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Galois Representations

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CONTENTS

Introduction

Let $\mathcal{G}_{\mathbf{Q}}$ the absolute Galois group of $\mathbf{Q}$, $\mathbf{A}$ the $\mathbf{Q}$-adèles and n a positive integer. The global Langlands conjecture sets up a bijection between (isomorphism classes of) *certain* representations of $\mathrm{GL}_n(\mathbf{A})$ and (isomorphism classes of) *certain* $\overline{\mathbf{Q}}_\ell$-valued representations of $\mathcal{G}_{\mathbf{Q}}$ of dimension n. The Langlands conjecture is formulated for any global field K. In this thesis however, we will only look at the case $K = \mathbf{Q}$.

Relative to all positive integers n, there are not so many proved cases of this conjecture. The case $n = 1$ is proved. Many cases for $n = 2$ are proved. For $n \geq 3$ only very little is proved. However, the reader should not underestimate the power of the results we already have. The case $n = 1$ implies class field theory. The case $n = 2$ implies the modularity of elliptic curves.

In chapters 1-4 of this thesis we will give a precise statement of the Langlands conjecture and the Fontaine-Mazur conjecture, setting up relations between automorphic forms, geometric Galois representations and étale cohomology of proper and smooth $\mathbf{Q}$-schemes. In chapter 5 we prove that the one dimensional case of this conjecture is equivalent to class field theory for $\mathbf{Q}$. In chapter 6 we give two examples of two dimensional Galois representations and the corresponding automorphic representation. In chapter 7 we explain the relation between the global Langlands conjecture and the local Langlands theorem.

This thesis is based on the article of Taylor [**39**], of which we treat only a very tiny piece. After reading this thesis, the reader is encouraged to look up this article.

I want to thank Bas Edixhoven, for helping me both with my bachelors thesis and my masters thesis (which you are looking at now) and answering many other mathematical questions. Also I want to thank him for advising me to go study in Orsay after my third year here in Leiden, which turned out very well. Next, I thank Hendrik Lenstra, Lenny Taelman and Laurent Clozel for their answers to my questions. I thank my entire family, especially my parents, sister and grandparents, for their support, interest and coming over to visit me several times during my stay in Paris, and finally my uncle for printing this thesis.

Chapter 1: Galois Representations

The global Langlands conjecture relates Galois representations with automorphic representations. In this chapter, and the next one, we introduce the "Galois side" of this correspondence.

1.1. ℓ-adic representations

Let $\ell \in \mathbf{Z}$ be a prime number and $\overline{\mathbf{Q}}_\ell$ be an algebraic closure of $\mathbf{Q}_\ell$. A finite dimensional $\mathbf{Q}_\ell$ vector space has a, up to equivalence unique, norm compatible with the norm on $\mathbf{Q}_\ell$, [**26**, XII, prop. 2.2]. In particular, any finite extension E of $\mathbf{Q}_\ell$ contained in $\overline{\mathbf{Q}}_\ell$ has a unique norm extending the norm on $\mathbf{Q}_\ell$. Therefore $\overline{\mathbf{Q}}_\ell$ is also equipped with an unique norm extending the norm on $\mathbf{Q}_\ell$. We will always normalise the norm on $\mathbf{Q}_\ell$ so that $\ell \in \mathbf{Q}_\ell$ has norm ℓ^{-1}.

Let E be a topological ring, M a topological E-module and G a topological group. A continuous G-representation in M is an $E[G]$-module structure on M such that the action $G \times M \to M$ is continuous. A morphism of continuous representations is a continuous morphism of $E[G]$-modules.

Assume E is a subfield of $\overline{\mathbf{Q}}_\ell$ containing $\mathbf{Q}_\ell$. In this case, we call a continuous, finite dimensional representation *ℓ-adic*. If moreover G is the absolute Galois group of some field, then we will speak of a *Galois representation*. In this entire thesis we fix an algebraic closure $\overline{\mathbf{Q}}$ of $\mathbf{Q}$, and we write $\mathcal{G}_{\mathbf{Q}} = \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$. For every prime number p we fix an algebraic closure $\overline{\mathbf{Q}}_p$ of $\mathbf{Q}_p$ and we write $\mathcal{G}_{\mathbf{Q}_p} = \mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$. We are mainly interested in the ℓ-adic representations of $\mathcal{G}_{\mathbf{Q}}$ and $\mathcal{G}_{\mathbf{Q}_p}$. Let us give some examples.

Example. — There is a unique continuous morphism of groups, $\chi_\ell \colon \mathcal{G}_{\mathbf{Q}} \to \mathbf{Z}_\ell^\times \subset \overline{\mathbf{Q}}_\ell^\times$, such that for all ℓ^n-th roots of unity $\zeta \in \boldsymbol{\mu}_{\ell^\infty}$ and all $\sigma \in \mathcal{G}_{\mathbf{Q}}$ we have $\sigma(\zeta) = \zeta^{\chi_\ell(\sigma)}$. This morphism is the *cyclotomic character*. Through the character χ_ℓ we may let $\mathcal{G}_{\mathbf{Q}}$ act on $\overline{\mathbf{Q}}_\ell$ via multiplication, this is an example of a one dimensional ℓ-adic $\mathcal{G}_{\mathbf{Q}}$-representation.

Example. — Let $E/\mathbf{Q}$ be an elliptic curve. We set $T_\ell(E) := \varprojlim_{n \in \mathbf{N}} E[\ell^n]$, where $E[\ell^n]$ is the ℓ^n-torsion subgroup of $E(\overline{\mathbf{Q}})$, and the morphisms of transition are given by $E[\ell^{n+1}] \to E[\ell^n]$, $x \mapsto \ell x$. The $\mathbf{Z}_\ell$-module $T_\ell(E)$ is free of rank 2; it is called the *Tate module* of E. We set $V_\ell(E) := \overline{\mathbf{Q}}_\ell \otimes_{\mathbf{Z}_\ell} T_\ell(E)$. More generally, for any commutative group scheme A of finite type over $\mathbf{Q}$ (or $\mathbf{Q}_p$) one may construct a Tate-module $T_\ell(A)$ and a $\mathcal{G}_{\mathbf{Q}}$-representation $V_\ell(A)$. For example $\mathbf{Z}_\ell(1) := T_\ell(\mathbf{G}_{m,\mathbf{Q}})$ is a free $\mathbf{Z}_\ell$-module of rank 1 on which $\mathcal{G}_{\mathbf{Q}}$ acts via χ_ℓ.

Let E be a subfield of $\overline{\mathbf{Q}}_\ell$ containing $\mathbf{Q}_\ell$. Suppose $\psi \colon \mathcal{G}_{\mathbf{Q}} \to E^\times$ is a continuous morphism. We write $E(\psi)$ for the $\mathcal{G}_{\mathbf{Q}}$-representation with space E and $\mathcal{G}_{\mathbf{Q}}$-action given by ψ. If $\psi = \chi^k$ for some $k \in \mathbf{Z}$ then we will write $E(k) := E(\chi^k)$. More generally, if V is an arbitrary ℓ-adic $\mathcal{G}_{\mathbf{Q}}$-representation, then we write $V(\psi) := V \otimes_E E(\psi)$ and $V(k) = V \otimes_E E(k)$. The $\mathcal{G}_{\mathbf{Q}}$-representation $V(\psi)$ is called a *twist* of V by ψ.

We call an ℓ-adic representation *irreducible*, or *simple*, if it has precisely two invariant subspaces. An ℓ-adic representation is *semi-simple* if it is a direct sum of simple representations. An ℓ-adic representation V can be made semi-simple in a functorial manner in the following way.

Because V is Artinian as $E[G]$-module it has non-zero submodules that are minimal for the inclusion relation. Such a module is a simple submodule. Therefore, any V has a simple submodule. An $E[G]$-submodule of V generated by two semi-simple submodules is again semi-simple. We define $\mathrm{soc}(V) \subset V$ to be the maximal semi-simple submodule of V, it is called the *socle* of V. The socle is also the $E[G]$-submodule of V generated by all simple $E[G]$-submodules. If $f\colon V \to V'$ is an $E[G]$-morphism, then $f(\mathrm{soc}(V))$ is semi-simple, and thus contained in $\mathrm{soc}(V')$. Therefore $\mathrm{soc}(\square)$ is a covariant endofunctor of the category of $E[G]$-modules.

The representation $\mathrm{soc}(V)$ is not the semi-simplification of V because $V/\mathrm{soc}(V)$ is not necessarily semi-simple. Therefore one takes the inverse image of $\mathrm{soc}(V/\mathrm{soc}(V))$ in V to obtain a submodule $V_2 \subset V$ containing $V_1 = \mathrm{soc}(V)$ such that V_1 and V_2/V_1 are semi-simple. Again, V_2 is functorial in V. One may continue in this manner, to make V into a $E[G]$-module equipped with a functorial filtration. Because V is finite dimensional, the module V_i equals to V for i large enough. The graded module $\mathrm{gr}(V) := \bigoplus_{i=0}^{\infty} V_{i+1}/V_i$ associated to this filtration is V^{ss}, the *semi-simplification* of V.

Proposition 1. — *Let Λ be an algebra over a field K of characteristic zero, and let ρ_1, ρ_2 be two Λ-modules of finite K-dimension. Assume that ρ_1 and ρ_2 are semi-simple and $\mathrm{Tr}_K(\rho_1(\lambda))$ equals $\mathrm{Tr}_K(\rho_2(\lambda))$ for all $\lambda \in \Lambda$. Then ρ_1 is isomorphic to ρ_2.*

Proof. — [**5**, chapter 8, §12, n° 1, prop. 3].

Proposition 1 has a variant for characteristic p coefficients.

***Theorem 2* (Brauer-Nesbitt).** — *Let G be a finite group, E a perfect field of characteristic p and ρ_1, ρ_2 two semi-simple $E[G]$-modules, of finite dimension over E. Then $\rho_1 \cong \rho_2$ if and only if the characteristic polynomials of $\rho_1(g)$ and $\rho_2(g)$ coincide for all $g \in G$.*

Proof. — [**12**, theorem 30.16].

Lemma 3. — *Let $\lambda\colon \overline{\mathbf{Q}} \to \overline{\mathbf{Q}}_\ell$ be a $\overline{\mathbf{Q}}$-prime lying above ℓ. Then the image of λ is a dense subfield of $\overline{\mathbf{Q}}_\ell$. In particular the morphism $\iota_\lambda\colon \mathcal{G}_{\mathbf{Q}_p} \to \mathcal{G}_{\mathbf{Q}}$ is injective.*

Proof. — The proof of this claim is an application of Krasner's lemma [**33**, 8.1.6], in the following manner. By Krasner's lemma, the completion of an algebraically closed field is still algebraically closed. Therefore $(\overline{\mathbf{Q}})_\lambda$ is complete and algebraically closed for any $\overline{\mathbf{Q}}$-prime λ lying above ℓ. Therefore, $(\overline{\mathbf{Q}})_\lambda$ contains $\mathbf{Q}_\ell$, consequently it contains an algebraic closure $\overline{\mathbf{Q}}_\ell$ of $\mathbf{Q}_\ell$ and thus also $\mathbf{C}_\ell$, which is the completion of $\overline{\mathbf{Q}}_\ell$. Thus $\mathbf{C}_\ell$ contains $\overline{\mathbf{Q}}$, is complete and is contained in $(\overline{\mathbf{Q}})_\lambda$. By the universal property of the completion, we find $\mathbf{C}_\ell = (\overline{\mathbf{Q}})_\lambda$. Therefore, $\lambda(\overline{\mathbf{Q}})$ is dense in $\mathbf{C}_\ell$, and in particular dense in $\overline{\mathbf{Q}}_\ell$.

Proposition 4. — *Let G be a profinite group. For any ℓ-adic representation (ρ, V) of G with $\overline{\mathbf{Q}}_\ell$-coefficients, there exists a finite extension $K \subset \overline{\mathbf{Q}}_\ell$ of $\mathbf{Q}_\ell$ and a $\mathcal{O}_K[G]$-submodule $L \subset V$ such that $\overline{\mathbf{Q}}_\ell \otimes_{\mathcal{O}_K} L = V$.*

Proof. — Fix a $\overline{\mathbf{Q}}$-prime λ lying above ℓ. By Lemma 3 the field $\overline{\mathbf{Q}}$ is a dense subfield of $\overline{\mathbf{Q}}_\ell$. If $\alpha \in \overline{\mathbf{Q}}$, and $\alpha_1, \ldots, \alpha_n$ are the $\mathcal{G}_{\mathbf{Q}_\ell}$-conjugates of α, then we may pick $\beta \in \overline{\mathbf{Q}}$ such that

$$|\alpha - \beta| < |\alpha - \alpha_i| \quad \text{for } i = 2, \ldots, n.$$

By Krasner's lemma [**33**, 8.1.6], one gets $\mathbf{Q}_\ell(\alpha) \subset \mathbf{Q}_\ell(\beta)$. Therefore, the number of finite extensions of $\mathbf{Q}_\ell$ contained in $\overline{\mathbf{Q}}_\ell$ is countable.

