SHORT COURSE ON $\mathcal{D}$-MODULES @UCHICAGO

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One of the goals of these talks is to cover the background needed for the upcoming workshop on Hodge ideals at UIC. A few of the sources I used to put these talks together are

(1) C. Schnell: "An overview of Morihiko Saito's theory of mixed Hodge modules." Excellent overview with exercises but almost no proofs.
(2) Saito's Modules de Hodge Polarisable '88 and Mixed Hodge Modules'90 papers.
(3) C. Elliott: *D*-modules. Excellent supplement for [HTT95].
(4) Hotta, Takeuchi, Tanisaki: [HTT95]. Extremely detailed and thorough. It does not contain any information on V-filtration or the Hodge filtration. But talks about good filtration which is a great starting point.
(5) M. Popa: "Kodaira–Saito vanishing and applications." Lots of examples, excellent source for motivations and applications.
(6) Notes from Mihnea's talks on Hodge modules and V-filtrations (not posted anywhere).
(7) N. Budur: Numerous surveys and lecture notes; e.g. "On the V-filtrations of $\mathcal{D}$-modules." Excellent survey, includes the proof of the uniqueness of the V-filtrations. For existence, see [Kas83].
(8) C. Schnell and C. Sabbah: "The MHM Project": This deals with the Hodge filtration by developing a theory over the Rees object assciated to the filtration.

1. Preliminaries

In this section we set basic notations, define the main protagonists and some of the operations we can do with them. The purpose of this section is to look at a lot of examples of $\mathcal{D}$-modules and create new ones using the $\mathcal{D}$-module operations.

Notation 1.1. Let X be a smooth algebraic variety over $\mathbb{C}$. We denote by Θ_X or $\mathcal{T}_X$ the sheafification of

$$\mathrm{Der}_{\mathbb{C}}(\mathcal{O}_X(U)) \coloneqq \{\theta \in \mathcal{E}nd_{\mathbb{C}}(\mathcal{O}_X)(U) | \theta(fg) = \theta(f)g + f\theta(g)\}.$$

This is the tangent sheaf of X. We identify the cotangent sheaf with $\Omega^1_X \coloneqq \mathcal{H}om_{\mathcal{O}_X}(\Theta_X, \mathcal{O}_X)$.

Definition 1.2 ($\mathcal{D}_X$)**.** The sheaf of differential operators $\mathcal{D}_X$-modules on X, is the $\mathbb{C}$ algebra generated by $\mathcal{O}_X$ and Θ_X. Locally, with local system of parameteres $t_1, \cdots, t_n$,

$$\mathcal{D}_X \overset{\text{loc}}{\simeq} \mathbb{C}[t_1, \cdots, t_n, \partial_1, \cdots, \partial_n]/\sim$$

where $\{\partial_i\}_i$ are the local sections of the sheaf Θ_X and $\sim$ denotes the relations

$$[t_i, t_j] = 0 = [\partial_i, \partial_j] \quad [t_i, \partial_j] = \delta_{ij}.$$

Properties 1.3. (1) The sheaf of differential operators is Noetherian.
(2) It is simple, i.e. there are no two-sided non-trivial two sided ideal of $\mathcal{D}_X$.

Lemma/Definition 1.4 (Dmodules)**.** An $\mathcal{O}_X$-modules M has a left (or, right) $\mathcal{D}_X$ action if and only if there exists a $\mathbb{C}$-linear morphism $\nabla\colon \Theta_X \to \mathcal{E}nd_{\mathbb{C}}(M)$ locally satisfying

- $\nabla_{f\theta} = f\nabla_\theta$
- $\nabla_\theta(fm) = \theta(f)m + f\nabla(m)$
- $\nabla_{[\theta_1,\theta_2]} = [\nabla_{\theta_1}, \nabla_{\theta_2}]$

where $\theta_i \in \Theta_X, m \in M, f \in \mathcal{O}_X$. Furthermore, the above is equivalent to having a $\mathbb{C}$-linear morphism $\nabla' : M \to \Omega^1_X \otimes_{\mathcal{O}_X} M$ locally satisfying

$$\nabla'(fm) = \sum_i dt_i \otimes \partial_i fm + f\nabla'(m) \quad \nabla'(Pm) = \sum_i dt_i \otimes \partial_i P(m) + P\nabla'(m).$$

Proof. The equivalence of the first two is rather straightforward, by denoting the action $\theta \cdot m := \nabla_\theta(m)$ (or, $m \cdot \theta := -\nabla_\theta(m)$ for the right action of $\mathcal{D}_X^{op}$.) For the last equivalence note that given a $\mathbb{C}$-linear morphism $\nabla : \Theta_X \to \mathcal{E}nd_{\mathbb{C}}(M)$ define

$$\nabla'(m) = \sum_i dt_i \otimes \nabla_{\partial_i}(m).$$

Conversely, given $\nabla' : M \to \Omega^1_X \otimes_{\mathcal{O}_X} M$ so that $m \mapsto \sum_i dt_i \otimes m_i$ define

$$\nabla_{\partial_i}(m) = m_i.$$

□

Definition 1.5 (Good/Coherent filtration)**.** Define $F_\bullet \mathcal{D}_X$ to be the coherent sheaves determined by sheafifying

$$F_i\mathcal{D}_X(U) = \begin{cases} \oplus_{I \in \mathbb{Z}^n_{\geq 0}} \mathcal{O}_U \partial^I & \text{if } i \geq 0 \\ \mathcal{O}_X & \text{otherwise.} \end{cases}$$

Note that $F_i\mathcal{D}_X \cdot F_j\mathcal{D}_X \simeq F_{i+j}\mathcal{D}_X$ for all i, j. A filtration on a quasi-coherent left (or right) $\mathcal{D}$-module M is an increasing exhaustive filtration $F_\bullet M$ by quasi-coherent sheaves, satisfying $F_1\mathcal{D}_X \cdot F_iM \subset F_{i+1}M$ and $F_iM = 0$ for $i \ll 0$. It is called a *good* or *coherent* filtration if F_iM are coherent sheaves and there exists a $i_0 \gg 0$ such that for all $i > i_0$ and for all k,

$$F_k\mathcal{D}_X \cdot F_iM = F_{i+k}M.$$

It is a small exercise that M admits a coherent filtration if and only if M is a coherent $\mathcal{D}_X$-module.

