# The fundamental group of the Hawaiian earring is not free

(International Journal of Algebra and Computation Vol. 2, No. 1 (1992), 33–37)

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**Abstract.** The Hawaiian earring is a topological space which is a countably infinite union of circles, that are all tangent to a single line at the same point, and whose radii tend to zero. In this note a short proof is given of a result of J.W. Morgan and I. Morrison that describes the fundamental group of this space. It is also shown that this fundamental group is not a free group, unlike the fundamental group of a wedge of an arbitrary number of circles.

**Key words:** Fundamental groups, Van Kampens theorem, free groups, projective limits.

**1980 Mathematics subject classification (1985):** 20E05, 20F34, 57M05.

## 1. Introduction.

For each positive integer $n$, let $C_n \subset \mathbb{C}$ be the the circle $\{z \in \mathbb{C} : |z - \frac{1}{n}| = \frac{1}{n}\}$. The union $\bigcup_{n=1}^{\infty} C_n$ is a topological space $H \subset \mathbb{C}$ called the *Hawaiian earring*. On a homework assignment to an algebraic topology course, M.W. Hirsch asked whether the fundamental group $\pi_1(H)$ of $H$ with base-point 0 is a free group. In the following an algebraic description of $\pi_1(H)$ is given, and it is shown that it is torsion free, but not free.

In a wedge $X$ of an arbitrary number of circles each loop can only go around finitely many circles, because a loop has a compact image, and consequently the fundamental group of $X$ is free on the obvious generators. In the case of the Hawaiian earring however, we have a coarser topology admitting more loops. For instance, let $f : [0,1] \to H$ be a loop that goes around $C_1$ on the middle third part of the unit interval, around $C_2$ on the middle third part of each of the remaining intervals, etcetera. This defines $f$ as a map $[0,1] \to H$, and $f$ is continuous because any open neighborhood of 0 in $H$ contains all $C_i$ with $i$ sufficiently large. Note that $f^{-1}(0)$ is the Cantor set in the unit interval.

The results in this note are not new, but the proofs are. The algebraic description of the group $\pi_1(H)$ was first given in [1], but more recently J.W. Morgan and I. Morrison found a flaw, and published an alternative proof (see [4]). In the purely group-theoretical paper [2, section 6], G. Higman mentions the same group, and shows that it is not free. A different argument, suggested by H.W. Lenstra, Jr., will be given for this in section 3.

For more background on free groups, free products and reduced words, we refer to [3].

## 2. An algebraic description.

The space $H_n = \bigcup_{i=1}^n C_n$ is a wedge of $n$ circles, and it follows from Van Kampen's theorem that the fundamental group $\pi_1(H_n)$ is a free group on $n$ generators $l_1, \ldots, l_n$, where $l_i$ is the homotopy class of a loop that wraps around $C_i$ once (so $l_i$ is a generator of $\pi_1(C_i)$).

For each $n \geq 0$ we have a retraction $H \to H_n$ collapsing the loops $C_i$ with $i > n$ to 0. Therefore we may view $F_n$ as a subgroup of $\pi_1(H)$, and the retraction induces a map from $\pi_1(H)$ to $F_n$ The restricted retractions $H_n \to H_{n-1}$ induce group homomorphisms $F_n \to F_{n-1}$ fixing $l_i$ for $i < n$, and mapping $l_n$ to 1. The groups $F_n$ form a projective system of groups, and we get a canonical group homomorphism

$$\varphi : \pi_1(H) \;\to\; F = \varprojlim_{n \geq 1} F_n.$$

Recall that the projective limit $F$ is the subgroup of the product $\prod_{i=1}^\infty F_i$ consisting of those elements $(f_i)_i$ for which the map $F_i \to F_{i-1}$ sends $f_i$ to $f_{i-1}$ for all $i > 1$. We will write down elements of $F$ by specifying each coordinate.