After the choice of a basis, we may assume $V = \overline{\mathbf{Q}}_\ell^n$. The field $\overline{\mathbf{Q}}_\ell$ is a filtered union of finite extensions E_i of $\mathbf{Q}_\ell$ where i ranges over some *countable* index set I. Similarly, we have $\mathrm{GL}_n(\overline{\mathbf{Q}}_\ell) = \bigcup_{i \in I} \mathrm{GL}_n(E_i)$. Recall that a topological space X is a *Baire space* if and only if given any countable collection of closed sets F_i in X, each with empty interior in X, their union has $\bigcup F_i$ has also empty interior. The image $\rho(G)$ of G in $\mathrm{GL}_n(\overline{\mathbf{Q}}_\ell)$ is compact and therefore complete as a metric space and in particular a Baire space, see [**32**, thm 48.2].

Let F_i be the closure of $\mathrm{GL}_n(E_i) \cap \rho(G)$ in the space $\rho(G)$. Then $\rho(G) = \bigcup_{i \in I} F_i$ has non-empty interior inside $\rho(G)$. Therefore, there exists an $i \in I$ such that F_i contains a non-empty open subset U of $\rho(G)$.

After translating and shrinking U, we may assume it is an open subgroup of $\rho(G)$. The quotient $\rho(G)/U$ is covered by the sets $\mathrm{GL}_n(E_j) \cap \rho(G) \mod U$, with j ranging over all elements of I such that $E_i \subset E_j$. Because the quotient $\rho(G)/U$ is finite, we need only a finite number of such j. The compositum K of the fields E_j is then finite over E_i, and we have found a finite extension K of $\mathbf{Q}_\ell$ such that $\rho(G) \subset \mathrm{GL}_n(K)$.

Pick any $\mathcal{O}_K$-lattice $L' \subset K^n \subset E^n$ (*e.g.*, the standard lattice) and let $(e_i)_{i=1}^n$ be a $\mathcal{O}_K$-basis of this lattice. The intersection of the stabiliser of the e_i is open in G. Therefore, the G-translates of L' are finite in number. The $\mathcal{O}_K$-module L generated by these translates is therefore of finite type over $\mathcal{O}_K$ and generates K^n as K-vector space, so it is a lattice. This lattice L has the desired property.

1.2. Ramification

The integral closure $\overline{\mathbf{Z}}_p$ of $\mathbf{Z}_p$ in $\overline{\mathbf{Q}}_p$ is a valuation ring with residue field $\overline{\mathbf{F}}_p$ and value group $\mathbf{Q}$. The kernel of the map $\mathcal{G}_{\mathbf{Q}_p} \to \mathrm{Gal}(\overline{\mathbf{F}}_p/\mathbf{F}_p)$ is the *inertia group* $\mathcal{I}_{\mathbf{Q}_p} \subset \mathcal{G}_{\mathbf{Q}_p}$. An ℓ-adic $\mathcal{G}_{\mathbf{Q}_p}$-representation (ρ, V) is *unramified* if $\mathcal{I}_{\mathbf{Q}_p} \subset \ker(\rho)$. In that case ρ factors over $\mathcal{G}_{\mathbf{Q}_p}/\mathcal{I}_{\mathbf{Q}_p} = \mathrm{Gal}(\overline{\mathbf{F}}_p/\mathbf{F}_p)$, and $\rho(\mathrm{Frob}_p)$ is well-defined, where Frob_p is the *geometric Frobenius*: $\mathrm{Frob}_p := (x \mapsto x^{1/p}) \in \mathrm{Gal}(\overline{\mathbf{F}}_p/\mathbf{F}_p)$.

The choice of a $\overline{\mathbf{Q}}$-prime $\mathfrak{p}$ lying above p induces an embedding $\mathcal{G}_{\mathbf{Q}_p} \to \mathcal{G}_{\mathbf{Q}}$ with image equal to the decomposition group $\mathcal{D}(\mathfrak{p}/p)$. The image of $\mathcal{I}_{\mathbf{Q}_p}$ in $\mathcal{G}_{\mathbf{Q}}$ is denoted $\mathcal{I}(\mathfrak{p}/p)$. We can thus restrict a $\mathcal{G}_{\mathbf{Q}}$-representation (ρ, V) to the $\mathcal{G}_{\mathbf{Q}_p}$-representation $\rho_{\mathfrak{p}} := \rho|_{\mathcal{D}(\mathfrak{p}/p)}$. We will often abuse notation and write $\rho_p = \rho_{\mathfrak{p}}$: The representation ρ_p depends on the choice of $\mathfrak{p}/p$, but only up to isomorphism. We say that ρ is *unramified* at p if $\mathcal{I}(\mathfrak{p}/p) \subset \ker(\rho_{\mathfrak{p}})$. In this case

$\rho_{\mathfrak{p}}(\mathrm{Frob}_p) \in \mathrm{GL}(V)$ is well-defined. The element $\rho_{\mathfrak{p}}(\mathrm{Frob}_p)$ depends on $\mathfrak{p}/p$, but only up to conjugacy. Usually we will abuse notation and language, and say "Frobenius at p" and write $\rho(\mathrm{Frob}_p) = \rho_{\mathfrak{p}}(\mathrm{Frob}_p)$.

Theorem 5. — *Let ρ, ρ' be two ℓ-adic $\mathcal{G}_{\mathbf{Q}}$-representations both unramified for nearly all primes. Their semi-simplifications are isomorphic if and only if* $\mathrm{Tr}(\rho(\mathrm{Frob}_p) = \mathrm{Tr}(\rho(\mathrm{Frob}_p)) \in \overline{\mathbf{Q}}_\ell$ *for nearly all prime numbers p.*

Proof. — Let S be the set of primes that are ramified in ρ or in ρ'. The field $K = \overline{\mathbf{Q}}^{\ker(\rho)\cap\ker(\rho')}$ is a Galois extension of $\mathbf{Q}$ which is unramified at nearly all primes. By the Chebotarev density theorem, the set of Frobenius elements in $\mathrm{Gal}(K/\mathbf{Q})$ is dense subset. So the theorem is a consequence of Proposition 1.

Recall that in [**34**, chap. IV], Serre equips the group $\mathcal{G}_{\mathbf{Q}_p}$ with a decreasing filtration $(\mathcal{G}_{\mathbf{Q}_p,i})_{i\in\mathbf{Z}}$ of normal subgroups of $\mathcal{G}_{\mathbf{Q}_p}$. We have $\mathcal{G}_{\mathbf{Q}_p,i} = \mathcal{G}_{\mathbf{Q}_p}$ for $i \leq -1$, and $\mathcal{G}_{\mathbf{Q}_p,0} = \mathcal{I}_{\mathbf{Q}_p}$. The wild-inertia subgroup $\mathcal{I}(\mathfrak{p}/p)^{\mathrm{wild}} \subset \mathcal{I}_{\mathbf{Q}_p}$ is the group $\mathcal{G}_{\mathbf{Q}_p,1}$. This group is pro-$p$, and in fact the p-Sylow subgroup of $\mathcal{I}_{\mathbf{Q}_p}$ because the quotient $\mathcal{I}_{\mathbf{Q}_p}/\mathcal{I}(\mathfrak{p}/p)^{\mathrm{wild}}$ is isomorphic to $\prod_{\ell\neq p}\mathbf{Z}_\ell$.

A $\mathcal{G}_{\mathbf{Q}}$-representation is *tamely ramified* at a prime p if $\mathcal{I}(\mathfrak{p}/p)^{\mathrm{wild}}$ is contained in the kernel of the representation.

Proposition 6. — *Let (ρ, V) be an ℓ-adic $\mathcal{G}_{\mathbf{Q}}$-representation. Then ρ is tamely ramified for nearly all primes p.*

Proof. — Let $I = \mathrm{Im}(\rho)$; then $I \subset \mathrm{GL}(L)$ for some $\mathcal{O}_K$-lattice $L \subset V$ and $K/\mathbf{Q}_\ell$ a finite extension. The group $H = \ker(\mathrm{GL}(L) \to \mathrm{GL}(L/\mathfrak{m}_K L))$ is pro-ℓ. Let $\sigma \in \mathcal{I}(\mathfrak{p}/p)^{\mathrm{wild}}$ be an element of the wild inertia group at a prime $\mathfrak{p}$ which does not divide $\ell \cdot \#\mathrm{GL}(L/\mathfrak{m}_K L)$. The image of σ in $\mathrm{GL}(L/\mathfrak{m}_K L)$ must be trivial, so $\rho(\sigma) \in H$. But the group H is pro-ℓ and σ lies in pro-p group, so $\rho(\sigma) = 1$.

Proposition 7. — *Let $\chi\colon \mathcal{G}_{\mathbf{Q}} \to \overline{\mathbf{Q}}_\ell^\times$ be a continuous morphism. Then χ is a product of a continuous morphism $\mathcal{G}_{\mathbf{Q}} \to \overline{\mathbf{Q}}_\ell^\times$ with finite image, and a continuous morphism $\mathcal{G}_{\mathbf{Q}} \to \overline{\mathbf{Q}}_\ell^\times$ which is unramified outside ℓ. In particular χ is unramified for nearly all primes.*

Proof. — Let $K \subset \overline{\mathbf{Q}}_\ell$ be a subfield such that the image of χ is contained in $K^\times$. Recall that $K^\times$ is the product of a finite group with a finite type, free $\mathcal{O}_K$-module M. A continuous morphism $\mathcal{G}_{\mathbf{Q}} \to \overline{\mathbf{Q}}_\ell^\times$ with finite image is unramified for nearly all primes. After twisting, we may thus assume that χ takes values in M. By class field theory, χ corresponds to a continuous morphism $\chi'\colon \hat{\mathbf{Z}}^\times \to M$. One has $\hat{\mathbf{Z}}^\times = S \times \hat{\mathbf{Z}}$ where $S = \mathbf{Z}/2\mathbf{Z} \times \prod_{p\neq 2}\mathbf{F}_p^\times$. The torsion submodule T of S is dense in S. Because M has no torsion, χ' is trivial on T, and by continuity also on S. We can decompose $\hat{\mathbf{Z}} = \mathbf{Z}_\ell \oplus \mathbf{Z}^{\{\ell\}}$. For every $p \neq \ell$, the morphism χ is trivial on $\mathbf{Z}_p \subset \mathbf{Z}^{\{\ell\}}$ (embedded on the corresponding axis), and χ' is trivial on $\bigoplus_{\lambda\neq\ell}\mathbf{Z}_\lambda \subset \mathbf{Z}^{\{\ell\}}$ and by density on $\mathbf{Z}^{\{\ell\}}$. We have reduced to a χ' which is only ramified at ℓ. This completes the proof.

Example. — Let us make a $\mathcal{G}_{\mathbf{Q}}$-representation which is ramified at infinitely many primes of $\mathbf{Q}$. Let $K \subset \overline{\mathbf{Q}}$ be the extension of $\mathbf{Q}$ obtained by adjoining all ℓ^n-th roots of all prime numbers p and all ℓ^n-th roots of unity, where n ranges over all elements of $\mathbf{Z}_{\geq 0}$. Then $\mathrm{Gal}(K/\mathbf{Q}) = \mathbf{Z}_\ell^{\mathbf{P}} \rtimes \mathbf{Z}_\ell^\times$, where $\mathbf{P}$ is the set of prime numbers. Fix a sequence $(x_p) \in \mathbf{Z}_\ell^{\mathbf{P}}$ such that $\lim_{p\to\infty} x_p = 0$ and $x_p \neq 0$ for all p. Consider the surjection

$$\mathbf{Z}_\ell^{\mathbf{P}} \rtimes \mathbf{Z}_\ell^\times \longrightarrow \mathbf{Z}_\ell \rtimes \mathbf{Z}_\ell^\times, \quad ((t_n)_{p\in\mathbf{P}}, y) \mapsto \left(\sum_{p=0}^{\infty} x_p t_p, y\right).$$

Note that $\mathbf{Z}_\ell^\times \rtimes \mathbf{Z}_\ell = \left(\begin{smallmatrix} \mathbf{Z}_\ell^\times & \mathbf{Z}_\ell \\ 0 & 1 \end{smallmatrix}\right) \subset \mathrm{GL}_2(\overline{\mathbf{Q}}_\ell)$, so we may compose

$$\mathcal{G}_{\mathbf{Q}} \longrightarrow \mathbf{Z}_\ell^{\mathbf{P}} \rtimes \mathbf{Z}_\ell^\times \longrightarrow \mathbf{Z}_\ell \rtimes \mathbf{Z}_\ell^\times \subset \mathrm{GL}_2(\overline{\mathbf{Q}}_\ell),$$

to obtain a two dimensional $\mathcal{G}_{\mathbf{Q}}$-representation which is ramified at all prime numbers.

