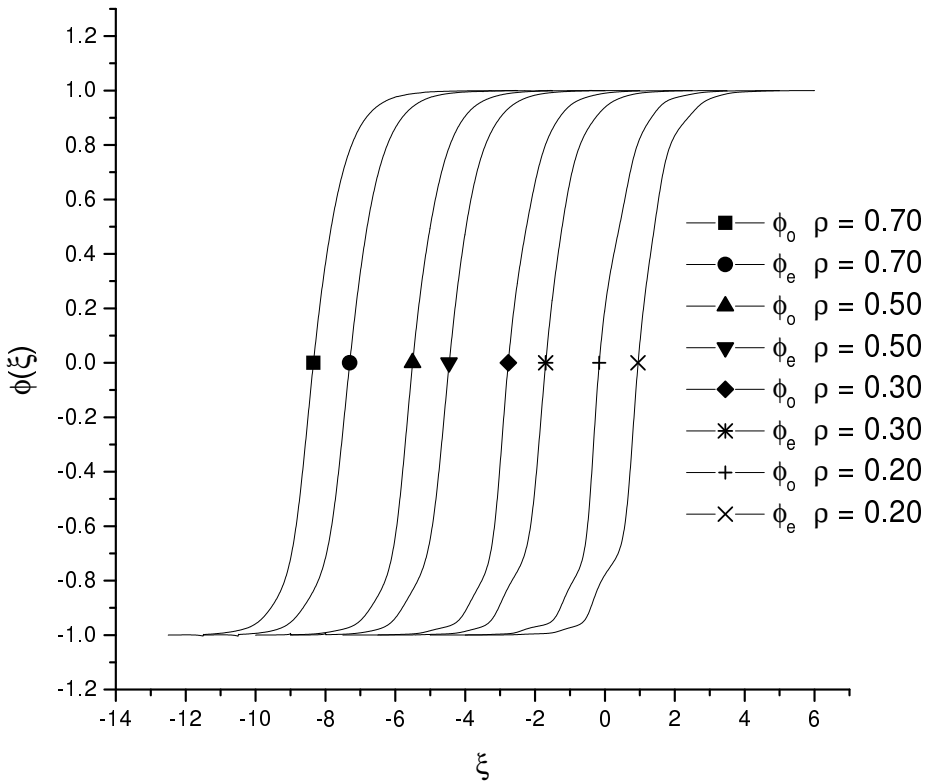


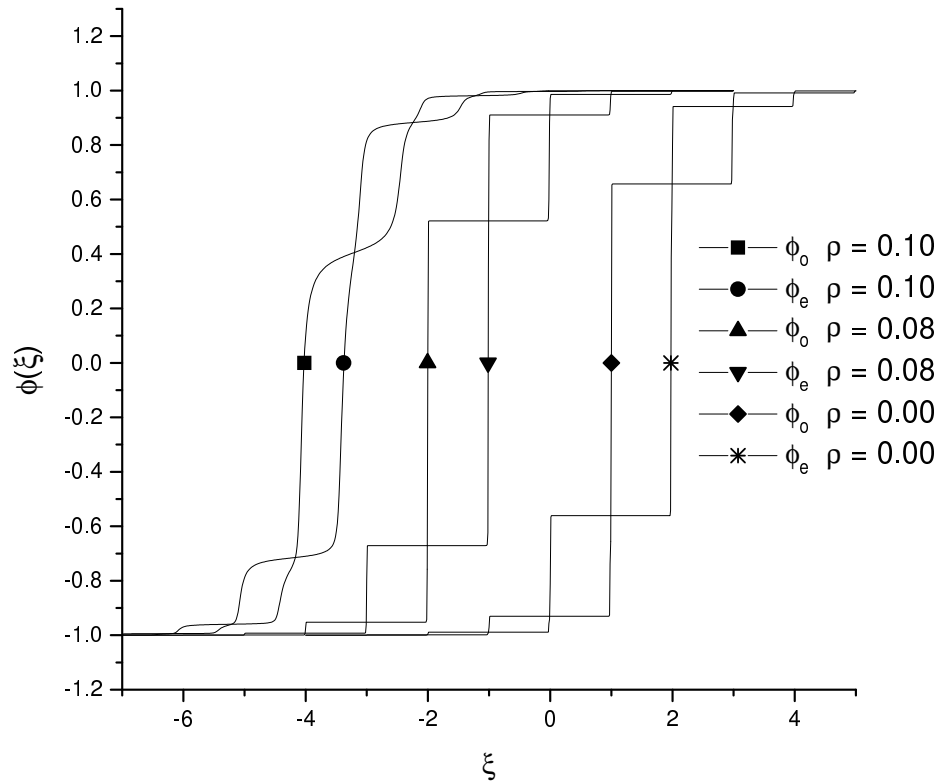
- Split all lattice points into two groups, even and odd.
- Two waveprofiles: ϕ_o and ϕ_e.
- Single wavespeed c.
- System becomes

$$\begin{cases} -c\phi_e'(\xi) = \alpha_e\big(\phi_o(\xi+1) + \phi_o(\xi-1) - 2\phi_e(\xi)\big) - f_{\text{cub}}(\phi_e(\xi), a) \\ -c\phi_o'(\xi) = \alpha_o\big(\phi_e(\xi+1) + \phi_e(\xi-1) - 2\phi_o(\xi)\big) - f_{\text{cub}}(\phi_o(\xi), a). \end{cases} \tag{19}$$

- Limits: $\phi_{o,e}(-\infty) = 0$, $\phi_{o,e}(\infty) = 1$ and $\phi_e(0) = a$.
- Complete analysis has been 1d in nature.
- Open Question: Existence + uniqueness of solution?
- Open Question: Will our algorithm always succeed?