It will turn out that $\varphi$ maps $\pi_1(H)$ isomorphically to a subgroup $\pi$ of $F$ which can be roughly described as the subgroup consisting of the elements $(f_i)_i \in F$ for which the following condition holds for each $j \geq 1$: *the number of times that $l_j$ occurs in the reduced word representation of $f_i$ is a bounded function of $i$.*

For a more precise formulation, define the *$j$-weight* $w_j(x)$ of an element $x \in F_i$ as follows: first write $x$ as a reduced word $x = g(1)^{a_1} g(2)^{a_2} \cdots g(s)^{a_s}$, where each $a_k$ is a non-zero integer and $g$ a map $\{1, 2, \ldots, s\} \to \{l_1, l_2, \ldots, l_i\}$ with $g(k) \neq g(k+1)$ for any $k$. Now put

$$w_j(x) = \sum_{g(k)=l_j} |a_k|.$$

Note that $w_j(x)$ is well defined as the representation of $x$ as a reduced word is unique. Now let $\pi$ be the subgroup of $F$ consisting of all elements $(x_i)_i \in F$ such that for every $j \in \mathbb{Z}_{\geq 1}$ the function $\mathbb{Z}_{\geq 1} \to \mathbb{Z}$ defined by $i \mapsto w_j(x_i)$ is bounded.

For $j > 0$ define $F_{\geq j} = \{(x_i)_i \in F : \; w_k(x_i) = 0 \text{ for all } i \text{ and all } k < j\}$, and put $\pi_{\geq j} = \pi \cap F_{\geq j}$. The next lemma will turn out to be an instance of the Van Kampen theorem, once we know that $\pi$ is $\pi_1(H)$, but at this point we have to prove it by group-theoretic means.

**Lemma.** *The group $\pi$ is the free sum $F_{j-1} * \pi_{\geq j}$ of its subgroups $F_{j-1}$ and $\pi_{\geq j}$.*

**Proof.** The subgroup of $\pi$ generated by $F_{j-1}$ and $\pi_{\geq j}$ is their free sum because of the unique reduced word representation in the free groups $F_i$. If $x = (x_i)_i \in F$ then the condition $x \in \pi$ implies that the number of occurrences of elements of $F_{j-1}$ in the reduced representation of $x_i$ is a bounded function of $i$. Again by uniqueness of the reduced word representation, it follows that $x$ is a finite product of elements of $F_{j-1}$ and of $\pi_{\geq j}$. □

**Proposition.** *The image of the homomorphism* $\varphi : \pi_1(H) \to F$ *is* $\pi$.

**Proof.** Let $f : [0,1] \to H$ be a loop at 0. By Van Kampens theorem, $\pi_1(H) = F_1 * \pi_1(C_{\geq 2})$, where $C_{\geq 2} = \bigcup_{n\geq 2} C_n$. So after a homotopy, we may assume that $f$ is a composition of loops $f_0 * e_1 * f_1 * \cdots * e_s * f_s$ where the $e_i$ are loops in $C_1$ and the $f_i$ are loops in $C_{\geq 2} = \bigcup_{n\geq 2} C_n$. We have $w_1(\varphi(f_i)) = 0$ and for $n > 1$ this implies $w_1(\varphi(f)_n) \leq \sum_{i=1}^{s} w_1(\varphi(e_i))$, which is a bounded function of $n$. Repeating the same argument shows that for every $i$ the function $w_2(\varphi(f_i)_n)$ of $n$ is bounded, so that $w_2(\varphi(f)_n)$ is also bounded. Using induction we obtain that $\varphi(f) \in \pi$, so the image of $\varphi$ lies in $\pi$.

Now let $x = (x_i)_i \in \pi$. We will construct a loop $f : [0,1] \to H$ such that the homotopy class of $f$ is mapped to $x$ by $\varphi$. By the lemma we can write $x$ as $y_0 e_1 y_1 e_2 y_2 \cdots e_s y_s$ with $e_i \in \{l_1, l_1^{-1}\}$ and $y_i \in \pi_{\geq 1}$. Accordingly, we divide the unit interval in $2s+1$ intervals $I_i = [\frac{i}{2s+1}, \frac{i+1}{2s+1}]$ where $0 \leq i \leq 2s$, and define a map $f_1 : [0,1] \to H$ to be zero on the intervals $I_{2i}$, and to go around $C_1$ in the appropriate direction on the interval $I_{2i-1}$, so that the homotopy class of the loop $f_1 | I_{2i-1}$ is $e_i$ for $i = 1, 2, \ldots, s$.