1.3. L-factors and L-functions

We recall the definition of invariants and co-invariants. Let G be a group, $H \subset G$ a subgroup and V a G representation over an arbitrary field. Then V^H is the *space of invariants* for the H-action on V

$$V^H = \{v \in V | \forall h \in H : hv = v\},$$

the group G acts on V^H if $H \subset G$ is normal. The space of *co-invariants* is defined as

$$V_H = V/\{v - h\cdot v | h \in H, v \in V\}.$$

If $H \subset G$ is normal, then this is a G-representation. Both constructions are covariant in V. The functor $V \mapsto V^H$ is left exact, and the functor $V \mapsto V_H$ is right exact. If $G = \mathcal{G}_{\mathbf{Q}_p}$, $H = \mathcal{I}_{\mathbf{Q}_p}$ and V is an ℓ-adic $\mathcal{G}_{\mathbf{Q}_p}$-representation, then $V^{\mathcal{I}_{\mathbf{Q}_p}}$ and $V_{\mathcal{I}_{\mathbf{Q}_p}}$ are two ways to make V into an unramified representation of $\mathcal{G}_{\mathbf{Q}_p}$.

Let (V, ρ) be an ℓ-adic $\mathcal{G}_{\mathbf{Q}_p}$-representation, where ℓ is not p. The L-factor of (V,ρ) is defined by

$$L_p(V, s) := \frac{1}{\det(1 - \rho|_{V^{\mathcal{I}_{\mathbf{Q}_p}}}(\mathrm{Frob}_p)\cdot p^{-s})} \in \overline{\mathbf{Q}}_\ell(p^{-s}),$$

where the symbol "p^{-s}" is transcendental over $\overline{\mathbf{Q}}_\ell$, and with the notation $\rho|_{V^{\mathcal{I}_{\mathbf{Q}_p}}}$ we mean the representation ρ restricted to the space of invariants $V^{\mathcal{I}_{\mathbf{Q}_p}}$ under the action of $\mathcal{I}_{\mathbf{Q}_p}$ on V.

In case of an ℓ-adic $\mathcal{G}_{\mathbf{Q}}$-representation (V,ρ), we define for primes $p \neq \ell$:

$$L_p(V, s) := L_p(V|_{\mathcal{D}(\mathfrak{p}/p)}, s) \in \overline{\mathbf{Q}}_\ell(p^{-s}), \tag{(I.1)}$$

which does not depend on the choice $\mathfrak{p}/p$. We should define the L-factor at ℓ also; unfortunately this is not so easy as the above, we will not give a definition of the factor at ℓ in this thesis. For the applications to the global $\mathcal{G}_{\mathbf{Q}}$-representations that we have in mind, it suffices to know the L-factors at nearly all primes anyway.

Let ρ, ρ' be two semi-simple ℓ-adic $\mathcal{G}_{\mathbf{Q}}$-representations which are unramified for nearly all primes. Then $\rho \cong \rho'$ if and only if nearly all their L-factors agree (Theorem 5).

We now want to multiply the L-factors, evaluate them in $s \in \mathbf{C}$ and talk about convergence. To do this, we see that we have a problem: The L-factors have coefficients in $\overline{\mathbf{Q}}_\ell$, but we want them to have coefficients in $\mathbf{C}$. To solve it, we choose an embedding $\iota\colon \overline{\mathbf{Q}}_\ell \to \mathbf{C}$, and set $L_p^{(\iota)}(\rho, s) := \iota(L_p(\rho, s)) \in \mathbf{C}(p^{-s})$. So now the problem is solved, and we can define the L-function as:

$$L^{(\iota)}(V, s) = L_\ell^{(\iota)}(V, s) \cdot \prod_{p \neq \ell} L_p^{(\iota)}(V, s) \quad \text{(formal product)}.$$

Conjecturally for a geometrical representation (we will define this notion in chapter 2), the L-factors have coefficients in $\overline{\mathbf{Q}}$ so the operation "applying ι" should not be as strange as it seems. Moreover, the L-function should converge in a right half-plane, have a meromorphic continuation, and satisfy a functional equation. For a precise statement, see [**39**, conjecture 2.1].

Example. — The L-function of the cyclotomic character is

$$L^{(\iota)}(\overline{\mathbf{Q}}_\ell(1), s) = L_\ell^{(\iota)}(\overline{\mathbf{Q}}_\ell(1), s) \cdot \prod_{p \neq \ell} \frac{1}{1 - 1/p \cdot p^{-s}} = \zeta(s+1), \tag{(I.2)}$$

where $\zeta(\cdot)$ is the Riemann-zeta function (we admit that the factor at ℓ equals $(1 - 1/\ell \cdot \ell^{-s})^{-1}$). To see this, one may apply class field theory. Otherwise, you can also just chase the definitions. Let us do it like this here. Let $p \neq \ell$, and consider $\mathrm{Frob}_p \in \mathcal{G}_{\mathbf{Q}}/\ker(\chi)$. Pick any prime $\mathfrak{p}$ of $K := \overline{\mathbf{Q}}^{\ker \chi}$ lying above p. The composition

$$\boldsymbol{\mu}_{\ell^\infty}(\mathcal{O}_K) \hookrightarrow \mathcal{O}_K^\times \longrightarrow (\mathcal{O}_K/\mathfrak{p})^\times, \tag{(I.3)}$$

is injective. To see this, assume $\zeta \equiv 1 \mod \mathfrak{p}$ where $\zeta \in \boldsymbol{\mu}_{\ell^\infty}(\mathcal{O}_K)$.

Let $\Phi_\zeta = \frac{X^{\ell^k} - 1}{X^{\ell^{k-1}} - 1} \in \mathbf{Z}[X]$ be the minimal polynomial of ζ at 1. Then $\Phi_\zeta(1) = \ell$ if $\zeta \neq 1$, and $\Phi_\zeta(1) = 0$ if $\zeta = 1$. Therefore, the norm $N_{L/\mathbf{Q}}(\zeta - 1)$ equals to ℓ if $\zeta \neq 1$ and equals to 0 otherwise. We assumed $\zeta - 1 \in \mathfrak{p}$, so its norm should be divisible by p, therefore $N_{L/\mathbf{Q}}(\zeta - 1) = 0$ and $\zeta = 1$, so the morphism in (I.3) is injective.

The automorphism Frob_p acts by $x \mapsto x^{1/p}$ on $\mathcal{O}_K/\mathfrak{p}$. By equation (I.3) we see it acts on $\boldsymbol{\mu}_{\ell^\infty}(\mathcal{O}_K)$ by $\zeta \mapsto \zeta^{1/p}$ and the eigenvalue of Frob_p is $1/p$. Thus the composition in equation (I.3) is indeed injective.

If you apply the formula (I.1) to compute the factor at $p = \ell$, then you get $L_\ell^{(\iota)}(\overline{\mathbf{Q}}_\ell(1), s) = 1$, which is wrong. Namely, if the factor at ℓ would be 1, then the L-function of χ_ℓ is $\zeta(s-1) \cdot (1 - \ell \cdot \ell^{-s})$ which depends on ℓ.

Conjecturally, irreducible geometric $\mathcal{G}_{\mathbf{Q}}$-representations (like χ_ℓ) should come in families, where ℓ varies, of ℓ-adic representations, with all the same L-factors, so the L-function should not depend on ℓ. The family of χ_ℓ is $\{\chi_\lambda | \lambda \text{ prime}\}$. The factor at ℓ may be computed using χ_λ, with λ a prime different from ℓ. This gives $L_\ell^{(\iota)}(\chi_\ell, s) = (1 - 1/\ell \cdot \ell^{-s})^{-1}$.

For characters $\chi \colon \mathcal{G}_{\mathbf{Q}} \to \overline{\mathbf{Q}}_\ell^\times$ with finite image the L-functions are classical. Let us explain this relation more explicitly. Via the Artin map of class field theory, χ corresponds to a character $\chi' \colon \hat{\mathbf{Z}}^\times \to \overline{\mathbf{Q}}_\ell^\times$. Under this correspondence we have

$$L_p^{(\iota)}(\chi, s) = \frac{1}{1 - \iota\chi(\mathrm{Frob}_p) \cdot p^{-s}} = \frac{1}{1 - \iota\chi'(\hat{p}) \cdot p^{-s}} \in \mathbf{C}(p^{-s}), \tag{(I.4)}$$

for unramified $p \neq \ell$, where $\hat{p} = (1, p^{-1}) \in \mathbf{Z}_p^\times \times \mathbf{Z}^{\{p\},\times}$.

Remark. — To see that $\hat{p} \in \hat{\mathbf{Z}}^\times/\mathbf{Z}_p^\times = \mathbf{Z}^{\{p\},\times}$ is the geometric Frobenius at p, prove that $\hat{p}$ acts as $\zeta \mapsto \zeta^{1/p}$ on roots of unity of prime-to-p-order, with a similar argument to the one in (I.3). Otherwise, use (geometrically normalised) class field theory in the following manner. The inclusion $\hat{\mathbf{Z}}^\times \hookrightarrow \mathbf{A}^\times$ induces an isomorphism $\hat{\mathbf{Z}}^\times \xrightarrow{\sim} \mathbf{Q}^\times\backslash\mathbf{A}^\times/\mathbf{R}_{>0}^\times$. The geometric Frobenius in $\mathbf{Q}^\times\backslash\mathbf{A}^\times/\mathbf{R}_{>0}^\times$ is the $\mathbf{Z}_p^\times$-class of the idèle in $\mathbf{A}^\times$ with p on the coordinate corresponding to p and 1 on all other coordinates. To put this class Frob_p in $\hat{\mathbf{Z}}^\times$ one should multiply it with $p^{-1} \in \mathbf{Q}^\times$, hence the definition of $\hat{p}$.

For the ramified $p \neq \ell$, the L-factor $L_p^{(\iota)}(\chi, s)$ equals 1. Let $N \in \mathbf{Z}_{>0}$ be minimal such that χ' factors over $(\mathbf{Z}/N\mathbf{Z})^\times \to \overline{\mathbf{Q}}_\ell^\times$. Extend χ' to a map $\mathbf{Z}/N\mathbf{Z} \to \overline{\mathbf{Q}}_\ell$, by setting $\chi'(x) = 0$ if $\gcd(x, N) \neq 1$. Then formul (I.4) is correct for every prime number $p \neq \ell$. We admit that for $p = \ell$ the above formula is also correct.

1.4. Example: The L-function of an elliptic curve

If V is an ℓ-adic $\mathcal{G}_{\mathbf{Q}}$-representation, then $V^\vee$ is the *dual* $\mathcal{G}_{\mathbf{Q}}$-representation. It is defined as follows. Apply the functor $\mathrm{Hom}_{\overline{\mathbf{Q}}_\ell[\mathcal{G}_{\mathbf{Q}}]}(\square, \overline{\mathbf{Q}}_\ell)$ to V. This functor is contravariant, so $\mathcal{G}_{\mathbf{Q}}$ acts (a priori) on the right on $V^\vee$. Compose the *anti-morphism* $\mathcal{G}_{\mathbf{Q}} \to \mathrm{GL}(V^\vee)$ with the *anti-morphism* $g \mapsto g^{-1}$ to obtain a *morphism* $\mathcal{G}_{\mathbf{Q}} \to \mathrm{GL}(V^\vee)$, this defines the dual representation.

Theorem 8. — *Let E be an elliptic curve over $\mathbf{Q}$. Fix a prime $\ell \in \mathbf{Z}$. Let $\tilde{E}/\mathbf{Z}$ be a Weierstrass model for E. Let $a_p(E) := 1 - \#\tilde{E}(\mathbf{F}_p) + p$ for all prime numbers p. Then for all primes p we have:*

$$L_p^{(\iota)}(V_\ell(E)^\vee, s) = \begin{cases} \frac{1}{1-a_p(E)p^{-s}+pp^{-2s}} & E \text{ has good reduction at } p \\ \frac{1}{1-a_p(E)p^{-s}} & E \text{ has bad reduction at } p. \end{cases} \tag{(I.5)}$$

Moreover, in case E has bad reduction at p, then

$$a_p(E) = \begin{cases} 1 & \text{if } E \text{ has split multiplicative reduction} \\ -1 & \text{if } E \text{ has non-split multiplicative reduction} \\ 0 & \text{if } E \text{ has additive reduction.} \end{cases} \tag{(I.6)}$$

We do not prove this entire theorem, only certain special cases.

Some remarks are in order. First, we have not defined the L-factor at ℓ of an ℓ-adic $\mathcal{G}_{\mathbf{Q}}$-representation, so the above statement does not (formally) have any sense for $p = \ell$. With the above we want to suggest that with the right definition of the factor at ℓ, this is what should come out.

Emphasising this once more: In case we directly apply formula (I.1) to compute the factor at ℓ, we obtain something wrong. For example if E has good, ordinary reduction at ℓ, then the L-factor is 1 over a linear polynomial in ℓ^{-s}, but it should be a degree 2 polynomial. If E has good, super singular reduction at ℓ, then the formula yields 1 as L-factor, this is also wrong.

Recall the criterion of Ogg-Tate-Shaverevich [**36**, p. 184] which states that E has good reduction at a prime $p \neq \ell$ if and only if $V_\ell(E)$ is an unramified $\mathcal{G}_{\mathbf{Q}}$-representation. The above theorem is stronger than this criterion. To see this, the degree of the polynomial in p^{-s} in the denominator of the L-factor $L_p(V_\ell(E), s)$ equals $\dim V_\ell(E) = 2$ if and only if the representation $V_\ell(E)$ is unramified.

