Why there exist no Division Algebras over $\mathbb{R}$ of uneven dimension greater than 1

Oliver Lenz

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Definition. *Identify the n-sphere S^n with all points in $\mathbb{R}^{n+1}$ of euclidean norm 1. The n-sphere is said to be* parallelisable *if there exist n continuous maps*

$$\phi_i : S^n \longrightarrow S^n$$

such that for every $a \in S^n$, $a, \phi_1(a), \phi_2(a), \dots, \phi_n(a)$ is linearly independent.

The concept of parallelisability is relevant to the existence of division algebras by way of the following implication:

Proposition. *Suppose that for $n \geq 0$, there exists an n-dimensional division algebra A over $\mathbb{R}$. Then the $(n-1)$-sphere is parallelisable.*

In the uneven-dimensional case we can then show that parallelisability of the n-sphere leads to a contradiction using the *Brouwer degree* of a map from S^n to S^n:

Claim. *Let $n \geq 0$. Let $f, g \in Mor(S^n, S^n)$. The Brouwer degree satisfies the following properties:*

1. *$deg(g \circ f) = deg(f)deg(g)$.*
2. *$deg(Const) = 0$.*
3. *$deg(Id_{S^n}) = 1$.*
4. *If $f \sim g$, then $deg(f) = deg(g)$.*
5. *Let $0 \leq i \leq n$, then $Refl_i$ is the map that sends a point $(v_0, v_1, \dots, v_i, \dots, v_n)$ to $(v_0, v_1, \dots, -v_i, \dots, v_n)$. We have $deg(Refl_i) = -1$.*

For the purpose of this talk, these properties will only be assumed, not proven, but it will be shown how this leads to a contradiction, and if there is time, the construction of the Brouwer degree will shortly be discussed.