Note that $gr^F\mathcal{D}_X \simeq \mathbb{C}[t_1, \cdots, t_n, \partial_1, \cdots, \partial_n]$ the polynomial ring in $2n$ generator. Indeed, $[t_i, \partial_i] = 0 \in \dfrac{F_1\mathcal{D}_X}{F_0\mathcal{D}_X}$ and therefore the associated graded ring must be commutative. Furthermore, let $\pi\colon T^*X \to X$ denote the bundle map, then, $gr^F\mathcal{D}_X \simeq \pi_*\mathcal{O}_{T^*X}$. It requires a bit of effort to show that, $F_\bullet M$ is a good filtration if and only $gr^F M$ is a coherent $\mathcal{O}_{T^*X}$-module.

Definition 1.6 (Characteristic Varieties)**.** Let M be a $\mathcal{D}$-module admitting a good filtration F, then define the characteristic variety associated to M as

$$\mathrm{Ch}(M) := \mathrm{Supp}_{T^*X}(\mathcal{O}_{T^*X} \otimes_{\pi^{-1}\pi_*\mathcal{O}_{T^*X}} \pi^{-1}gr^F M) \subseteq T^*X.$$

Note that $\pi_*(\mathcal{O}_{T^*X} \otimes_{\pi^{-1}\pi_*\mathcal{O}_{T^*X}} \pi^{-1}gr^F M) \simeq gr^F M$.

Example 1.7.

(1) $\mathcal{D}_X$ is both a left and a right module over itself. Moreover, $\mathrm{Ch}(\mathcal{D}_X) \simeq T^*X$.

(2) $\mathcal{O}_X$ is naturally a left $\mathcal{D}$-module which has a trivial coherent filtration

$$F_i\mathcal{O}_X = \begin{cases} 0 & \text{if } i < 0 \\ \mathcal{O}_X & \text{otherwise.} \end{cases}$$

and $\mathrm{Ch}(\mathcal{O}_X) \simeq T^*_X X$, the zero section of X in T^*X.

(3) ω_X is a right $\mathcal{D}$-module, under the action $\omega \cdot \theta = -\mathrm{Lie}_\theta(\omega)$, where

$$\mathrm{Lie}_\theta(\omega)(\theta_1 \wedge \cdots \theta_n) := \theta(\omega(\theta_1 \wedge \cdots \wedge \theta_n) - \sum_{i=1}^n \omega(\theta_1 \wedge \cdots [\theta, \theta_i] \wedge \cdots \theta_n).$$

This can also be equipped with a good filtration

$$F_i\omega_X = \begin{cases} 0 & \text{if } i < -n \\ \mathcal{O}_X & \text{otherwise.} \end{cases}$$

and again $\mathrm{Ch}(\omega_X) \simeq T^*_X X$.

(4) Let M be a $\mathcal{D}$-module then for any divisor $D \overset{\text{loc}}{=} (f = 0)$, $M(*D) \overset{\text{loc}}{\simeq} M[\frac{1}{f}]$ is also a $\mathcal{D}_X$-module with the usual action satisfying Leibnitz rule. This is not a coherent $\mathcal{O}_X$-module. If $M = \mathcal{O}_X$ (or any coherent $\mathcal{O}_X$-module), the pole-order filtration $P_k(\mathcal{O}_X(*D)) :=$ $\mathcal{O}_X((k+1)D)$ is a F-filtration. It is good, if D is smooth. Indeed, when $D \overset{\text{loc}}{\simeq} (t_1 = 0)$, then, $\partial_1(\dfrac{1}{t_1^k}) \simeq -\dfrac{k}{t_1^{k+1}}$ generates $\mathcal{O}_X((k+1)D)$.

(5) Let $X = pt$, and L a $\mathbb{Q}$ vector space with an underlying Hodge structure of weight k, i.e. there exists a decomposition
$$L \otimes \mathbb{C} \simeq \bigoplus_{p+q=k} L^{p,q}$$
such that $\overline{L^{p,q}} \simeq L^{q,p}$. Equivalently, there is a decreasing filtration $F^\bullet L$ satisfying
$$F^p L_{\mathbb{C}} \cap \overline{F^{k-p+1}} L_{\mathbb{C}} \simeq 0 \text{ and } F^p L_{\mathbb{C}} \oplus F^{n-p+1} L_{\mathbb{C}} \simeq L_{\mathbb{C}}.$$
Such filtrations are called the Hodge filtration on L. This is $\mathcal{D}$-module with trivial action and the filtration taken as $F_p L_{\mathbb{C}} \simeq F^{-p} L_{\mathbb{C}}$ we obtain that the filtration is trivially a filtration of D-modules.

