SUBJECT: LIE GROUPS

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Introduction

A Lie group is a group which is also a differentiable manifold, with the property that the group operations are compatible with the manifold structure. They are named after the nineteenth century Norwegian mathematician Sophus Lie, who laid the foundations of the theory of continuous transformation groups. Lie groups represent the best developed theory of continuous symmetry of mathematical objects and structures, which makes them indispensable tools for many parts of contemporary mathematics, as well as for modern theoretical physics: one of the first spectacular uses of group theory in theoretical physics was Wigner's insight of 1940's, which relates "symmetries of spacetime" to "things in it" (particules).

Presentation

Since Lie groups are manifolds, they can be studied using differential calculus, in contrast with the case of more general topological groups. One of the key ideas in the theory of Lie groups, due to Sophus Lie, is to replace the global object, the group, with its local or linearised version, which Lie himself called its "infinitesimal group" and which has since become known as its Lie algebra.

A (real) Lie group is a group which is also a finite-dimensional real smooth manifold, and in which the group operations of multiplication and inversion are smooth maps.

The basic example of a real Lie group is the $\mathrm{GL}_n(\mathbf{R})$ of $n\times n$ invertible matrices with coefficients in $\mathbf{R}$, for an integer $n \geqslant 1$. The *usual* groups of matrices are actually Lie groups: $\mathrm{SL}_n(\mathbf{R})$ the group of $n \times n$ matrices with determinant 1, the group of orthogonal matrices $\mathrm{O}_n(\mathbf{R})$ and so on. We can introduce the notion of complex Lie groups and as example just cite $\mathrm{GL}_n(\mathbf{C})$, $\mathrm{SL}_n(\mathbf{C})$ ...

To every Lie group G, we can associate a Lie algebra, whose underlying vector space is the tangent space of G at the identity element, which completely captures the local structure of the group. Informally we can think of elements of the Lie algebra as elements of the group that are "infinitesimally close" to the identity, and the Lie bracket is something to do with the commutator of two such infinitesimal elements. If we require that the Lie group be connected and simply connected, then the global structure is determined by its Lie algebra: for every finite dimensional Lie algebra $\mathfrak{g}$, there is a connected and simply connected Lie group G with $\mathfrak{g}$ as Lie algebra, unique up to isomorphism. Moreover every homomorphism between Lie algebras lifts to a unique homomorphism between the corresponding connected and simply connected Lie groups.

The introduction of quaternions allows us to prove that the universal cover of SO(3) is SU(2). Wigner related representations of a certain Lie groups and particules. So mathematically, an interesting question may be to compute representations of Lie groups.

An algebraic approach allows us to define "algebraic groups" a generalization of Lie groups over an arbritrary field.

What can be done in a Bachelor thesis

The notion of Lie groups has to be learned. A background in calculus and basic algebra is required. Then there are different possibilities. The first is to compute the representations of SO(3), and perhaps SO(4). Or, one can prove the Peter-Weyl theorem: the space of square integrable functions on a compact Lie group may be decomposed. Or, one can go to a more algebraic thesis by studying the classification of the complex simple Lie algebras.

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