TEP: The heart of the matter

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Let A and B denote random variables whose joint probability distribution encapsulates our uncertainty as to the actual amounts a and b in the two envelopes. I do not need assume here that A is half or twice B. I just assume that A and B are always different and that their distribution is symmetric under exchange. The following facts can therefore be used for two envelopes (all symmetric versions), two neckties, two-sided cards; with or without subjective probability, with or without finite expectations. The derivation is elementary. The results are not surprising. The point is that they are general results. Many solutions take a particular prior distribution by way of example and show that certain of these facts are true. That is a bit unsatisfactory because it doesn't prove that the results always have to be true, hence leaves a doubt in the mind of the reader. For example, this is why Martin Gardner felt that neither Kraitchik's problem nor TEP were properly solved at the time when he wrote about them. He had only seen particular examples but this does not prove that what we see in those examples always has to be true.

Theorem

(1) Under symmetry, $E(A) = E(B)$.

(2) Under symmetry, if $E(A)$ is finite, then it is impossible that $E(B|A = a) > a$ for all a.

(3) Under symmetry, it is impossible that

$$P(A < B|A = a) = 1/2$$

for all a.

Proof

(1) is obvious (symmetry!)

(2): proof by contradiction with (1). If $E(B|A) > A$ then $E(B) > E(A)$ or both are infinite or undefined.

(3): proof by symmetry of *stochastic independence* between r.v. A and event $\{A < B\}$. Because if $P(A < B|A = a) = 1/2$ for all a, then the event $\{A < B\}$ is independent of the random variable A. Now replace A and B by $A' = g(A)$, $B' = g(B)$ where g is a strictly increasing function from the real line into a bounded interval of the real line (for instance, the arc tangent function). All the assumptions we made about A and B also hold for the transformed versions, but now we can be certain that expectation values are finite. Till further notice, forget the prime and just write A and B for these transformed versions. Consider the trivial inequality $E(A-B|A-B > 0) > 0$. By finite expectation values, this can be rewritten as

$$E(A|A > B) > E(B|A > B) = E(A|B > A)$$

where the last equality uses symmetry. This inequality shows that A is statistically dependent on the event $\{A < B\}$, hence the event $\{A < B\}$ is statistically dependent on the random variable A. Transforming back to the original variables this remains true.

Corollary 1 (an exercise for connoisseurs/students of probability theory). Let g be a strictly increasing function and let $A' = g(A)$, $B' = g(B)$. Then the theorem also applies to the pair A' and B'. Extend to not necessarily strictly increasing g by approximating by strict and going to the limit (strict inequalities need no longer be strict in the limit). We find

(4) The probability distributions of $A|A < B$, of A, and of $A|A > B$ are strictly *stochastically ordered* (from small to large).

These facts take care of the main variants of the two envelopes problem as well as all its predecessors two neckties, two-sided cards. The only way to escape the facts is to assume improper distributions. But they are ... improper. In fact, they are: ludicrous, according to Schrödinger, Littlewood, Falk, and just about everyone.

Some writers like to escape TEP paradoxes associated with infinite expectation values by transforming amounts of money to the utility of money, supposed to be bounded. This is covered by the following restatement of (2) and (4):

Corollary 2 Let g be increasing and define $A' = g(A)$. Suppose $Eg'(A)$ is finite. Then

$$E(g(A')|A < B) \leq Eg(A') \leq E(g(A')|A > B).$$

If g is moreover one-to-one, the inequalities are strict.