Structure of Complete Discrete Valuation Rings II, Unequal charasteristic case.

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Outline

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Notation

In this entire talk:

- A is a complete d.v.r. with char $A = 0$.
- K it's fraction field.
- m it's maximal ideal
- $\overline{K}$ it's residue field with $p = \text{char}\,\overline{K} \neq 0$
- $f : \overline{K} \to A$ the system of multiplicative representatives

Ramification

Since $\mathbb{Z} \in A$ and $p\mathbb{Z} \subseteq m$ we have $Z_p \subseteq A$ and $Q_p \subseteq K$.

Definition (absolute ramification index)

$e = v(p)$ is the absolute ramification index.
if $e = 1$ we say A is absolutely unramified.

Note if $Q_p \subseteq K$ is a finite extension, then this is just the ramification of $Q_p \subseteq K$

Outline

The Absolute Unramified Case

Unramified d.v.r. are completely determinded by their residue field

Theorem (3)

Let k be a perfect field with $\operatorname{char} k \neq 0$ then there is an absolute unramified d.v.r. $W(k)$ with residue field k. This $W(k)$ is unique up to a unique isomorphism.

The Ramified Case

Ramified d.v.r.'s are finite extensions of unramified ones

Theorem (4)

Let A be a $d.v.r.$ of char 0 then there is a unique f such that:

$$\begin{array}{ccc} W(\overline{K}) & \xrightarrow{f} & A \\ & \searrow \quad \swarrow & \\ & \overline{K} & \end{array}$$

commutes. This f is injective and makes A into a $W(\overline{K})$ module of degree e over $W(\overline{K})$.

Outline

The Definition of p-Rings

A generalisation of the $p - adic's$

Definition (Filtration)

Let R be a ring then a chain of ideals $a_0 \supseteq a_1 \supseteq \ldots$ is a filtration if $a_i a_j \subseteq a_{i+j}$

Definition (p-Ring)

A ring R with filtration is a p-Ring if:

- R is complete and hausdorf
- R/a_0 is a perfect ring

and a strict p-Ring if $a_n = p^n R$

Example of a p ring

Let X_a be a family of indeterminates.
Define $S = \mathbb{Z}\left[X_a^{p^{-\infty}}\right]$
Take $\widehat{S} = \widehat{\mathbb{Z}}\left[X_a^{p^{-\infty}}\right]$ the completion w.r.t. $p^i S$.
$\widehat{S}$ is a p-ring.
It's residue field is $\mathbb{F}_p\left[X_a^{p^{-\infty}}\right]$

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Discription of $\times, +, -$ in terms of representatives

Let $X_0, X_1, \ldots, Y_0, Y_1, \ldots$ be indeterminates, and $\widehat{S}$ as before.
Define $x := \sum X_i p^i$, $y := \sum Y_i p^i$
For $* \in \{\times, +, -\}$ there $Q_i^* \in \mathbb{F}_p\left[X_i^{p^{-\infty}}, Y_i^{p^{-\infty}}\right]$ such that
$x * y = \sum f(Q_i^*) p^i$.

Proposition (9)

Let A be a p-Ring $\alpha = \{\alpha_i\}$ *and* $\beta = \{\beta_i\}$ *be two sequinces of elements of* $\overline{K}$*. and define* $\gamma_i = Q_i^*(\alpha, \beta)$ *then:*
$\sum f(\alpha_i) p^i * \sum f(\beta_i) p^i = \sum f(\gamma_i) p^i$

Define $\theta : \widehat{\mathbb{Z}}\left[X_i^{p^{-\infty}}, Y_i^{p^{-\infty}}\right] \to A$ by $\theta(X_i) = f(\alpha_i)$, $\theta(Y_i) = f(\beta_i)$
then $\sum f(\alpha_i) p^i * \sum f(\beta_i) p^i = \theta(x * y) = \sum \theta(f(Q_i^*)) p^i =$
$\sum f(\theta(Q_i^*)) p^i = \sum f(\gamma_i) p^i$

The residue field gives all info

Proposition (10)

Let A, A' be p-rings and A strict. Then for all $\phi : \overline{K} \to \overline{K'}$ there is a unique g such that the following commutes

$$\begin{array}{ccc} A & \xrightarrow{\quad g \quad} & A' \\ \downarrow & & \downarrow \\ \overline{K} & \xrightarrow{\quad \phi \quad} & \overline{K'} \end{array}$$

Proof: take $a = \sum f(\alpha_i)p^i \in A$ and use
$g(a) = \sum g(f_A(\alpha_i))p^i = \sum f'_A(\phi(\alpha_i))p^i$

Corollary

Strict p-rings with isomorphic residue fields are isomorphic.

A lemma for constucting $W(k)$

Lemma (2)

Let $\phi : \overline{K} \to \overline{K'}$ be a surjective homomorphism of perfect rings. Then every strict p ring A over $\overline{K}$ gives rise to a strict p ring A' over $\overline{K'}$

Sketch of proof: Define A' as a quotient of A by defining $a := \sum f(\alpha_i)p^i \sim b := \sum f(\beta_i)p^i$ if and only if $\phi(a_i) = \phi(b_i)$. Proposition 9 shows that A' is ring. Now verify that A' is a p-ring

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A generalization of theorem 3

Theorem (5)

For every perfect ring k there exists a unique strict p-ring $W(k)$ with residue ring k.

Proof: unicity is the corrolarry of proposition 10. Existence: Every perfect ring of char p is of the form A/I with $A = \mathbb{F}_p\left[X_a^{p^{-\infty}}\right]$ A is the residue ring of $\widehat{S} = \widehat{\mathbb{Z}}\left[X_a^{p^{-\infty}}\right]$. Apply the lemma to get the theorem.

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The Theorem

Just to refresh your memory

Theorem (4)

Let A be a d.v.r. of char 0 then there is a unique f such that:

$$\begin{array}{ccc} W(\overline{K}) & \xrightarrow{f} & A \\ & \searrow \quad \swarrow & \\ & \overline{K} & \end{array}$$

commutes. This f is injective and makes A into a $W(\overline{K})$ module of degree e over $W(\overline{K})$.

The Proof

Existence and uniqueness are essentially proposition 10. To see that it's a module of degree $e = v(p)$ over $W(\overline{K})$ take $\pi \in A$ a uniformizer and any $a \in A$
Define a_{ij} recursively as follows.
Suppose $a \equiv \sum_{i+j \leq k} f(a_{ij})\pi^i p^j \mod \pi^k$ then take a_{mn} s.t. $m < e$ and $a \equiv \sum_{i+j \leq k+1} f(a_{ij})\pi^i p^j mod \pi^{k+1}$.
This shows $a = \sum_{i+j \leq k} f(a_{ij})\pi^i p^j \mod \pi^k$ hence $1, \pi, \pi^2, \ldots, \pi^{e-1}$ generate A as a $W(\overline{K})$ module.

Summary

- Absolute unramified complete d.v.r.'s are determined by their residue field.
- Ramified complete d.v.r.'s are finite extensions of the above.
- Residue field's say a lot about complete d.v.r.'s

- Next Time:
    - Witt Vectors: A discription of the Q_i^*
    - Begin of chapter III: Discriminant and Different