(6) **A glimpse to the Riemann-Hilbert Correspondence** A variation of Hodge structure on X of weight k is a tuple $(\mathcal{V}, F^\bullet, \mathbb{L})$ where $\mathbb{L}$ is a local system with fibres L, $\mathbb{Q}_X$-vector space on X, $\mathcal{V} \simeq \mathbb{L} \otimes_{\mathbb{Q}} \mathcal{O}_X$ equipped with a decreasing Hodge filtration $F^p\mathcal{V}$ of subbundles satisfying the following conditions:
1. At each point $y \in Y$, the filtered vector space $(L \otimes_{\mathbb{Q}} \mathbb{C}, F^\bullet)$ is a Hodge structure of weight k on the stalk $\mathbb{L}_y \simeq L$.
2. (Griffiths Transversality) The local system $\mathbb{L}$ induces an $\mathcal{O}_X$-vector bundle with integrable connection $(\mathcal{V}, \nabla)$ on X. This connection should satisfy the condition
$$\nabla(F^p\mathcal{V}) \subset \Omega^1_X \otimes_{\mathcal{O}_X} F^{p-1}\mathcal{V}$$
This is a good filtration.
Therefore $\mathcal{V}$ is a $\mathcal{D}$-module admitting a filtration given by $F_p\mathcal{V} := F^{-p}\mathcal{V}$. Note that Griffiths transversality can be interpreted as $F_1\mathcal{D}_X \cdot F_p\mathcal{V} \subseteq F_{p+1}\mathcal{V}$.

Another good filtration on $\mathcal{V}$ is the trivial filtration off-set at 0. Then $gr^F\mathcal{V} \simeq \mathcal{V}$ supported along the copy of X in T^*X namely, $T^*_X X$. Therefore, $\mathrm{Ch}(\mathcal{V}) \simeq T^*_X X$. We will see later in 1.9 that this is sufficient to ensure that the $\mathcal{D}$-module is a variation of Hodge structure.

1.0.1. *Left-right correspondence.* There is an equivalence of categories
$$\mathrm{Mod}(\mathcal{D}_X) \to \mathrm{Mod}(\mathcal{D}_X^{op})$$
via $M \mapsto \omega_X \otimes_{\mathcal{O}_X} M$ where $(\omega \otimes m) \cdot \theta = -\mathrm{Lie}_\theta\, \omega \otimes m - \omega \otimes \theta \cdot m$ and the inverse is given by $M' \mapsto M' \otimes_{\mathcal{O}_X} \omega_X^{-1} \simeq \mathcal{H}om_{\mathcal{O}_X}(\omega_X, M')$ via the action $\theta(\phi)(\omega) = -\phi(\omega) \cdot \theta + \phi(-Lie_\theta \omega)$.

Remark 1.8. This operation is the same as taking formal adjoints of the differntital since in terms of local coordinates $(f dt_1 \wedge \cdots \wedge dt_n) \cdot \theta = \theta(f) dt_1 \wedge \cdots \wedge dt_n$ we obtain
$$\mathcal{D}_X^{op} \simeq \omega_X \otimes_{\mathcal{O}_X} \mathcal{D}_X \otimes_{\mathcal{O}_X} \omega_X^{-1}.$$

Using the tensor product filtration we obtain
$$F_p(M \otimes \omega_X^{-1}) \simeq F_{p-n} M \otimes_{\mathcal{O}_X} \omega_X^{-1}.$$
For example, $F_0\mathcal{O}_X \simeq F_{-n}\omega_X \otimes \omega_X^{-1} \simeq \mathcal{O}_X$.

Proposition 1.9. *A coherent $\mathcal{D}_X$-module M is (1) a coherent $\mathcal{O}_X$-module if and only if it is (2) a vector bundle with integrable connection if and only if (3) $\mathrm{Ch}(M) \simeq T^*_X X$.*

Proof. (1)⇒ (2): For $x \in X$, let $M_x \simeq \mathcal{O}_{X,x} < m_1, \cdots, m_k >$ is minimally generated via lifting the basis

$$M \otimes \kappa(x) \simeq \bigoplus_{i=1}^{k} \mathbb{C} m_i$$

Then, suppose there exists a relation

$$\sum_i f_i m_i = 0$$

for some $f_i \in \mathcal{O}_{X,x}$. Since M is a $\mathcal{D}_X$-module,

$$\partial_j(\sum_i f_i m_i) = \sum_i(\partial_j f m_i + f_i \partial_j \cdot m_i) = \sum_i(\partial_j f_i + \sum_\ell f_\ell g_i^\ell) m_i$$

where $\partial_j m_i = \sum_\ell g_\ell^i m_\ell$. Now, $\operatorname{ord}(\partial_j f_i + \sum_\ell f_\ell g_i^\ell) = \min(\operatorname{ord}(\partial_j f_i), \operatorname{ord}(\sum_\ell f_\ell g_i^\ell))$ For some j, $\operatorname{ord}\partial_j f_i < \operatorname{ord} f_i$. Hence we can induct on the order to conclude that, the relation reduces to

$$\sum_i c_i m_i = 0 \text{ for } c_i \in \mathbb{C}.$$

But this is a contradiction.

(2)⇒ (3): Discussed above.

(3)⇒ (1): Since M is coherent, it has a good filtration, i.e. for there exists $i_0 \gg 0$ so that $F_k\mathcal{D}_X \cdot F_i M = F_{k+i}M$ for all k and for all $i \geq i_0$. Now since $\operatorname{Ch}(M) = T^*_X X$, we have that, we have an inclusion of ideal sheaves, locally $(\operatorname{Ann}(gr^F_\bullet))^{m_0} \subseteq (\partial_1, \cdots, \partial_n) gr^F_\bullet \mathcal{D}_X$ for some $m_0 \gg 0$. This means that, for all i, $\partial^{m_0} \cdot F_i M = 0 \in gr^F_{i+m_0} M$, i.e. $F_{m_0}\mathcal{D}_X \cdot F_i M \subseteq F_{i+m_0-1}$. When $i \geq i_0$, this implies

$$F_{m_0}\mathcal{D}_X \cdot F_i M = F_{m+i} M \subseteq F_{i+m_0-1}.$$

Hence, $F_{i_0+m_0-1}M = F_{i_0+m_0}M = F_{i_0+m_0+1}M = \cdots$. Then, M is generated over $\mathcal{O}_X$ by the coherent $\mathcal{O}_X$-modules $F_1 M, \cdots, F_{i_0+m_0-1}M$ and hence is coherent.