For an abelian variety A over $\mathbf{Q}$ the L-factor at a prime p of good reduction may be defined without using Tate modules, which implies the independence of ℓ at those primes. Moreover, it implies that the factors have coefficients in $\mathbf{Z}$. Let us explain this very briefly. Let p be a prime where A has good reduction and let $\tilde{A}$ be a $\mathbf{Z}_p$-model of $A \times \mathbf{Q}_p$. This means that $\tilde{A}$ is an abelian variety over $\mathbf{Z}_p$ such that its generic fiber is isomorphic to $A \times \mathbf{Q}_p$ (the isomorphism is part of the data). Such a model $\tilde{A}$ is unique up to isomorphism. One shows that for any $\alpha \in \operatorname{End}(\tilde{A} \times \mathbf{F}_p)$ there exists a polynomial $P_\alpha(X) \in \mathbf{Q}[X]$ of degree $2\dim(A)$, such that for all $r \in \mathbf{Q}$ we have $\deg(1 - [r] \cdot \alpha) = P_\alpha(r)$. One takes α the Frobenius endomorphism of $\tilde{A} \times \mathbf{F}_p$, then the L-factor at p is given by $1/P_\alpha(p^{-s})$. For details see [**27**, prop. 12.9] or in the case of elliptic curves, [**37**, remark 10.1].

By the theorem, the L-factors are completely independent of ι. This is also true for abelian varieties. More generally (but only conjecturally) also for the Galois representations in the étale cohomology spaces of proper smooth $\mathbf{Q}$-schemes. However, we will also consider subquotients of representations occurring in cohomology. For those representations it is no longer true that the coefficients of the L-factors lie in $\mathbf{Z}$. For example, take $\phi\colon \mathcal{G}_{\mathbf{Q}} \to \overline{\mathbf{Q}}_\ell^\times$ a continuous morphism with finite image of cardinality at least 3. Then the coefficients of L-factors of ϕ do not lie in $\mathbf{Z}$ (they lie in $\mathbf{Z}[\boldsymbol{\mu}_{\#\phi(\mathcal{G}_{\mathbf{Q}})}]$).

Also, the coefficients of the L-factors of the cyclotomic character are not integral over $\mathbf{Z}$.

To know E up to $\mathbf{Q}$-isogeny, is the same as to know (nearly) all its L-factors. This is a consequence of (1) the isomorphism

$$\mathbf{Q}_\ell \otimes \operatorname{Hom}_{\mathbf{Q}}(E_1, E_2) \xrightarrow{\sim} \operatorname{Hom}(V_\ell(E_1), V_\ell(E_2))^{\mathcal{G}_{\mathbf{Q}}},$$

for any pair of elliptic curves E_1, E_2 over $\mathbf{Q}$ [**17**], and the facts (2) two semi-simple Galois representations are isomorphic if and only if nearly all their L-factors agree (Theorem 5), and (3) that $V_\ell(E)$ is irreducible for all elliptic curves $E/\mathbf{Q}$ (Theorem 44), so a non-zero $\mathbf{Q}_\ell[\mathcal{G}_{\mathbf{Q}}]$-morphism $V_\ell(E_1) \to V_\ell(E_2)$ exists if and only if the representations are isomorphic.

Alternatively, if you know the L-function of an elliptic curve then the modularity theorem associates to the L-function of E a modular form f, and in turn to f an elliptic curve which is isogeneous to E [**15**, p. 362, p. 241].

Proof of Theorem 8, for $p \neq \ell$ and E good reduction at p. — Let $E_{\mathbf{Q}_p} = E \times_{\mathbf{Q}} \mathbf{Q}_p$, $\tilde{E}/\mathbf{Z}_p$ the minimal Weierstrass model for $E_{\mathbf{Q}_p}$, and $\tilde{E}_{\mathbf{F}_p}$ the special fiber of $\tilde{E}$. Fix an algebraic closure $\overline{\mathbf{Q}}_p$ of $\mathbf{Q}_p$. Consider the algebraic closure of $\mathbf{Q}$ inside $\overline{\mathbf{Q}}_p$, let $V_\ell(E_{\mathbf{Q}_p})$ (resp. $V_\ell(E)$) be constructed relative to this choice of $\overline{\mathbf{Q}}_p$ (resp. $\overline{\mathbf{Q}}$). Then $V_\ell(E_{\mathbf{Q}_p}) = V_\ell(E)$ as $\overline{\mathbf{Q}}_\ell[\mathcal{G}_{\mathbf{Q}_p}]$-modules.

Let $\phi\colon \tilde{E}_{\mathbf{F}_p} \to \tilde{E}_{\mathbf{F}_p}$ be the Frobenius morphism sending (in projective coordinates) points $(x, y, z) \in \tilde{E}(\overline{\mathbf{F}}_p)$ to (x^p, y^p, z^p). We have an isomorphism

$$V_\ell(E_{\mathbf{Q}_p}) \xrightarrow{\sim} V_\ell(\tilde{E}_{\mathbf{F}_p}). \qquad ((\text{I}.7))$$

Let us define this morphism and prove it is an isomorphism. Let $\tilde{E}[\ell^n]$ be the kernel group scheme of multiplication by ℓ^n on $\tilde{E}$. By definition this means, for any $\mathbf{Z}_p$-scheme S we have $\tilde{E}[\ell^n](S) = \ker([\ell^n](S))$. By the valuative criterion of properness, [**21**, II, thm 4.7], $\tilde{E}[\ell^n](\overline{\mathbf{Q}}_p)$ equals to $\tilde{E}[\ell^n](\overline{\mathbf{Z}}_p)$. We have a morphism $\varphi_n\colon \tilde{E}[\ell^n](\overline{\mathbf{Z}}_p) \to \tilde{E}[\ell^n](\overline{\mathbf{F}}_p)$. The scheme $\tilde{E}[\ell^n]/\mathbf{Z}_p$ is étale [**28**, thm 7.2]. By [**20**, cor I.6.2], the natural map

$$\mathrm{Hom}_{\mathbf{Z}_p}(A, B) \longrightarrow \mathrm{Hom}_{\mathbf{F}_p}(A, B \otimes \mathbf{F}_p),$$

is bijective for every finite $\mathbf{Z}_p$-algebra B. On applying this for all finite $\mathbf{Z}_p$-subalgebras B of $\overline{\mathbf{Z}}_p$, we find that

$$\varphi_n\colon\ \mathrm{Hom}_{\mathbf{Z}_p}(A, \overline{\mathbf{Z}}_p) \longrightarrow \mathrm{Hom}_{\mathbf{F}_p}(A, \overline{\mathbf{F}}_p),$$

is bijective for all n. To obtain the isomorphism in (I.7), it suffices to take the limit over all n and to tensor with $\mathbf{Q}$.

On the left side in equation (I.7), $V_\ell(E(\overline{\mathbf{Q}}_p))$, the Frobenius Frob_p^{-1} acts, and on the right side, $V_\ell(\tilde{E}(\overline{\mathbf{F}}_p))$, the endomorphism ϕ acts, and these actions correspond under the isomorphism in (I.7). In particular, the characteristic polynomials of these operators coincide.

The characteristic polynomial of ϕ acting on $V_\ell(\tilde{E}_{\mathbf{F}_p})$ is of the form $X^2 + aX + b \in \mathbf{Q}_\ell[X]$. Note that $1 - \phi$ is a separable isogeny, so

$$\#\tilde{E}(\mathbf{F}_p) = \#\ker(1-\phi) = \deg(1-\phi) = (1-\phi)\circ(1-\hat{\phi}) = 1 - (\phi + \hat{\phi}) + p \in \mathbf{Z},$$

where $\hat{\phi}\colon E \to E$ is the *dual isogeny* of ϕ, and deg is the *degree* of ϕ. Therefore, a equals to $1 - \#\tilde{E}(\mathbf{F}_p) + p$. Because $\phi \circ \hat{\phi} = [p]$ as endomorphism of $\tilde{E}_{\mathbf{F}_p}$, we get $b = p$.

In the above we have calculated the characteristic polynomial of ϕ which is of the form $\det(X - \phi)$. The L-factor at p is given by $1/\det(1 - \phi p^{-s})$. Therefore we get the formula in the theorem.

Proof of Theorem 8 in case E has split multiplicative reduction at p, and p is different from ℓ.

With these hypothesis, we have $E(\overline{\mathbf{Q}}_p) \cong \overline{\mathbf{Q}}_p^\times/q^{\mathbf{Z}}$ for some $q \in \mathbf{Q}_p^\times$ with $|q| < 1$, where the isomorphism is as $\mathbf{Z}[\mathcal{G}_{\mathbf{Q}_p}]$-modules [**31**]. Let $n \geq 1$, then the ℓ^n-torsion of the group $\overline{\mathbf{Q}}_p^\times/q^{\mathbf{Z}}$ is given by the elements of the form

$$\zeta_{\ell^n}^a \cdot (q^{1/\ell^n})^b \in \overline{\mathbf{Q}}_p^\times/q^{\mathbf{Z}},$$

where ζ_ℓ^n is a primitive ℓ^n-th root of unity, $q^{1/\ell^n} \in \overline{\mathbf{Q}}_p$ a root of the polynomial $X^{\ell^n} - q \in \mathbf{Z}_p[X]$, $a \in \mathbf{Z}$ and $b \in \mathbf{Z}_{\geq 0}$.

By definition, the canonical surjection $\overline{\mathbf{Q}}_p^\times \to \overline{\mathbf{Q}}_p^\times/q^{\mathbf{Z}}$ is $\mathcal{G}_{\mathbf{Q}_p}$-equivariant. Let $\alpha \in (\overline{\mathbf{Q}}_p^\times/q^{\mathbf{Z}})_{\ell^n}$ and lift α to an element $\tilde{\alpha} \in \overline{\mathbf{Q}}_p^\times$. Then $\tilde{\alpha}$ is a root of the polynomial of the form $f_{n,b} = X^{\ell^n} - q^b \in \mathbf{Z}_\ell[X]$ for some $b \geq 0$. Such a polynomial $f_{n,b}$ has only a single root modulo p, so it is totally ramified. The $\mathcal{I}_{\mathbf{Q}_p}$-orbits of the roots of $f_{n,b}$ in $\overline{\mathbf{Q}}_p$ correspond to the irreducible factors of $f_{n,b} \in \mathbf{Z}_p[X]$. This implies that Frob_p acts trivially on the set of these $\mathcal{I}_{\mathbf{Q}_p}$-orbits, or equivalently, trivially on the irreducible factors of $f_{n,b} \in \mathbf{Z}_p[X]$. Moreover, the map $x \mapsto \zeta_{\ell^n} x$ induces $\mathcal{I}_{\mathbf{Q}_p}$-equivariant bijections between the $\mathcal{I}_{\mathbf{Q}_p}$-orbits of the roots of $f_{n,b}$, so these orbits have the same cardinality, and the degrees of the roots of $f_{n,b}$ are all equal and thus a power of ℓ. Let $d(n)$ be the degree of the irreducible factors of $f_{n,1}$. We see

$$\mathcal{I}_{\mathbf{Q}_p}\tilde{\alpha} = \alpha\boldsymbol{\mu}_{\deg(\tilde{\alpha})} \subset \alpha\boldsymbol{\mu}_{d(n)},$$

where $\deg(\tilde{\alpha})$ is the degree of $\tilde{\alpha}$ over $\mathbf{Q}_p$. Any element $\alpha \in (\overline{\mathbf{Q}}_p^\times/q^{\mathbf{Z}})_{\ell^n}$ can be written as $\zeta_{\ell^n}^a \cdot (q^{1/\ell^n})^b \mod q^{\mathbf{Z}}$. The inertia group $\mathcal{I}_{\mathbf{Q}_p}$ acts trivially on ζ_{ℓ^n}, so we conclude

$$\left\{ \frac{\alpha}{g\alpha} | \alpha \in (\overline{\mathbf{Q}}_p^\times/q^{\mathbf{Z}})_{\ell^n}, g \in \mathcal{I}_{\mathbf{Q}_p} \right\} = \boldsymbol{\mu}_{d(n)}.$$

Therefore,

$$E(\overline{\mathbf{Q}}_p)_{\ell^n,\mathcal{I}_{\mathbf{Q}_p}} \cong (\overline{\mathbf{Q}}_p^\times/q^{\mathbf{Z}})_{\ell^n}/\boldsymbol{\mu}_{d(n)},$$

as $\mathbf{Z}_\ell[\mathcal{G}_{\mathbf{Q}_p}]$-modules.

We already remarked that Frob_p acts trivially on the $\mathcal{I}_{\mathbf{Q}_p}$-orbits of the polynomials $f_{n,b}$. Therefore we have an exact sequence

$$0 \longrightarrow \boldsymbol{\mu}_{\ell^n}/\boldsymbol{\mu}_{d(n)} \longrightarrow (\overline{\mathbf{Q}}_p^\times/q^{\mathbf{Z}})_{\ell^n}/\boldsymbol{\mu}_{d(n)} \longrightarrow \mathbf{Z}/\ell^n\mathbf{Z} \longrightarrow 0,$$

where $\mathcal{G}_{\mathbf{Q}_p}$ acts trivially on $\mathbf{Z}/\ell^n\mathbf{Z}$.