Next we break up each $y_i \in \pi_{\geq 2}$ into elements $l_2^{\pm 1}$ and elements of $\pi_{\geq 3}$, and divide the interval $I_{2i}$ accordingly. Then we define a map $f_2$, that only differs from $f_1$ on subintervals of $I_{2i}$, where it gives the $l_2$-pattern in $y_i$. This way we get a uniformly convergent sequence $f_1, f_2, \ldots$ of loops in $H$, so they converge to a *continuous* loop $f$ in $H$. As the image of $\varphi(f)$ in $F_n$ is $x_n$, it follows that $\varphi(f) = x$. □

**Proposition.** *The canonical map* $\varphi : \pi_1(H) \to F$ *is injective.*

**Proof.** Let $f : [0,1] \to H$ be a loop, whose image in $F$ is 1. We want to construct null-homotopy for $f$, i.e., a homotopy of $f$ with the constant loop. As in the previous proof, we first homotop $f$ to a composition of loops $f_0 * e_1 * f_1 * \cdots * e_s * f_s$ where the $e_i$ are loops in $C_1$ and the $f_i$ are loops in $C_{\geq 2}$. Then $\varphi(f_i) \in \pi_{\geq 2}$ and $\varphi(e_i) \in F_1$, and since $\varphi(f) = 1$, the word $\varphi(f_0)\varphi(e_1)\varphi(f_1) \cdots \varphi(e_s)\varphi(f_s)$ in the free sum $F_1 * \pi_{\geq 2}$ can be reduced to 1. We can now make a null-homotopy for $f$ by following the cancellation steps to reduce this word to 1, provided that we can find a homotopy to the constant loop for loops in $C_{\geq 2}$ that are in the kernel of $\varphi$. In other words we can define the homotopy on part of $[0,1]\times[0,1]$, and on the remaining parts we still need to fill in a null-homotopy for certain loops in $C_{\geq 2}$ that we know to be null-homotopic. But to construct such a homotopy we follow the same procedure to obtain a word in $\pi_1(C_2) * \pi_{\geq 3}$ that can be reduced to 1, etcetera. Taking the union of the sequence of "partial homotopies" obtained by this inductive procedure, we obtain the required null-homotopy for $f$. □

Since $F$ is torsion free, this shows that $\pi_1(H)$ is torsion free.

## 3. Proof that the fundamental group is not free.

In this section, a group-theoretical proof is given that $\pi$ is not free. First note that $\pi$ is uncountable. An uncountable free group $F$ is free on uncountably many generators, and therefore $\mathrm{Hom}(F,\mathbb{Z})$ is uncountable. So it suffices to show that $\mathrm{Hom}(\pi,\mathbb{Z})$ is countable. We start with a lemma.

**Lemma.** *For each positive integer $j$ let $x^{(j)} = (x_i^{(j)})_i$ be an element of $F$ such that $x_i^{(j)} = 1$ for all $i < j$. Then there is a homomorphism $f : F \to F$ sending $l_j$ to $x^{(j)}$ for all $j$. If $x^{(j)} \in \pi_{\geq j}$ for all $j$, then $f(\pi) \subset \pi$.*

**Proof.** As $F_n$ is a free group on $l_1,\dots,l_n$, there are unique group homomorphisms $f_n : F_n \to F_n$ sending $l_j$ to $x_n^{(j)}$ for $j \leq n$. As we have $x_i^{(j)} = 1$ for $i < j$, we have a commutative diagram:

$$\begin{array}{ccccccc} F_1 & \twoheadleftarrow & F_2 & \twoheadleftarrow & F_3 & \twoheadleftarrow & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \\ F_1 & \twoheadleftarrow & F_2 & \twoheadleftarrow & F_3 & \twoheadleftarrow & \cdots \end{array}$$

This induces a homomorphism $f : F \to F$ of the projective limits, that satisfies the conditions.

Now suppose $y = (y_i)_i \in F$ and fix $k \geq 1$ then

$$w_k(f(y)_n) = w_k(f_n(y_n)) \leq \sum_{j=1}^{n} w_j(y_n) w_k(f_n(l_j)) = \sum_{j=1}^{n} w_j(y_n) w_k(x_n^{(j)}).$$

If $x^{(j)} \in \pi_{\geq j}$ for each $j$, then the terms with $j > k$ on the right hand side vanish. If in addition $y \in \pi$ then the remaining $k$ terms are all bounded functions of $n$, so that $w_k(f(y)_n)$ is a bounded function of $n$ and $f(y) \in \pi$. This shows the last statement. □

**Proposition.** *Let $f$ be a homomorphism from $\pi$ to $\mathbb{Z}$. Then $f(l_i) = 0$ for all sufficiently large $i$.*

**Proof.** Suppose that $f(l_{i_1}), f(l_{i_2}), \dots$ are non-zero for some $i_1 < i_2 < \cdots$. By the lemma there is a homomorphism $g : \pi \to \pi$ mapping $l_j$ to $l_{i_j}^{\pm 3}$ where the sign in the exponent is chosen to be the sign of $f(l_{i_j})$. By replacing $f$ by $f \circ g$ we may now assume that $f(l_i) \geq 3$. Put $a_i = f(l_i)$.