Nerve Conduction Theory: Periodic Node Lengths





- Results with $\alpha_o \neq \alpha_e$.
- Notice propagation failure!
- Notice $\phi_o \approx \phi_e$ away from propagation failure.

Higher Dimensional Systems

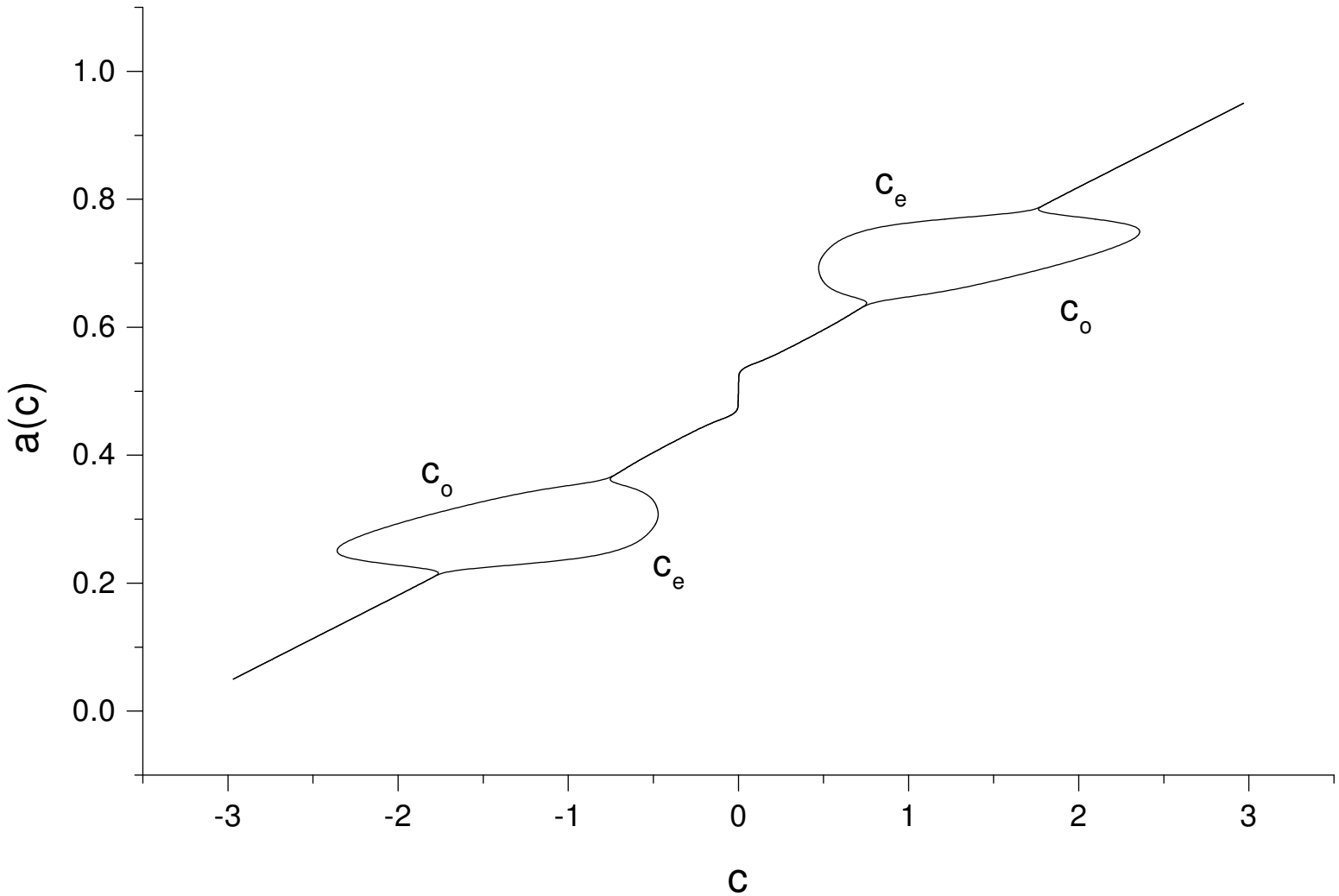
Now introduce two wavespeeds c_o and c_e.

$$\begin{cases} -c_e\phi_e'(\xi) = \alpha_e\big(\phi_o(\xi+1) + \phi_o(\xi-1) - 2\phi_e(\xi)\big) - f_{\mathrm{cub}}(\phi_e(\xi), a) \\ -c_o\phi_o'(\xi) = \alpha_o\big(\phi_e(\xi+1) + \phi_e(\xi-1) - 2\phi_o(\xi)\big) - f_{\mathrm{cub}}(\phi_o(\xi), a). \end{cases} \tag{20}$$

The solutions were normalized to have $\phi_e(0) = a$ and $\phi_o(0) = a$. If we choose $c_e = c_o$ and $\phi_o(\xi) = \phi_e(\xi)$, the system (20) reduces to a one dimensional problem which has a unique solution.

However, solutions to (20) are NOT unique.

Period Two Bifurcation - Solution Is No Longer Unique



The End