On taking the projective limit over all n, we find that $T_\ell(E)_{\mathcal{I}_{\mathbf{Q}_p}}$ surjects onto $\mathbf{Z}_\ell(0)$ with finite kernel. Therefore, $V_\ell(E)_{\mathcal{I}_{\mathbf{Q}_p}}$ is isomorphic to the trivial representation and $a_p(E) = 1$.

Because E has split multiplicative reduction, the cardinality $\#\tilde{E}(\mathbf{F}_p)$ equals to $\#\mathbf{G}_m(\mathbf{F}_p)$ plus one, because the smooth locus of E is isomorphic to $\mathbf{G}_m$ as group scheme, and the plus one comes from the singular point. Therefore $\#\tilde{E}(\mathbf{F}_p) = (p-1)+1 = p$. This completes the proof.

Remark. — From the above calculation we see that the $\mathcal{G}_{\mathbf{Q}_p}$-surjection $T_\ell(E)_{\mathcal{I}_{\mathbf{Q}_p}} \to \mathbf{Z}_\ell(0)$ is surjective with finite kernel. This morphism is not always an isomorphism. For example, if q is an ℓ-th power in $\mathbf{Q}_p^\times$ then it is not an isomorphism. Only *after* extending scalars to $\mathbf{Q}_\ell$ it becomes an isomorphism.

Chapter 2: Geometric representations

In this chapter we will introduce the notion of "geometric representation", and state the Fontaine-Mazur conjecture.

2.1. Étale cohomology

Let X, Y be two schemes, a morphism $X \to Y$ is *étale* if it is smooth of relative dimension zero.

Let X be a scheme. The *étale site of* X, notation $X_{\text{ét}}$, is the category whose objects are étale morphisms $U \to X$ and whose morphisms are the (étale) X-morphisms. The category $X_{\text{ét}}$ is endowed with a *Grothendieck topology*. This is to say that open covers are prescribed: Let $U \in X_{\text{ét}}$, a set of objects $\mathcal{U}$ of $U_{\text{ét}}$ is an open cover of U if $\bigcup_{V \in \mathcal{U}} \text{Im}(V \to U) = U$ (set theoretical equality).

A *presheaf of abelian groups* $\mathcal{F}$ on X for the étale-topology is a contravariant functor $X_{\text{ét}} \to \underline{\text{Ab.Groups}}$. The presheaf $\mathcal{F}$ is a *sheaf* if for each $U \to X$ in $X_{\text{ét}}$ and each étale covering $\mathcal{U}$ of U the diagram

$$\mathcal{F}(U) \longrightarrow \prod_{V \in \mathcal{U}} \mathcal{F}(V) \rightrightarrows \prod_{V', V'' \in \mathcal{U}} \mathcal{F}(V' \times_U V''),$$

is exact.

A *morphism of presheaves* is a natural transformation of functors, and a *morphism of sheaves* is a morphism of presheaves. We obtain the category $\underline{\text{Ab}}(X_{\text{ét}})$ of sheaves on $X_{\text{ét}}$. The categories of sheaves and presheaves on $X_{\text{ét}}$ are abelian categories [**29**, prop 7.8].

The inclusion functor from $\underline{\text{Ab}}(X_{\text{ét}})$ to the category of presheaves on $X_{\text{ét}}$ admits a left adjoint functor [**29**, prop 7.15]. This is the *sheafification functor* $\mathcal{P} \mapsto \mathcal{P}^+$.

The category $\underline{\text{Ab}}(X_{\text{ét}})$ has enough injectives and the global sections functor $\Gamma(X, \square)$ is left exact [**29**, prop 8.12]. Therefore, it has right derived functors $H^q(X_{\text{ét}}, \square)$.

The cohomology $H^q(X_{\text{ét}}, \mathbf{Q}_\ell)$ with $\mathbf{Q}_\ell$-coefficients is not defined as the cohomology of the sheaf associated to the constant presheaf $U \mapsto \mathbf{Q}_\ell$: The definition is slightly more complicated. First $H^q(X_{\text{ét}}, \mathbf{Z}/\ell^n\mathbf{Z})$ is the cohomology of the sheaf associated to the presheaf $U \mapsto \mathbf{Z}/\ell^n\mathbf{Z}$. The cohomology $H^q(X_{\text{ét}}, \mathbf{Z}_\ell)$ is by definition $\varprojlim_{n \in \mathbf{N}} H^q(X_{\text{ét}}, \mathbf{Z}/\ell^n\mathbf{Z})$, and $H^q(X_{\text{ét}}, \mathbf{Q}_\ell) = \mathbf{Q}_\ell \otimes_{\mathbf{Z}_\ell} H^q(X_{\text{ét}}, \mathbf{Z}_\ell)$.

Let k be a field, k^s/k a separable closure, and X a proper k-scheme. Define $X_{k^s} := X \times_k k^s$ and assume that all of the irreducible components of X_{k^s} have dimension at most $n \in \mathbf{Z}_{\geq 0}$. The $\mathbf{Z}_\ell$-module $H^q(X_{k^s,\text{ét}}, \mathbf{Z}_\ell)$ is finitely generated and the spaces $H^q(X_{k^s,\text{ét}}, \mathbf{Z}_\ell)$ are 0 for $q > 2n$ [**29**]. The space $H^q(X_{k^s,\text{ét}}, \mathbf{Z}_\ell)$ is contravariant in X. Therefore, if G is a group acting on X, the cohomology is a G-representation. Let $G = \text{Gal}(k^s/k)$ be the absolute Galois group of k, the k^s-scheme X_{k^s} is equipped with a G-action, and the spaces $H^q(X_{k^s,\text{ét}}, \mathbf{Q}_\ell)$ are ℓ-adic representations of G.

Example. — Let A/k be an abelian variety, and assume $\text{char}(k) \neq \ell$. Then $H^q(A_{k^s,\text{ét}}, \mathbf{Q}_\ell) = \bigwedge^m V_\ell(A)^\vee$; see [**27**, 15.1].

The idea of étale cohomology originates from cohomology of analytical spaces. Let us explain this in precise terms. An analytical space is defined as follows. Let $U \subset \mathbf{C}^n$ be an open subset, $\mathcal{O}_U$ be the sheaf of holomorphic functions on U, $I \subset \mathcal{O}_U(U)$ a finitely generated ideal, $V(I) \subset U$ the subspace on which all functions of I vanish. On $V = V(I)$ we put the structure sheaf $\mathcal{O}_V = \mathcal{O}_U/I$, making $(V, \mathcal{O}_V)$ into a locally ringed space. An *analytical space* is a locally ringed space $(X, \mathcal{O}_X)$ over $\mathrm{Spec}(\mathbf{C})$ which is locally isomorphic to a locally ringed space of the form $(V, \mathcal{O}_V)$, where V is constructed as above.

On an analytical space X one has Betti cohomology. This it is defined as follows. The category of sheaves of abelian groups on X is abelian, has enough injectives, and the global sections functor $\Gamma(X, \Box)$ is a left exact functor to the category of abelian groups. The right derived functors $H^n_{\mathrm{Betti}}(X, \Box)$ of $\Gamma(X, \Box)$ are the Betti cohomology on X.

A morphism between two analytical spaces $X \to Y$ is *étale* if it is locally an open immersion. The site X_{cl} is the site of all étale morphisms of analytic spaces $U \to X_{\mathrm{cl}}$. Betti cohomology on X^{an} is cohomology on the site X_{Betti} of open immersions $U \to X$. In the analytical setting, every étale morphism can be covered by open immersions, so the toposses of abelian sheaves on X_{Betti} and X_{cl} are equivalent; in particular the two cohomologies “Betti” and “cl” coincide.

Let X be a scheme, locally of finite type over $\mathbf{C}$. Consider the functor which associates to an analytic space $\mathfrak{X}$ the set of morphisms $\mathfrak{X} \to X$ of locally ringed spaces on $\mathbf{C}$-algebras. This functor is representable by an analytic space X^{an}, the *analytification* of X, or the *analytic space associated to* X, see [**20**, exposé XII].

A morphism of varieties $X \to Y$ over $\mathbf{C}$ is étale if and only if the morphism of analytic spaces $X^{\mathrm{an}} \to Y^{\mathrm{an}}$ is if it is locally an open immersion, i.e. étale in the analytical sense [*loc. cit*].

As we already mentioned before, any étale morphism $U \to X$ induces an étale morphism $U^{\mathrm{an}} \to X^{\mathrm{an}}$. This induces a morphism of sites $X_{\mathrm{cl}} \to X_{\text{ét}}$, and, by pull back, one may associate to any étale sheaf $\mathcal{F}$ an analytic sheaf $\mathcal{F}^{\mathrm{an}}$ on X_{cl}. The morphism $\Gamma(X_{\text{ét}}, \mathcal{F}) \to \Gamma(X_{\mathrm{cl}}, \mathcal{F}^{\mathrm{an}})$ induces morphisms on the cohomology $H^q(X_{\text{ét}}, \mathcal{F}) \to H^q(X^{\mathrm{an}}_{\mathrm{cl}}, \mathcal{F}^{\mathrm{an}})$. The comparison theorem [**1**, exp. VI, thm 4.1] states that these morphisms are isomorphisms, provided $\mathcal{F}$ is a torsion sheaf, and either $X/\mathbf{C}$ is proper or $\mathcal{F}$ is constructible.

Therefore, étale cohomology provides an extension of Betti cohomology to arbitrary schemes. It coincides for varieties over $\mathbf{C}$. There is one point of caution: The sheaves in question should be torsion sheaves, for non-torsion sheaves, e.g. $\mathcal{F}$ the constant sheaf $\mathbf{Z}$ on a curve [**1**, exp. XI], the comparison theorem is not true. This is why one defines cohomology with $\mathbf{Z}_\ell$-coefficients as a projective limit of cohomology with $\mathbf{Z}/\ell^n\mathbf{Z}$-coefficients, so that one has the comparison theorem on each stage in this projective limit.

2.2. Representations coming from geometry

Let X be a smooth and proper $\mathbf{Q}$-scheme. There exists a number $N \in \mathbf{Z}$ such that X admits a proper and smooth *model* $\mathfrak{X}$ over $S = \mathrm{Spec}(\mathbf{Z}[1/N])$. With this we mean the following data, (1) a proper and smooth morphism $\pi\colon \mathfrak{X} \to S$, and (2) an isomorphism $\mathfrak{X} \times_S \mathbf{Q} \xrightarrow{\sim} X$.

Theorem 9. — *Let X be a smooth and proper $\mathbf{Q}$-scheme. Let $N \in \mathbf{Z}$ be a natural number such that X admits a smooth and proper model over $\mathbf{Z}[1/N]$, then $H^q(X_{\overline{\mathbf{Q}},\text{ét}}, \overline{\mathbf{Q}}_\ell)$ is unramified outside $N \cdot \ell$.*

Proof. — Let p be prime different from ℓ, which does not divide N. Pick a $\overline{\mathbf{Q}}$-prime $\mathfrak{p}$ lying above p and let T be the spectrum of the localisation of $\overline{\mathbf{Z}}$ at $\mathfrak{p}$. Denote with s be the special point of T and with η the generic point of T. Let $\mathcal{F}$ be the constant sheaf $\mathbf{Z}/\ell^n\mathbf{Z}$ on $\mathfrak{X}_T$.

The morphism $\eta\colon \mathfrak{X}_T \to S_T$ is proper and smooth. By smooth base change, the sheaf $(R^q\eta_*)\mathcal{F}$ is locally constant and constructible. Because both the residue field and the fraction field of T are separably closed, we find

$$H^q(\mathfrak{X}_s, \mathbf{Z}/\ell^n\mathbf{Z}) = H^q(\mathfrak{X}_s, \mathcal{F}|_{\mathfrak{X}_s}) = H^q(\mathfrak{X}_{\overline{\mathbf{Q}}}, \mathcal{F}|_{\mathfrak{X}_{\overline{\mathbf{Q}}}}) = H^q(\mathfrak{X}_{\overline{\mathbf{Q}}}, \mathbf{Z}/\ell^n\mathbf{Z}),$$

functorially in $\mathfrak{X}$. Hence, $H^q(\mathfrak{X}_s, \mathbf{Z}/\ell^n\mathbf{Z}) = H^q(\mathfrak{X}_{\overline{\mathbf{Q}}}, \mathbf{Z}/\ell^n\mathbf{Z})$ as $\mathbf{Z}_\ell[\mathcal{G}_{\mathbf{Q}_p}]$-modules. But $\mathcal{I}_{\mathbf{Q}_p}$ acts trivially on $\mathfrak{X}_s$, and so also trivially on $H^q(\mathfrak{X}_{\overline{\mathbf{Q}}}, \mathbf{Z}/\ell^n\mathbf{Z}) = H^q(X_{\overline{\mathbf{Q}}}, \mathbf{Z}/\ell^n\mathbf{Z})$.