□

1.1. **Regular Holonomic $\mathcal{D}$-modules.**

Theorem 1.10 (Sato–Kashiwara–Kawai, Beilinson–Bernstein, Gabber, Kashiwara–Schapira)**.** *Let M be a $\mathcal{D}$-module on a smooth algebraic variety X of dimension n, then the characteristic variety is involutive with respect to the symplectic structure on T^*X and therefore $\dim \Lambda \geq n$ for any irreducible component $\Lambda \subseteq \operatorname{Ch}(M)$.*

Definition 1.11. A coherent $\mathcal{D}$-module M is said to be holonomic if $\dim \operatorname{Ch}(M) = n$

It follows from geometry that $\operatorname{Ch}(M)$ is a conic Lagrangian submanifolds of T^*X. It was shown by Kashiwara that for any holonomic $\mathcal{D}$-module there exists a Whitney stratification $X = \sqcup X_\alpha$ such that $\operatorname{Ch}(M) \subseteq \sqcup_\alpha T^*_{X_\alpha} X$.

Corollary 1.12. *Let M be a holonomic $\mathcal{D}_X$-module. Then there exists an open dense subset $U \subseteq X$ such that $M|_U$ is coherent over $\mathcal{O}_U$. In other words, $M|_U$ is a locally free $\mathcal{O}_X$-module with an integrable connection (or, simply integrable connections).*

Proof. If $\operatorname{Ch}(M) = T^*_X X$, then by Proposition 1.9 M is an integrable connection. Otherwise, let $S \subseteq X$ denote the subvariety of X on which the image of $\operatorname{Ch}(M) \setminus T^*_X X$ is supported. Since M is holonomic $\dim(S) \leq n-1$. Then $\operatorname{Ch}(M|_{X\setminus S}) = T^*_U U$. Letting $U = X \setminus S$, we get $M|_U$ is coherent over $\mathcal{O}_U$ by Proposition 1.9. □

Remark 1.13. In the setting when M is a polarised (mixed) "Hodge module", Saito proved that there exists an open set such that $M|_U$ is a polarised variation of (mixed) Hodge structure.

Definition 1.14 (regular)**.** The definition of regularity comes from moderate growth of solutions to PDE. A holomorphic function $f : \mathbb{C}^n \to \mathbb{C}$ is said to have moderate growth at 0, if for all $z \in \{z = (r, \theta) | 0 < r < \epsilon \text{ and } \delta_1 < \theta < \delta_2\}$,

$$|f(z)| < \frac{c}{|z|^j}$$

for some $c > 0$ and $j \in \mathbb{Z}_{\geq 0}$. I will not define it explicitly here, but I will give an example. Let $X = \mathbb{P}^1$ with local coordinate t around 0, then $\mathcal{D}_X/(t^2\partial_t + 1)\mathcal{D}_X$ is *not* regular. Indeed, $\partial_t f/f = -\frac{1}{t^2}$ results in the solution $f = e^{\frac{1}{t}}$ which has exponential growth at 0.

Notation 1.15. We denote by $\mathrm{Mod}_{rh}(\mathcal{D}_X) \subset \mathrm{Mod}_c(\mathcal{D}_X)$ the category of regular holonomic $\mathcal{D}$-modules as a subcategory of coherent $\mathcal{D}_X$-modules and by $D^b_{rh}(\mathcal{D}_X)$ its corresponding bounded derived categories.

An algebraic theory of regular-holonomic $\mathcal{D}$-modules were developed by Beilinson and Bernstein. In their language regular holonomic $\mathcal{D}$-modules are "made-up" of middle extension of local systems under pushforward from a locally closed subvariety Y of X, so that the embedding morphism $Y \hookrightarrow X$ is affine. Indeed, a-posteriori Riemann-Hilbert correspondence these correspond to intermediate extensions of local systems. See [HTT95, §3] for a detailed enough exposition.

1.2. **De Rham complexes and RH-correspondence for $Mod_{rh}(\mathcal{D}_X)$ and $D^b_{rh}(\mathcal{D}_X)$.** Consider the following resolution of ω_X by locally free $\mathcal{D}_X$-modules

$$C^\bullet : [0 \to \mathcal{D}_X \to \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{D}_X \to \cdots \omega_X \otimes_{\mathcal{O}_X} \mathcal{D}_X \to \omega_X \to 0]$$

with maps given by $d(\omega \otimes P) \overset{\text{loc}}{=} dw \otimes P + \sum_i dt_i \wedge \omega \otimes \partial_i P$

Claim 1.16. *The above resolution is filtered exact, i.e.*

$$F_k(C^\bullet) : [0 \to F_k\mathcal{D}_X \to \Omega^1_X \otimes_{\mathcal{O}_X} F_{k+1}\mathcal{D}_X \to \cdots \omega_X \otimes_{\mathcal{O}_X} F_{k+n}\mathcal{D}_X \to F_{k+n}\omega_X \to 0]$$

where $F_k\omega_X$ is as it was defined in 1.7.