For every $j \geq 1$ define the element $x^{(j)} = (x_i^{(j)})_i \in \pi$ by $x_i^{(j)}$=1 if $i < j$, and for $i \geq j$:

$$x_i^{(j)} = l_j(l_{j+1}(\cdots(l_{i-1}l_i^{a_{i-1}})^{a_{i-2}}\cdots)^{a_{j+1}})^{a_j}.$$

The integer $f(x^{(1)})$ now satisfies congruence conditions that are too strong to hold for any integer. First note that $x^{(j)} = l_j(x^{(j+1)})^{a_j}$, so $f(x^{(j)}) = a_j + a_j f(x^{(j+1)})$. If follows that

$$f(x^{(1)}) = a_1 + a_1a_2 + \cdots + a_1a_2\cdots a_n + a_1a_2\cdots a_n f(x^{(n)}).$$

Put $b_n = a_1 + a_1a_2 + \cdots + a_1a_2\cdots a_{n-1}$ and $c_n = a_1a_2\cdots a_n$, then $f(x^{(1)}) \equiv b_n \bmod c_n$. Now $b_n < c_n$ and $b_n$ tends to infinity. If $f(x^{(1)}) \geq 0$ then $f(x^{(1)}) \geq b_n$ for all $n$, which is a contradiction. If $f(x^{(1)}) < 0$, then $f(x^{(1)}) \leq b_n - c_n$, and we get a contradiction too, because it follows from the fact that $a_i \geq 3$ that $c_n - b_n$ also tends to infinity. □

**Proposition.** *Let $g$ be a homomorphism from $\pi$ to $\mathbb{Z}$ such that $g(l_i) = 0$ for all $i$. Then $g = 0$.*

**Proof.** Suppose that $g(x) \neq 0$ for some $x \in \pi$, then we can write $x = f_1y_1f_2y_2\cdots f_sy_s$ with $f_i \in F_{j-1}$ and $y_i \in \pi_{\geq j}$. Now put $x_{\geq j} = y_1y_2\cdots y_s$, then $g(x_{\geq j}) = g(x) \neq 0$ as $g(f_i) = 0$ for $i = 1, \ldots, s$. By the lemma there is a homomorphism $\pi \to \pi$ mapping $l_j$ to $x_{\geq j}$. Composing with $g$, we get a homomorphism $\pi \to \mathbb{Z}$ mapping all $l_i$ to non-zero integers, contradicting the previous proposition. □

These two propositions imply that the homomorphism $\mathrm{Hom}(\pi, \mathbb{Z}) \to \prod_{i\geq 1}\mathbb{Z}$ that sends a homomorphism $g$ to the tuple $(g(l_1), g(l_2), \ldots)$, maps the group $\mathrm{Hom}(\pi, \mathbb{Z})$ injectively to $\bigoplus_{i\geq 1}\mathbb{Z}$, which is countable. This concludes the proof that $\pi$ is not free.

As subgroups of free groups are free, this implies that the projective limit $F$ of free groups can not be free either. This can also be shown by replacing $\pi$ with $F$ is the above proof, with a slight adaptation in the proof of the last proposition (cf. [2]).

It follows from the proof that all homomorphisms from $\pi$ to $\mathbb{Z}$ factor over $F_n$ for some $n$ (in other words, all algebraic homomorphism are continuous if we give $\mathbb{Z}$ and $F_n$ the discrete topology and $\pi$ the induced topology from $\prod_{n\geq 1} F_n$). Again, the same is true with $F$ instead of $\pi$.

The first homology group of the Hawaiian earring (the abelianized group $\pi^{\mathrm{ab}}$) is uncountable too, and as it has the same group of homomorphisms to $\mathbb{Z}$ as $\pi$, the proof also implies that $\pi^{\mathrm{ab}}$ is not free abelian.

**References.**


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