In the coming sections we will be deriving, or stating, certain properties of irreducible subquotients V of the spaces $H^q(X_{\overline{\mathbf{Q}},\text{ét}}, \overline{\mathbf{Q}}_\ell)$ as Galois representation, so that eventually we will have enough to give some sort of (conjectural) representation theoretic classification of these representations. This is what the Fontaine-Mazur conjecture is about.

Let V be an irreducible subquotient of $H^q(X_{\overline{\mathbf{Q}},\text{ét}}, \overline{\mathbf{Q}}_\ell)$. Apart from the condition that V is unramified locally at nearly all prime numbers, there is another important condition: V is de Rham at ℓ. In the coming sections we define what this means, but let us first explain by example why more conditions should be imposed on the representations.

We have the following example. Let $a \in \mathbf{Z}_\ell$. Let $\chi_\ell^{(a)}$ be the composition

$$\mathcal{G}_{\mathbf{Q}} \xrightarrow{\chi_\ell} \mathbf{Z}_\ell^\times = \mathbf{F}_\ell^\times \times (1 + \ell\mathbf{Z}_\ell) \longrightarrow 1 + \ell\mathbf{Z}_\ell \xrightarrow{x \mapsto x^a} 1 + \ell\mathbf{Z}_\ell \longrightarrow \mathbf{Z}_\ell^\times, \tag{(II.8)}$$

and let $\overline{\mathbf{Q}}_\ell(\chi_\ell^{(a)})$ be the $\mathcal{G}_{\mathbf{Q}}$-representation with space $\overline{\mathbf{Q}}_\ell$, and $\mathcal{G}_{\mathbf{Q}}$-action given by $\chi_\ell^{(a)}$. The L-factor of $\chi_\ell^{(a)}$ at a prime $p \neq \ell$ is given by $(1 - \iota(\chi_\ell^{(a)}(\hat{p})) \cdot p^{-s})^{-1} \in \mathbf{C}(p^{-s})$. If $a \notin \mathbf{Z}$, then this character does not come from geometry (for the argument, see below). One may believe this, by looking at the L-function,

$$L^{(\iota)}(\chi_\ell^{(a)}, s) = L_\ell^{(\iota)}(\chi_\ell^{(a)}, s) \times \prod_{p \neq \ell} \frac{1}{1 - \iota(\chi_\ell^{(a)}(\hat{p})) \cdot p^{-s}} \qquad \text{(formal product!)}, \tag{(II.9)}$$

which appears to be really bad. If $a \notin \mathbf{Q}$, I expect $\chi_\ell^{(a)}(\hat{p})$ to be transcendental over $\mathbf{Q}$, at least for one prime number p. So in this case, I think the L-function even depends on ι in an awful manner.

We give a complete argument now, but we use notions and theorems that we will define and prove only later. Assume that $\chi^{(a)}$ occurs in the étale cohomology of some proper and smooth $\mathbf{Q}$-scheme. The global Langlands Conjecture 40 is true for one-dimensional representations. Therefore $\chi^{(a)}$ corresponds to an automorphic representation π of $\mathrm{GL}_1(\mathbf{A})$. Such a π is a twist of a continuous homomorphism $\phi'\colon \hat{\mathbf{Z}}^\times \to \mathbf{C}^\times$ with a k-th power of the adelic norm $|\cdot|\colon \mathrm{GL}_1(\mathbf{A}) \to \mathbf{C}^\times$ where $k \in \mathbf{Z}$, see formula (V.22). Under the global Langlands conjecture this automorphic representation corresponds to $\phi \otimes \chi_\ell^k$, where ϕ is obtained from ϕ' by composing with the isomorphism of global class field theory (geometrically normalised). Therefore $\chi^{(a)} = \phi \otimes \chi_\ell^k$. It is easy to check that this implies $a = k$, hence a is an integer.

2.3. Hodge-Tate representations

Let $\mathbf{C}_\ell$ be the completion of $\overline{\mathbf{Q}}_\ell$; it is an algebraically closed field and the Galois action of $\mathcal{G}_{\mathbf{Q}_\ell}$ on $\overline{\mathbf{Q}}_\ell$ extends to $\mathbf{C}_\ell$ by continuity. The *Hodge-Tate ring* $\mathrm{B_{HT}}$ is the $\mathbf{Z}$-graded $\mathbf{C}_\ell$-algebra $\mathbf{C}_\ell[t, 1/t] = \bigoplus_{r \in \mathbf{Z}} \mathbf{C}_\ell \cdot t^r$. The group $\mathcal{G}_{\mathbf{Q}_\ell}$ acts on $\mathbf{C}_\ell \subset \mathrm{B_{HT}}$ via the Galois action, and on $t \in \mathrm{B_{HT}}$ via χ_ℓ, so for all $\sigma \in \mathcal{G}_{\mathbf{Q}_\ell}$ one has $\sigma(t) = \chi_\ell(\sigma)t$.

***Theorem 10* (Tate).** — *For all subfields $K \subset \overline{\mathbf{Q}}_\ell$ containing $\mathbf{Q}_\ell$ the ring $\mathrm{B}_{\mathrm{HT}}^{\mathrm{Gal}(\overline{\mathbf{Q}}_\ell/K)}$ is the completion of K.*

Proof. — [**10**, thm 2.14, p. 31].

Let $\mathbf{D}_{\mathrm{HT}}$ be the functor that associates to an ℓ-adic $\mathcal{G}_{\mathbf{Q}_\ell}$-representation V (over $\overline{\mathbf{Q}}_\ell$) the $\mathbf{Z}$-graded $\overline{\mathbf{Q}}_\ell$-vector spaces $(\mathrm{B_{HT}} \otimes_{\mathbf{Q}_\ell} V)^{\mathcal{G}_{\mathbf{Q}_\ell}}$ with grading defined by $\mathrm{gr}^r \mathbf{D}_{\mathrm{HT}}(V) := (\mathrm{gr}^r \mathrm{B_{HT}} \otimes_{\mathbf{Q}_\ell} V)^{\mathcal{G}_{\mathbf{Q}_\ell}}$.

To an ℓ-adic $\mathcal{G}_{\mathbf{Q}_\ell}$-representation V over E we associate *Hodge-Tate numbers* $HT(V)$. It is the multiset of integers in which an integer $r \in \mathbf{Z}$ occurs with multiplicity $\dim_{\overline{\mathbf{Q}}_\ell} \mathrm{gr}^{-r} \mathbf{D}_{\mathrm{HT}}(V)$.

To avoid confusion, let us define the concept 'multiset'. A *multiset* is a set X together with a map $f \colon X \to \mathbf{Z}_{\geq 0}$. An $x \in X$ is called an *element* and $f(x)$ is the *multiplicity* of x. We will always suppress f from the notation. When we say a multiset of integers, or complex numbers, etc we mean that $X = \mathbf{Z}$, $X = \mathbf{C}$, etc. We say that the multiset has finite cardinality if f has finite support, and in that case we define the cardinality of the multiset as $\sum_{x \in X} f(x)$. A good example for this notion is the *multiset of roots* of a polynomial $f \in K[X]$ over an algebraically closed field K. The advantage is that the cardinality of the multiset of roots is always equal to the degree of the polynomial, whereas the cardinality of the *set of roots* equals to the degree of f if and only if f is separable. Later on in this thesis we will also see multisets of complex numbers encoding in a natural way the $\mathbf{C}$-algebra morphisms of $\mathbf{C}[X_1, \ldots, X_n]^{\mathfrak{S}_n}$ into $\mathbf{C}$ (see Corollary 37).

Remark. — Let X be a proper and smooth $\mathbf{Q}_\ell$-scheme. Consider the $\mathcal{G}_{\mathbf{Q}_\ell}$-representation

$$\mathbf{C}_\ell \otimes_{\mathbf{Q}_\ell} H^n(X_{\overline{\mathbf{Q}}_\ell, \text{ét}}, \mathbf{Q}_\ell), \tag{(II.10)}$$

where $\mathcal{G}_{\mathbf{Q}_\ell}$-acts on $\mathbf{C}_\ell$ via the Galois action, on $H^n(X_{\overline{\mathbf{Q}}_\ell, \text{ét}}, \mathbf{Q}_\ell)$ via pull-back functoriality, and on the tensor product via the diagonal action. Next, consider the $\mathcal{G}_{\mathbf{Q}_\ell}$-representation

$$\bigoplus_{q \in \mathbf{Z}} (\mathbf{C}_\ell(-q) \otimes_{\mathbf{Q}_\ell} H^{n-q}(X, \Omega^q_{X/\mathbf{Q}_\ell})), \tag{(II.11)}$$

where $\mathcal{G}_{\mathbf{Q}_\ell}$ acts on $\mathbf{C}_\ell(-q)$ via the Galois action, twisted by the $(-q)$-th power of the cyclotomic character, trivially on $H^{n-q}(X, \Omega^q_{X/\mathbf{Q}_\ell})$, and on the tensor product via the diagonal action. Faltings theorem states that there is an $\mathbf{Q}_\ell[\mathcal{G}_{\mathbf{Q}_\ell}]$-isomorphism between the representation in (II.10) and the representation in (II.11). This isomorphism is functorial in X. Therefore, the multiplicity of an integer $q \in \mathbf{Z}$ in $HT(H^n(X_{\overline{\mathbf{Q}}_\ell,\text{ét}}, \mathbf{Q}_\ell))$ equals to the dimension of $H^{n-q}(X, \Omega^q_{X/\mathbf{Q}_\ell})$. In particular, if one starts with a proper and smooth $\mathbf{Q}$-scheme, the Hodge-Tate multiset of $H^n(X_{\overline{\mathbf{Q}}_\ell,\text{ét}}, \mathbf{Q}_\ell)$ does not depend on ℓ, and is just the multiset coming from the de Rham cohomology of X.

Let V be an ℓ-adic $\mathcal{G}_{\mathbf{Q}}$-representation. Consider the morphism $\alpha_V : \mathrm{B}_{\mathrm{HT}} \otimes_{\mathbf{Q}_\ell} \mathbf{D}_{\mathrm{HT}}(V) \to \mathrm{B}_{\mathrm{HT}} \otimes_{\mathbf{Q}_\ell} V$, defined by $\lambda \otimes (\sum_{i=1}^n \mu_i \otimes x_i) \mapsto \sum_{i=1}^n \lambda\mu_i \otimes x_i$.

Theorem & Definition 11. — *The map α_V is injective and the following are equivalent:*

1. *The map α_V is surjective;*
2. *The dimension of V equals to the dimension of $\mathbf{D}_{\mathrm{HT}}(V)$;*
3. *There exists a map $\mathrm{B}_{\mathrm{HT}} \otimes_{\mathbf{Q}_\ell} V \to \mathrm{B}_{\mathrm{HT}}^{\dim(V)}$ which is an isomorphism of B_{HT}-modules and $\mathcal{G}_{\mathbf{Q}_\ell}$-sets.*

We call the representation V Hodge-Tate *if one of the above conditions on V is true. Moreover, the property "Hodge-Tate" is stable under tensor products, duals, direct sums and subquotients of ℓ-adic $\mathcal{G}_{\mathbf{Q}_\ell}$-representations.*

Proof. — [**18**, 2.13].

Example. — The continuous morphism $\chi_\ell^{(a)} : \mathcal{G}_{\mathbf{Q}_\ell} \to \mathbf{Z}_\ell^*$ (see formula (II.8)) is Hodge-Tate if and only if $a \in \mathbf{Z}$. This follows from theorem 41.

2.4. de Rham representations

The notion "de Rham representation" is similar to "Hodge-Tate representation", only the de Rham ring B_{dR} is more complicated than B_{HT}. Let us recall briefly its construction. See [**10**, 2.2.2] for more details.

Let $\widetilde{\mathbf{E}}^+ = \varprojlim_{n\in\mathbf{N}} \mathcal{O}_{\mathbf{C}_\ell}/\ell$, as topological ring, with the quotient topology on $\mathcal{O}_{\mathbf{C}_\ell}/\ell$ and with respect to the maps $x \mapsto x^\ell$. The group $\mathcal{G}_{\mathbf{Q}_\ell}$ acts on $\widetilde{\mathbf{E}}^+$. Let $\tilde{\mathbf{A}}^+$ be the ring of Witt vectors of $\widetilde{\mathbf{E}}^+$. The $\mathcal{G}_{\mathbf{Q}_\ell}$-action and the Frobenius automorphism on $\widetilde{\mathbf{E}}^+$ lift to $\tilde{\mathbf{A}}^+$. The ring $\tilde{\mathbf{A}}^+$ is equipped with a surjective morphism of rings $\theta\colon \tilde{\mathbf{A}}^+ \to \mathcal{O}_{\mathbf{C}_\ell}$. This morphism is also $\mathcal{G}_{\mathbf{Q}_\ell}$-equivariant. This ring $\tilde{\mathbf{A}}^+$ also has a Teichmuller character $[\cdot]\colon \widetilde{\mathbf{E}}^+ \to \tilde{\mathbf{A}}^+$. On applying the functor $\square[\frac{1}{\ell}]$ to θ we obtain a morphism $\tilde{\mathbf{A}}^+[\frac{1}{\ell}] \to \mathbf{C}_\ell$, also denoted by θ. The kernel $I \subset \tilde{A}^+[\frac{1}{\ell}]$ of θ is principal and $\mathcal{G}_{\mathbf{Q}_\ell}$-stable; the ring $\mathrm{B}_{\mathrm{dR}}^+$ is the completion of $\tilde{\mathbf{A}}^+[\frac{1}{\ell}]$ with respect to the I-adic topology. The ring B_{dR} is the quotient field of $\mathrm{B}_{\mathrm{dR}}^+$.