Indeed if $P \in F_k\mathcal{D}_X$, $\partial_i P \in F_{k+1}\mathcal{D}_X$ and therefore $F_{-n}C^\bullet = [\omega_X \overset{\simeq}{\to} \omega_X]$, then the claim follows from the fact that $gr^F_k \mathcal{D}_X \simeq S^k\mathcal{T}_X$ and the Koszul resolution

$$L^\bullet \colon [0 \to \mathcal{O}_{T^*X} \otimes \wedge^n \pi^*\Theta_X \to \cdots \to \mathcal{O}_{T^*X} \otimes \wedge^0 \pi^*\Theta_X \to \mathcal{O}_{T^*_X X}]$$

where $X \simeq T^*_X X$ is the zero section of $\Gamma(T^*X, \pi^*\Omega^1_X)$. Since $\pi : T^*X \to X$ is affine, we obtain an acyclic complex which we tensor by ω_X,

$$[0 \to gr^F\mathcal{D}_X \to \Omega^1_X \otimes gr^F\mathcal{D}_X \cdots \to \omega_X \otimes gr^F\mathcal{D}_X \to \omega_X]$$

Therefore, by induction, $F_kC^\bullet$ are exact for all k. This proves the claim.

Definition 1.17 (deRham functor)**.** The de Rham functor

$$\mathrm{DR} : D^b(\mathcal{D}_X) \to D^b(\mathbb{C}_X)$$

is defined by sending $M^\bullet \mapsto \omega_X \otimes^L_{\mathcal{D}_X} M^\bullet$. Using the left resolution of ω_X, one can then write

$$\mathrm{DR}(M) = [0 \to M \to \Omega^1_X \otimes_{\mathcal{O}_X} M \to \cdots \omega_X \otimes_{\mathcal{O}_X} M \to 0]$$

considered in degree $-n$ to 0 with the morphisms given by

$$d(\omega \otimes m) = d\omega \otimes m + \sum_i dt_i \otimes \partial_i m.$$

In terms of filtration

$$F_k \,\mathrm{DR}(M) = [0 \to F_kM \to \Omega^1_X \otimes_{\mathcal{O}_X} F_{k+1}M \to \cdots \omega_X \otimes_{\mathcal{O}_X} F_{k+n}M \to 0]$$

Theorem 1.18 (The Riemann-Hilbert Correspondence <Kashiwara (an), Mebkhout (an), Beilinson–Bernstein (alg), Brylinski (an $\Rightarrow$ alg)>)**.** *Let X be a smooth algebraic variety. Then there is an equivalence of categories*

$$rat\colon D^b_{rh}(\mathcal{D}_X) \to D^b_c(\mathbb{C}_X)$$

given by the de Rham functor. Furthermore, if $f\colon X \to Y$ is a morphism of smooth varieties, the de Rham functor is compatible with the functors $f_*, f^*, f_!, f^!, \mathcal{H}om$, $\otimes$ and hence in particular with ψ_f and φ_f. Moreover, rat is an equivalence on $Mod_{rh}(\mathcal{D}_X)$ identified as a subcategory of $D^b_{rh}(\mathcal{D}_X)$ and maps to the category of perverse sheaves, $Perv(X)$ under this functor.

Example 1.19.
(1) On a smooth variety X, $\mathcal{O}_X$ corresponds to $\mathbb{C}_X[n]$, whereas $\mathcal{O}_X(*D)$ corresponds to $j_*\mathbb{C}_{X\setminus D}$.
(2) Any local system $\mathbb{L}[n]$ corresponds to $\mathbb{L}\otimes_{\mathbb{C}}\mathcal{O}_X$. When D is simple normal crossing $j_*\mathbb{L}_U[n]$ can be described using Deligne extensions and for more general D, in terms of **the** V-filtrations. $j_{!*}$ is very hard to describe in general and corresponds to simple regular holonomic $\mathcal{D}$-modules under the RH-correspondence.
(3) Let D be a reduced effective divisor on X and $\mu : X' \to X$ be a log resolution of $D \subset X$ and $E = \mu^{-1}(D)_{\rm red}$ is simple normal crossing, so that $\mu : X'\setminus E \to X\setminus D$ is an isomorphism, then by proper base change of Abelian sheaves, $\mu_* j'_*\mathbb{C}_U[n] \simeq j_*\mathbb{C}_U[n]$ and hence $\mu_+\mathcal{O}_{X'}(*E) \simeq \mathcal{O}_X(*D)$. I will describe the pushforward functor μ_+ in the next section.

1.3. **Direct images.** Let $f : X \to Y$ be a morphism of smooth varieties. We will not talk about the inverse image functor in great detail. However, the following object is of central importance:

$$f^*\mathcal{D}_X \simeq \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_X} f^{-1}\mathcal{D}_X.$$

This object has a right-$f^{-1}\mathcal{D}_X$ structure via the right multiplicand. A Left $\mathcal{D}_X$ structure on the transfer module is given by the tensor product operation, namely

$$\theta\cdot(\varphi\otimes P) \overset{\rm loc}{=} \sum_i \theta(y_i\circ\varphi)\otimes\partial_i P + \theta(\varphi)\otimes P$$

where (y_i, ∂_i) are local coordinates and derivations of Y. Let $\iota\colon X \hookrightarrow Y$ be a closed immersion, then choosing local coordinates $(x_1,\cdots,x_r,y_{r+1},\cdots,y_n)$ of Y and partials $(\partial_1,\cdots,\partial_n)$, we obtain locally

$$\iota^*(\mathcal{D}_Y) \simeq \mathbb{C}[x_1,\cdots,x_r]\otimes_{\mathbb{C}[x_1,\cdots,y_n]}\mathcal{D}_Y \simeq \mathcal{D}_X[\partial_{r+1},\cdots,\partial_n].$$

Using this the naïve pushforward of a left $\mathcal{D}$-module M is defined to be

$$f_*M \simeq f_*(\omega_X\otimes_{\mathcal{D}_X} M\otimes_{\mathcal{D}_X} f^*\mathcal{D}_Y)\otimes_{\mathcal{O}_Y}\omega_Y^{-1}$$

Note that if M already was a right $\mathcal{D}$-module, tensoring by the canonical sheaf would be unnecessary. In that case we describe the $\mathcal{D}_Y$ action on f_*M locally as follows, Let (y_i, ∂_{y_i}) denote the local coordinates on Y. Then, for $P\in\mathcal{D}_Y$, and $m\in M$

$$(m\otimes P)\cdot\partial_{y_i} = \sum_j \partial_{x_j}(y_i\circ f)\partial_{x_j} m\otimes P + m\otimes P\partial_{y_i}.$$

To simplify notation denote,

$$\mathcal{D}_{Y\leftarrow X} \coloneqq \omega_Y\otimes_{\mathcal{O}_X} f^*\mathcal{D}_Y\otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\omega_Y^{-1}$$

and call this the *transfer module*. The right exact nature of tensor product and the left exact nature of f_* prohibits this operation to behave nicely. As an example of this bad behaviour note that $(fg)_*$ is not necessarily f_*g_*. This is an added reason to work in the derived category and consider the derived push forward. Define $f_+ : D^b(\mathcal{D}_X) \to D^b(\mathcal{D}_Y)$ by

$$f_+M^\bullet \coloneqq Rf_*(\mathcal{D}_{Y\leftarrow X}\otimes^L_{\mathcal{D}_X} M^\bullet)$$

When f is proper, Saito [Sai88, §2.3] has defined a filtered version of direct image between the categories of filtered coherent $\mathcal{D}$-modules equipped with good filtration, i.e. $f_+ : D^b_c(F\mathcal{D}_X) \to$

$D^b_c(F\mathcal{D}_Y)$ For the ease of writing down the filtration on the pushforward we resort to right $\mathcal{D}$-modules for a moment. We additionally assume that $f\colon X \to Y$ is so that $f_+(M, F)$ is strict as an object in $D^b(F\mathcal{D}_Y)$, i.e. the differentials of the complex f_+M satisfies, $d(F_pM^i) \simeq d(M^i) \cap F_pM^{i+1}$ This means that

$$\mathcal{H}^i(F_kf_+(M;F)) \to \mathcal{H}^if_+(M;F)$$

is injective for all integers i, p . Then the cohomologies of f_+ are also filtered $\mathcal{D}$-modules. Saito's definition of the filtration on the direct image gives us a way to compute the Hodge filtration on the pushforward

$$F_p\mathcal{H}^if_+M \simeq R^if_*(F_k(M \otimes^L_{\mathcal{D}_X} f^*\mathcal{D}_Y).$$

Caveat: This phenomenon is not true with any random choice of good filtration on f_+M. For instance, for a birational morphism $f\colon X' \hookrightarrow X$ that are isomorphism outside a singular divisor $D \subseteq X$ and call $E = f^{-1}(D)$. Then if we endow $\mathcal{O}_{X'}(*E)$ with the Hodge filtration, one can deduce after the workshop that even though $f_+\mathcal{O}_{X'}(*E) \simeq \mathcal{O}_X(*D)$, endowing $\mathcal{O}_X(*D)$ with the good filtration $F_i\mathcal{O}_X(*D) := F_i\mathcal{D}_X \cdot \mathcal{O}_X(D)$ is not the filtration we obtain from via the pushforward.

If $(M;F)$ underlies a (mixed) Hodge module and f is proper, it is known that $\mathcal{H}^if_+$ are (mixed) Hodge modules and the induced filtration serves as the Hodge filtration. This is more involved and constitutes a large part of [Sai88].

We now apply this knowledge of direct image functor to create more $\mathcal{D}$-modules out of the examples we already discussed.

Example 1.20. (1) Let $\iota : D \overset{\text{loc}}{=} (t = 0) \to X$ be a smooth hypersurface in X. Let M be a right $\mathcal{D}$-module. Since

$$F_k\iota_+M \simeq R^i\iota_*(F_k(M \otimes^L_{\mathcal{D}_X} \iota^*\mathcal{D}_X)) \simeq \iota_*F_k(M \otimes_{\mathcal{D}_D} \mathcal{D}_D[\partial_t]) \simeq \bigoplus_{i+j=k} \iota_*F_iM \otimes \partial_t^j.$$

And therefore $\mathcal{H}^i\iota_+M = 0$ for all $i \neq 0$.

(2) Let $j\colon U \hookrightarrow X$ be an open immersion so that $X \setminus U = D$, a divisor. Then $\mathcal{H}^ij_+\mathcal{O}_U = 0$ for all $i \neq 0$, moreover, $j_+\mathcal{O}_U \simeq \mathcal{O}_X(*D)$. It is a good exercise to check that $j_*\mathbb{C}_U$ is a perverse sheaf and therefore it corresponds to a regular holonomic $\mathcal{D}$-module.

(3) Let $M = \omega_X$ with trivial filtration. Using the left resolution of ω_X we determine that

$$F_k\mathcal{H}^if_+\omega_X \simeq R^if_*(F_k\,\mathrm{DR}(\omega_X))$$

Therefore, for $k = -n$, we obtain

$$F_{-n}\mathcal{H}^if_+\omega_X \simeq R^if_*\omega_X$$

2. Kashiwara-Malgrange V-filtrations

Having seen the statement of Riemann-Hilbert correspondence in the previous talk, I would like to show the $\mathcal{D}$-module side of the story of nearby and vanishing cycles using V-filtration. This was the primary motivation behind the definition. The definition is pretty technical, but once we walk past the definitions and some crucial properties I would try to motivate its important by describing some of its consequence in birational geometry. Besides, this will also come up in Mustaţă's talk where he will talk about V-filtration interpretation of Hodge ideals. V-filtration is defined with respect to some closed subvariety $Z \subset X$ of a smooth variety X. When we assume that Z is a divisor, we will denote it by D.