The rings $\mathrm{B}_{\mathrm{dR}}^+$ and B_{dR} have the following properties:

$d\mathcal{R}_1$ $\mathrm{B}_{\mathrm{dR}}^+$ is a complete discrete valuation ring;

$d\mathcal{R}_2$ $\mathrm{B}_{\mathrm{dR}}^+$ and B_{dR} are $\overline{\mathbf{Q}}_\ell$-algebras;

$d\mathcal{R}_3$ The group $\mathcal{G}_{\mathbf{Q}_\ell}$ acts on $\mathrm{B}_{\mathrm{dR}}^+$ and with the induced action on B_{dR};

$d\mathcal{R}_4$ Denote with $\mathrm{Fil}^r\mathrm{B}_{\mathrm{dR}}$ the r-th power of the maximal ideal of $\mathrm{B}_{\mathrm{dR}}^+$. Then $\{\mathrm{Fil}^r\mathrm{B}_{\mathrm{dR}} : r \in \mathbf{Z}\}$ make B_{dR} into a $\mathbf{Z}$-filtered ring. We have $\mathrm{Fil}^r\mathrm{B}_{\mathrm{dR}}/\mathrm{Fil}^{r+1}\mathrm{B}_{\mathrm{dR}} = \mathrm{gr}^r\mathrm{B}_{\mathrm{HT}}$ for all $r \in \mathbf{Z}$;

$d\mathcal{R}_5$ For all subfields $K \subset \overline{\mathbf{Q}}_\ell$ containing $\mathbf{Q}_\ell$ the ring $\mathrm{B}_{\mathrm{dR}}^{\mathcal{G}_K}$ is the completion of K.

(These properties do not characterise the ring uniquely.)

Let $\mathbf{D}_{\mathrm{dR}}$ be the functor that associates to an ℓ-adic $\mathcal{G}_{\mathbf{Q}_\ell}$-representation over $\overline{\mathbf{Q}}_\ell$ the $\mathbf{Z}$-filtered $\overline{\mathbf{Q}}_\ell$-vector space $(\mathrm{B}_{\mathrm{dR}} \otimes_{\mathbf{Q}_\ell} V)^{\mathcal{G}_{\mathbf{Q}_\ell}}$ with filtration $\mathrm{Fil}^r \mathbf{D}_{\mathrm{HT}}(V) := (\mathrm{Fil}^r \mathrm{B}_{\mathrm{dR}} \otimes_{\mathbf{Q}_\ell} V)^{\mathcal{G}_{\mathbf{Q}_\ell}}$. Define the morphism $\alpha_V : \mathrm{B}_{\mathrm{dR}} \otimes_{\mathbf{Q}_\ell} \mathbf{D}_{\mathrm{dR}}(V) \to \mathrm{B}_{\mathrm{dR}} \otimes_{\mathbf{Q}_\ell} V$, $\lambda \otimes (\sum_{i=1}^n \mu_i \otimes x_i) \mapsto \sum_{i=1}^n \lambda\mu_i \otimes x_i$.

Theorem & Definition 12. — *The map α_V is injective and the following are equivalent:*

1. *The map α_V is surjective;*
2. *The dimension of V equals to the dimension of $\mathbf{D}_{\mathrm{dR}}(V)$;*
3. *There exists a map $\mathrm{B}_{\mathrm{dR}} \otimes_{\mathbf{Q}_\ell} V \to \mathrm{B}_{\mathrm{dR}}^{\dim(V)}$ which is an isomorphism of B_{dR}-modules and $\mathcal{G}_{\mathbf{Q}_\ell}$-sets.*

We call the representation V de Rham *if one of the above conditions on V is true. Moreover, the property "de Rham" is stable under tensor products, duals, direct sums and subquotients of ℓ-adic $\mathcal{G}_{\mathbf{Q}_\ell}$-representations.*

Proof. — [**18**, 2.13].

Assume V is an ℓ-adic de Rham representation. By property $d\mathcal{R}_4$ we have an exact sequence $0 \to \mathrm{Fil}^{r+1}\mathrm{B}_{\mathrm{dR}} \to \mathrm{Fil}^r \mathrm{B}_{\mathrm{dR}} \to \mathbf{C}_\ell(r) \to 0$. Tensor it with V, take $\mathcal{G}_{\mathbf{Q}_\ell}$-invariants, and sum over all r to obtain an injection

$$\varphi\colon \bigoplus_{r\in\mathbf{Z}} \mathrm{Fil}^r \mathbf{D}_{\mathrm{dR}}(V)/\mathrm{Fil}^{r+1}\mathbf{D}_{\mathrm{dR}}(V) \longrightarrow \bigoplus_{r\in\mathbf{Z}} \mathrm{gr}^r \mathbf{D}_{\mathrm{HT}}(V) = \mathbf{D}_{\mathrm{HT}}(V).$$

The map φ is a surjection because the left hand side has dimension $\dim V$, and $\dim \mathbf{D}_{\mathrm{HT}}(V) \leq \dim(V)$.

We conclude that the diagram of functors

$$\begin{array}{ccc} \{\ell\text{-adic de Rham } \mathcal{G}_{\mathbf{Q}_\ell}\text{-representations over } \overline{\mathbf{Q}}_\ell\} & \xrightarrow{\mathbf{D}_{\mathrm{dR}}} & \{\mathbf{Z}\text{-filtered } \overline{\mathbf{Q}}_\ell\text{-vector spaces}\} \\ & \searrow^{\mathbf{D}_{\mathrm{HT}}} & \downarrow \mathrm{gr} \\ & & \{\mathbf{Z}\text{-graded } \overline{\mathbf{Q}}_\ell\text{-vector spaces}\} \end{array}$$

is essentially commutative. In particular we have the following proposition.

Proposition 13. — *Let V be a de Rham representation of $\mathcal{G}_{\mathbf{Q}_\ell}$, then V is Hodge-Tate.*

Theorem 14. — *Let V be an ℓ-adic $\mathcal{G}_{\mathbf{Q}}$-representation isomorphic to a subquotient of the étale cohomology of a smooth proper $\mathbf{Q}$-scheme, then V is de Rham at ℓ.*

Proof. — The proof of this theorem is difficult. First one needs the results of Tsuji [**40, 41**], proving the conjecture C_{st} of Fontaine which states the following. Let K be a finite extension of $\mathbf{Q}_\ell$, and Y a proper and semi-stable scheme over $\mathcal{O}_K$, then $H^i(Y_{\overline{K},\text{ét}}, \mathbf{Q}_\ell)$ is a semi-stable representation of $\mathrm{Gal}(\overline{K}/K)$. For the notion semi-stable morphism of schemes, see [**3**, 1.3], for the notion semi-stable Galois representation, see [**11**] or [**18**].

Next, one turns to the conjecture C_{pst}, also of Fontaine. This conjecture states that if X is a proper and smooth K-scheme, then $H^i(X_{\overline{K},\text{ét}}, \overline{\mathbf{Q}}_\ell)$ is a potentially semi-stable representation. A Galois representation is potentially semi-stable, when it is semi-stable when restricted to an open subgroup of the absolute Galois group, see [**11**] or [**18**]. To prove this conjecture, it suffices that for each X in the conjecture C_{pst} there exists a Y as in the conjecture C_{st} such that X is the generic fiber of Y. This follows from a result of de Jong, [**25**, thm 4.5].

Finally, one applies another conjecture of Fontaine, stating that an ℓ-adic $\mathcal{G}_{\mathbf{Q}_\ell}$-representation is de Rham if and only if it is potentially semi-stable. This conjecture is proved in [**18**, section 6.5.2, prop A on page 164] and [**2**].

Definition 15. — Let V be an ℓ-adic $\mathcal{G}_{\mathbf{Q}}$-representation, unramified for nearly all primes, and de Rham at ℓ. Then V is called *geometric*.

Notice that we now have two notions of geometric, namely, "geometric" as in the above definition, and "coming from geometry".

2.5. The Fontaine-Mazur conjecture

Conjecture 16. — *Let V be an irreducible geometric $\mathcal{G}_{\mathbf{Q}}$-representation. There exists a smooth projective $\mathbf{Q}$-scheme X, integers $i, r \in \mathbf{Z}$ with $i \geq 0$ such that V is isomorphic to an irreducible subquotient of $H^i(X_{\overline{\mathbf{Q}},\text{ét}}, \overline{\mathbf{Q}}_\ell(r))$.*

We only deal with the base field $K = \mathbf{Q}$ in this thesis, to simplify matters. All of the above is also formulated for general number fields. After 'the de Rham' condition has been defined for all p-adic fields, the Fontaine-Mazur conjecture states that any ℓ-adic $\mathcal{G}_K$-representation which is de Rham locally at all primes dividing ℓ and unramified for nearly all primes, occurs as a subquotient in the étale cohomology of some smooth projective K-scheme, maybe twisted with some power of the cyclotomic character. Assume the conjecture is true for $\mathbf{Q}$, let us indicate how it can be deduced for a number field K. Let V be an irreducible geometric $\mathcal{G}_K$-representation, then $V' = \mathrm{Ind}_{\mathcal{G}_K}^{\mathcal{G}_{\mathbf{Q}}}(V)$, is a geometric $\mathcal{G}_{\mathbf{Q}}$-representation, so V' occurs in $H^i(X_{\overline{\mathbf{Q}},\text{ét}}, \overline{\mathbf{Q}}_\ell(r))$ for some smooth projective $\mathbf{Q}$-scheme X. By Frobenius duality, the representation V' contains the representation V, so V occurs in the cohomology of $X \times_{\mathbf{Q}} K$.

Let $\rho\colon \mathcal{G}_{\mathbf{Q}} \to \mathrm{GL}(V)$ be an irreducible ℓ-adic $\mathcal{G}_{\mathbf{Q}}$-representation over $\overline{\mathbf{Q}}_\ell$ with finite image, then ρ factors over $G := \mathrm{Gal}(K/\mathbf{Q})$ where $K/\mathbf{Q}$ is some finite Galois extension. One has $H^0(\mathrm{Spec}(K)_{\overline{\mathbf{Q}},\text{ét}}, \overline{\mathbf{Q}}_\ell) \cong \overline{\mathbf{Q}}_\ell[G]$ as G-representations, and $\overline{\mathbf{Q}}_\ell[G]$ decomposes as the direct sum of all (modulo isomorphism) irreducible G-representations. So the conjecture is true for representations with finite image.

The abelian case of the conjecture is also true. Any geometric irreducible abelian representation is of the form $\chi_\ell^r \otimes \phi$, where $\phi\colon \mathcal{G}_{\mathbf{Q}} \to \overline{\mathbf{Q}}_\ell$ is a continuous morphism with finite image and $r \in \mathbf{Z}$ (Theorem 41). Note that both these instances have only little to do with the general statement. To illustrate this: in both cases the $\mathbf{Q}$-scheme can be chosen to be finite over $\mathbf{Q}$.

Chapter 3: Smooth Representations

In the previous chapters we introduced the Galois side of the Langlands correspondence, in the following two we will explain the automorphic side.

3.1. Restricted products

Let Σ be an index set and assume we are given for each $v \in \Sigma$ a locally compact group G_v, a finite set $\Sigma_\infty \subset \Sigma$, and for all $v \in \Sigma \backslash \Sigma_\infty$ a maximal compact open subgroup $K_v \subset G_v$.

For each finite subset $S \subset \Sigma$ with $\Sigma_\infty \subset S$ we set $G(S) = \prod_{v \in \Sigma \backslash S} K_v \times \prod_{v \in S} G_v$, as topological groups with the product topology. The set of the $G(S)$ when S varies, together with the inclusions $G(S) \subset G(S')$, is an inductive system of topological groups. The inductive limit $\prod'_{v \in \Sigma} G_v := \varinjlim_S G(S)$ of this system is the *restricted product* of the G_v with respect to the K_v. The group $G := \prod'_{v \in \Sigma} G_v$ is locally compact.

Note that for every S we have an inclusion $G(S) \to \prod_{v \in \Sigma} G_v$, and thus an inclusion $G \to \prod_{v \in \Sigma} G_v$. The image of this inclusion is the set of $(x_v)_{v \in \Sigma} \in \prod_{v \in \Sigma} G_v$ with $x_v \in K_v$ for nearly all v. It is important to know that the topology on G is not necessarily the topology induced from the inclusion $G \to \prod_{v \in \Sigma} G_v$.