2.1. Z is smooth.

Definition 2.1. The V-filtration of $\mathcal{D}_X$ along $Z \overset{\text{loc}}{=} (t_{r+1} = \cdots = t_n = 0)$ is a decreasing filtration indexed by rational numbers α and is locally defined by

$$(1) \qquad V^\alpha \mathcal{D}_X = \bigoplus_{|I|-|J| \geq \lfloor \alpha \rfloor} \mathcal{D}_Z t^I \partial^J \overset{\text{globally}}{=} \left\{ P \in \mathcal{D}_X | P \cdot I_Z^{\lfloor \alpha \rfloor} \subseteq I_Z^{\lfloor \alpha \rfloor + 1} \right\}.$$

In particular, $V_Z^0 \mathcal{D}_X \simeq \mathbb{C}[t_1, \cdots, t_n, \partial_1, \cdots, \partial_r, t_{r+1}\partial_{r+1}, \cdots, t_n \partial_n]/ \sim$.

Note that V is essentially indexed by the integers. Defining it for rational numbers will be useful for the V-filtration of a $\mathcal{D}$-module may or may not be integrally indexed. Note also that $V_D^0 \mathcal{D}_X / V_D^1 \mathcal{D}_X \simeq \mathcal{D}_D[t\partial_t]$. This has a $\mathcal{D}_Z$-module structure.

Definition 2.2. The filtration V along D, on a coherent $\mathcal{D}$-module $M \in Mod_c(\mathcal{D}_X)$ is an exhaustive decreasing $\mathbb{Q}$-indexed filtration of coherent $V^0\mathcal{D}_X$-submodules satisfying

- $\{V^\alpha\}_\alpha$ is indexed discretely and $V^{\alpha-\epsilon} \simeq V^\alpha$ for $0 < \epsilon \ll 1$.
- $V^i \mathcal{D}_X \cdot V^\alpha M \subseteq V^{\alpha+i} M$. In particular, $t \cdot V^\alpha M \subset V^{\alpha+1} M$ with equality when $\alpha > 0$. and $\partial_t \cdot V^\alpha M \subseteq V^{\alpha-1} M$.
- Defining $gr_V^\alpha M := \dfrac{V^\alpha M}{V^{>\alpha} M}$ with $V^{>\alpha} M := \cup_{\beta > \alpha} V^\beta M$, we have $\partial_t t - \alpha : gr_V^\alpha M \to gr_V^\alpha M$ is nilpotent.

Remark 2.3.
(1) The last condition is what makes V-filtration unique [Kas83, Theorem 1] if it exists.
(2) The second condition ensures that $V^i \mathcal{D}_X \cdot V^\alpha M = V^{\alpha+i} M$ if $i, \alpha \geq 0$ and $i, \alpha \leq 0$. Indeed, when $i \geq 0$ $t \cdot V^\alpha M = V^{\alpha+1} M$. For the other side, pick $0 < \beta \ll 1$, then $V^i \mathcal{D}_X \cdot V^\alpha M \subseteq V^{\alpha+i} M$ for $i < 0$. If equality does not hold, this results in an increasing chain of $\mathcal{D}_D[t, t\partial_t]$-modules (Noetherian) which must stabilise and therefore $V^\beta M$-generates M over $\mathcal{D}_X$. Define $V'^\alpha = V^\alpha$ if $\alpha \geq \beta$ and $V^i \mathcal{D}_X \cdot V^{\alpha+i} M$ for i such that $\beta + 1 > \alpha + i \geq \beta$ otherwise. Then by the uniqueness of V-filtration we obtain the other equality.
(3) When $\alpha \neq 1$, $\partial_t : gr_V^\alpha \overset{\simeq}{\to} gr_V^{\alpha-1}$ and when $\alpha \neq 0$, $\partial_t : gr_V^\alpha \overset{\simeq}{\to} gr_V^{\alpha-1}$ are isomorphisms.

Example 2.4 (Non-characteristic embeddings)**.** . Let D be a smooth hypersurface in X so that D intersects the support of the characteristic variety of M transversely. Then:
(1) $(M; F)$ is regular and quasi-unipotent along D.
(2) The V -filtration on M is given by
$$V^j M = O_X(-jH) \cdot M$$
for $j > 0$ and M otherwise.

Corollary 2.5. *Let $u \colon M \to N$ be a morphism of regular holonomic $\mathcal{D}$-modules quasi-unipotent along D. Then u is strict with respect to the respective V-filtrations i.e. $u(V_D^\alpha M) = u(M) \cap V_D^\alpha N$.*

Corollary 2.6. *If M is a coherent $\mathcal{D}_X$-module supported on D, $M \simeq \iota_+ gr_V^0 M \simeq \iota_+ \operatorname{Ker}(t : M \to M)$.*

Proof. As $\mathcal{O}_X$-module, $M \simeq i_* M_0$, therefore, $M \simeq i_* M_0 \otimes_{\mathbb{C}} \mathbb{C}[\partial_t]$. One can check by checking the properties of the V-filtration that $V^{-j} M \simeq \bigoplus_{i=1}^j M_0 \otimes \partial_t^j$. In particular $V^j M = 0$ if $j > 0$.

From the action of t on the pushforward described below in Definition 2.9 we have, $t(m_0 \otimes 1) = 0$, and hence $M_0 \simeq \operatorname{Ker}(t : M \to M) \simeq gr_V^0 M$. □

Corollary 2.7. *Let $u : M \to N$ be a morphism of $\mathcal{D}_X$-modules such that $u|_U$ is an isomorphism, where $U = X \setminus D$. Then $u : V^\alpha M \overset{\simeq}{\to} V^\alpha N$ for all $\alpha > 0$. In particular $V^{>0} M$ depends only on the restriction of M to U.*

Proof. Since $\operatorname{Ker} u$ and $\operatorname{Coker}(u)$, $\mathcal{D}_X$-modules, are supported along D, we know how its V-filtration looks like from the previous lemma. Hence the corollary. □

2.2. **Nearby and Vanishing cycles.** Since $\partial_t t - \alpha$ acts nilpotently on $gr_V^\alpha M$, we have

$$\partial_t : gr_V^\alpha M \xrightarrow{\simeq}: gr_V^\alpha M$$

when $\alpha \neq 1$ and

$$t : gr_V^\alpha M \xrightarrow{\simeq}: gr_V^\alpha M$$

when $\alpha \neq 0$

Theorem 2.8 (Kashiwara-Malgrange, Saito)**.** *If M is regular holonomic and* $\mathrm{DR}(M)$ *has quasi-unipotent monodromy along D, then $V_D^\alpha M$ exists. Further we have the Jordan decomposition $T = T_sT_u$ of the monodromy T of $\psi_f K$ and $\varphi_f K$. Denote*

$$\psi_\lambda K := \operatorname{Ker}(T_s - \lambda : \psi_f K \to \psi_f K)$$

We have isomorphisms

$$\mathrm{DR}_D(gr_V^\alpha M) = \begin{cases} \varphi_\lambda K & \text{for all } 0 \leq \alpha < 1 \\ \psi_\lambda K & \text{for all } 0 < \alpha \leq 1 \end{cases}$$

where $\lambda = e^{2\pi i\alpha}$ and such that $\partial_t t - \alpha$ identified with $\log T$*.*

Furthermore, the morphism between the unipotent nearby and the unipotent vanishing cycle functors translates as

$$t = var : gr_V^0 M \to gr_V^1 M$$

and

$$\partial_t = can : gr_V^1 M \to gr_V^0 M$$

If M underlies a Hodge module so do the unipotent nearby and vanishing cycles.

2.3. D **is not smooth.** This can of course be done for Z, however our interest is only when Z is a hypersurface.

Definition 2.9. Let $D = (f = 0)$ be a hypersurface on X (not necessarily smooth), let $i_f : X \hookrightarrow X \times \mathbb{C}$ be the graph embedding. Then, for any $\mathcal{D}$-module M the V-filtration on $M_f := i_{f_+} M$ along $t = 0$, determines the V-filtration along M. In other words,

$$V^\alpha M := V_t^\alpha M_f \cap (M \otimes 1)$$

Here $(i_{f_+} M, F) \simeq \bigotimes_{i=0}^k F_i M \otimes \partial_t^{k-1}$ and the actions of $\mathcal{D}_{X\times\mathbb{C}}$ can remembers the graph embedding as follows:

$$\partial_{x_i}(m \otimes \partial_t^i) = \partial_{x_i} m \otimes \partial_t^i - (\partial_{x_i} f) m \otimes \partial_t^{i+1},$$

$$t \cdot (m \otimes \partial_t^i) = fm \otimes \partial_t^i - im \otimes \partial_t^{i-1},$$

$$\partial_t \cdot (m \otimes \partial_t^i) = m \otimes \partial_t^{i+1}.$$

Note that $D = i_f^{-1}(X \times 0) = (t \circ i_f)^{-1}(0)$ and $M \simeq i_f^* M_f$ via the $\mathcal{D}$-module pull back. Everything we discussed above also work for these V-filtration. It is a few more steps to check that the definition does not depend on f.

The work of Budur–Mustaţă–Saito relates these to the so called multiplier ideals, another invariant of singularities.

2.3.1. *Multiplier Ideals.* Everything in this section can be done in general for subvarieties of any dimension. Let D be an effective hypersurface of a smooth variety X. Let $\mu : Y \to X$ be a log resolution of D. Write $K_Y = \mu^* K_X + E_1$ and $\mu^* D = \tilde{D} + E_2$. Then

$$K_Y - \mu^* K_X - \mu^*(cD) = E_1 - c\tilde{D} - cE_2 =: F.$$

Then

$$J(cD) \coloneqq \mu_* \mathcal{O}_Y(K_{Y/X} - \lfloor \mu^*(cD)) \rfloor) = \mu_* \mathcal{O}_Y(\lceil F \rceil).$$

Example: Cusp and its lct.

Theorem 2.10 ([BMS06])**.** *For $\alpha > 0$, $V^\alpha \mathcal{O}_X = J((\alpha - \epsilon)D)$, where $0 < \epsilon \ll 1$ and the filtration $V^\bullet$ of $\mathcal{O}_X$ is taken along D.*

Remark 2.11. As a consequence of this or directly from the definition multiplier ideals change discretely i.e. there exists $\eta_1 < \eta_2 < \cdots \eta_k < 1$ such that for all $\eta_i \leq c < \eta_{i+1}$,

$$J(\eta_{i+1} D) \subsetneq J(cD) = J(\eta_i D).$$

Furthermore,

$$J((c+1)D) = J(cD)(-D).$$

Theorem 2.12 ([ELSV05])**.** *Let $D = (f = 0)$ be hypersurface of a smooth variety $\mathbb{C}^n$. Let $b_f(s)$ denote the minimal polynomial of the action of $\partial_t t$ on $\dfrac{V^0 \mathcal{D}_{X \times \mathbb{C}} \cdot (t - f)}{V^1 \mathcal{D}_X \cdot (t - f)}$ considering f as function on $\mathcal{O}_{X \times 0}$ (Existence of $b_f(s)$ follows from the work of Kashiwara). Let η be a jumping coefficient of f on $\mathbb{C}^n$ which is lying in the interval $(0, 1]$, then η is a root of $b_f(s)$.*

This polynomial can be defined more generally and are known as the Bernstein-Sato polynomials or simply b-functions.

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