If $\Omega \subset \Sigma$ is a finite subset, then

((III.12))

$$\prod_{v \in \Omega} G_v \times \prod'_{v \in \Sigma \backslash \Omega} G_v = \prod_{v \in \Omega} G_v \times \varinjlim_{S \subset (\Sigma \backslash \Omega)} G(S) = \varinjlim_{S \subset (\Sigma \backslash \Omega)} \left(\prod_{v \in \Omega} G_v \right) \times G(S) = G.$$

In particular, if Σ is finite, we see that finite restricted products are just products.

In case the groups G_v are topological rings and K_v compact open subrings, then G is a topological ring. The *ring of adèles* $\mathbf{A}$ of $\mathbf{Q}$ is the restricted product of the rings $\mathbf{Q}_v$ over all places v with respect to the subrings $\mathbf{Z}_v \subset \mathbf{Q}_v$ for finite v. The *ring of finite adèles* of $\mathbf{Q}$, $\mathbf{A}^\infty$, is the same product, but taken over the *finite* places. The group $\mathrm{GL}_n(\mathbf{A})$ is the restricted product $\prod'_{\text{all } v} \mathrm{GL}_n(\mathbf{Q}_v)$ with respect to the subgroups $\mathrm{GL}_n(\mathbf{Z}_v) \subset \mathrm{GL}_n(\mathbf{Q}_v)$ for finite v.

We return to the general case, where the G_v are locally compact groups. Let Π_v be for each $v \in \Sigma$ a complex vector space together with a representation $G_v \to \mathrm{GL}(\Pi_v)$. Let $S \subset \Sigma$ be a finite subset with $\Sigma_\infty \subset S$ and define $T(S) = \bigotimes_{v \in \Sigma \backslash S} \Pi_v^{K_v} \otimes \bigotimes_{v \in S} \Pi_v$, then $T(S)$ is a $G(S)$-representation. Define $T := \varinjlim_S T(S)$ where S ranges over all finite subsets $S \subset \Sigma$ with $\Sigma_\infty \subset S$. Then, for each S, $G(S)$ acts on T. By passage to the inductive limit, G acts on T. We define the *restricted tensor product* $\bigotimes'_v \Pi_v$ as the space T together with the morphism $G \to \mathrm{GL}(T)$ defined above.

If $\Pi_v^{K_v}$ is 0 for infinitely many v, then ${\bigotimes'}_v \Pi_v = 0$. By definition, a G_v-representation Π_v is called *unramified* if $\Pi_v^{K_v} \neq 0$. When forming a restricted tensor product, we will usually assume that Π_v is unramified for nearly all v.

If $\Omega \subset \Sigma$ is a finite subset then

$$\bigotimes_{v\in\Omega}\Pi_v \otimes \bigotimes'_{v\in\Sigma\setminus\Omega}\Pi_v = \bigotimes_{v\in\Omega}\Pi_v \otimes \varinjlim_{S\subset\Sigma\setminus\Omega}\left(\bigotimes_{v\in\Sigma\setminus(S\cup\Omega)}\Pi_v^{K_v}\otimes\bigotimes_{v\in S}\Pi_v\right)$$

$$= \varinjlim_{S\subset\Sigma\setminus\Omega} T(\Omega\cup S) = \bigotimes'_{v\in\Sigma}\Pi_v. \tag{(III.13)}$$

In particular, if Σ is finite then the restricted tensor product is a usual product.

Proposition 17. — *Let Π_v (resp. Π'_v) for each $v \in \Sigma$ be an irreducible G_v-representation, unramified for nearly all v, such that $\bigotimes'_{v\in\Sigma}\Pi_v \cong \bigotimes'_{v\in\Sigma}\Pi'_v$. Then $\Pi_v \cong \Pi'_v$ for every $v \in \Sigma$.*

Proof. — By equation (III.13), it suffice to do the case where $\Sigma = \{1,2\}$ contains only two elements, so a classical tensor product and a classical product of groups. There is a non-zero $G(1)$-composition $\Pi_1 \to \Pi_1^{(\dim\Pi_2)} \xrightarrow{\sim} \Pi_1'^{(\dim\Pi'_2)} \to \Pi'_1$, hence $\Pi_1 \cong \Pi'_1$. The case $i = 2$ is left to the reader.

Remark. — Usually in the literature the following alternative definition of restricted tensor product is used. Let $\Sigma, \Sigma_\infty, G_v, K_v, \Pi_v$ be as above. Let moreover be given for nearly all $v \in \Sigma\setminus\Sigma_\infty$, an invariant vector $x_v \in \Pi_v^{K_v}$. Then $\bigotimes'_v \Pi_v$ can also be defined as the subspace of the full tensor product $\bigotimes_v \Pi_v$ generated by elementary tensors of the form $\bigotimes_v y_v$ with $y_v = x_v$ for nearly all v. In case $\dim\Pi_v^{K_v} = 1$ for nearly all v, this notion of restricted tensor product coincides with the notion of restricted tensor product defined above, in the sense that the two objects are G-isomorphic. For the groups $\mathrm{GL}_n(\mathbf{Q}_p)$, it is true that $\dim\Pi_p^{\mathrm{GL}_n(\mathbf{Z}_p)} \in \{0,1\}$, for any irreducible smooth-admissible $\mathrm{GL}_n(\mathbf{Q}_p)$-representation Π_p. As we will see, this follows from the fact that the Hecke algebra $\mathcal{H}(\mathrm{GL}_n(\mathbf{Q}_p) : \mathrm{GL}_n(\mathbf{Z}_p))$ is commutative.

The ultimate generality in which one wants to work with restricted tensor products is when G is a reductive algebraic group over a global field F, and where $G_v = G(F_v)$ for all v. One lets $\mathcal{O}_F$ be the ring of integers of F in case it is a number field; in case F has characteristic p, one picks a random F-place v and one lets $\mathcal{O}_F$ be the ring of $x \in F$ which have non-negative valuation for all F-places $w \neq v$. Then G has a model $\tilde{G}$ over some open subscheme U of $\mathrm{Spec}(\mathcal{O}_F)$. One may now define $K_v = \tilde{G}(F_v)$ for nearly all F-places $v \in U$.

I expect (but cannot prove) that $\mathcal{H}(G_v : K_v)$ is commutative for nearly all v, in the above described generality. If this is so, then the two notions of tensor products will coincide for restricted tensor products of irreducible, smooth-admissible G_v-representations.

In case this is not so for a certain reductive group over F, problems with both definitions will appear. With the definition of restricted tensor product that we gave, I think it is not true (in general) that the restricted tensor product of smooth-admissible representations is again smooth-admissible. With the definition that is given in the literature, I think the isomorphism class of a restricted tensor product will (in general) depend on the choice of the vectors x_v.

In this thesis, we will quickly start assuming $G_p = \mathrm{GL}_n(\mathbf{Q}_p)$ and $K_p = \mathrm{GL}_n(\mathbf{Z}_p)$, which avoids all of the above problems and suffices for the applications we have in mind.

3.2. Locally profinite groups

Let G be a totally disconnected locally compact topological group, then G is called *locally profinite*. The next lemma explains the terminology.

Lemma 18. — *Let G be a topological group. Then G is locally profinite group if and only if there exists a profinite open subgroup $K \subset G$.*

Proof. — The implication "$\Leftarrow$" is true because profinite groups are totally disconnected and compact. For the other implication, assume G is totally disconnected and locally compact. Let $U_1 \subset G$ be open such that $1 \in U_1$ and the closure $\overline{U}_1 \subset G$ in G is compact. Because G is totally disconnected there exists a closed and open subset $U_2 \subset U_1$, with $1 \in U_2$. Because $\overline{U}_1$ is compact, it follows that U_2 is compact. Therefore, $U_2 \subset G$ is compact open and contains 1. We define:

$$K = \{g \in U_2 | \forall h \in U_2 : g^{-1}h \in U_2\}.$$

We claim that K is a profinite open subgroup of G. Let us verify this. If $g, g' \in K$ and $h \in U_2$ then $(gg')^{-1}h = (g')^{-1}g^{-1}h \in U_2$, so $gg' \in K$. Because also $1 \in K$, we see that K is a subgroup of G. Let us prove that K is also open in G. Define

$$\tilde{m} \colon U_2 \times U_2 \longrightarrow G, \quad (g, h) \mapsto g^{-1}h.$$

Let $p_2 \colon U_2 \times U_2 \to U_2, (g, h) \mapsto h$ be the projection on the second coordinate. Then p_2 is closed and open. If $S \subset G$ is a subset, then we denote with $S^{\complement}$ the complement of S in G. We have the equality

$$p_2\tilde{m}^{-1}(U_2^{\complement}) = \{g \in U_2 | \exists h \in U_2 : g^{-1}h \notin U_2\},$$

so $K = U_2 \cap (p_2\tilde{m}^{-1}(U_2^{\complement}))^{\complement}$ is open in G.

The group K is a closed and contained in U_2, so K is compact. We conclude that K is profinite and open in G.

Example. — Let F be a non-Archimedian local field and X a finite type F-scheme. Then $X(F)$ is locally compact and totally disconnected. In particular if $\underline{G}$ is a finite type F-group scheme, then $\underline{G}(F)$ is a locally profinite group. The Weil group (see Section 7.1) of F is locally profinite.

Let Σ be an index set, G_v a locally profinite group for every $v \in \Sigma$, and $K_v \subset G_v$ a compact open subgroup in G_v for all $v \in \Sigma$. Then $\prod_v K_v$ is a profinite open subgroup of $\prod'_v G_v$, therefore the restricted product of locally profinite groups is again a locally profinite group.

Therefore, if $\underline{G}$ is a finite type $\mathbf{Q}$-group scheme, then $\underline{G}(\mathbf{A}^\infty)$ is a locally profinite group.

3.3. Smooth representations

Let G be a locally profinite group and Π be a complex vector space, together with a representation $G \to \mathrm{GL}(\Pi)$. Then Π is *smooth* as G-representation if for every $v \in \Pi$ the stabiliser $\mathrm{stab}(v) \subset G$ is open. The smooth representation Π is *smooth-admissible* if for all open subgroups $H \subset G$ the space of invariants Π^H is finite dimensional. A *morphism* of smooth or smooth-admissible G-representations is a G-equivariant morphism of vector spaces.

Lemma 19. — *Let G_1 and G_2 two groups and Π_1 (resp. Π_2) a complex G_1 (resp. G_2) representation. Then $(\Pi_1 \otimes \Pi_2)^{G_1 \times G_2} = \Pi_1^{G_1} \otimes \Pi_2^{G_2}$.*

Proof. — Consider the exact sequence

$$0 \longrightarrow \Pi_1^{G_1} \longrightarrow \Pi_1 \longrightarrow \bigoplus_{g \in G_1} \Pi_1, \qquad \text{((III.14))}$$

where the last map in this exact sequence is given by $(x \mapsto (x - gx)_{g \in G_1})$. Tensor the sequence (III.14) with Π_2 and use $\left(\bigoplus_{g\in G_1} \Pi_1\right) \otimes \Pi_2 = \bigoplus_{g \in G_1} (\Pi_1 \otimes \Pi_2)$, to find the exact sequence

$$0 \longrightarrow \Pi_1^{G_1} \otimes \Pi_2 \longrightarrow \Pi_1 \otimes \Pi_2 \longrightarrow \bigoplus_{g \in G_1} (\Pi_1 \otimes \Pi_2)\,.$$

The last map in this sequence is given by $(x \mapsto (x-gx)_{g \in G_1})$. Therefore, $(\Pi_1 \otimes \Pi_2)^{G_1} = \Pi_1^{G_1} \otimes \Pi_2$. Moreover,

$$(\Pi_1 \otimes \Pi_2)^{G_1 \times G_2} = \left((\Pi_1 \otimes \Pi_2)^{G_1}\right)^{G_2} = (\Pi_1^{G_1} \otimes \Pi_2)^{G_2},$$

and, by the same argument as above,

$$(\Pi_1^{G_1} \otimes \Pi_2)^{G_2} = \Pi_1^{G_1} \otimes \Pi_2^{G_2}.$$

Proposition 20. — *Let A and B be two unitary associative algebras over an algebraically closed field Ω. Let M resp. N be a simple module over A resp. B. Then $M \otimes N$ is simple as $A \otimes B$-module. Furthermore, let $\mathcal{C}(A)$ (resp. $\mathcal{C}(B)$, $\mathcal{C}(A \otimes B)$) be the set of isomorphism classes of simple modules over A (resp. B, $A \otimes B$), which are finite dimensional over Ω. The map $\mathcal{C}(A) \times \mathcal{C}(B) \to \mathcal{C}(A \otimes B)$, $(M, N) \mapsto M \otimes N$ is a bijection.*

Proof. — [**5**, p. 94, chap. 8].

We need to fix some notations. Let $(\Sigma, \Sigma_\infty, G_v, K_v, \Pi_v)$ be given by

$$\begin{aligned}
\Sigma &:= \text{an index set;}\\
\Sigma_\infty &:= \emptyset;\\
G_v &:= \text{a locally profinite group for all } v \in \Sigma;\\
K_v &:= \text{a maximal compact open subgroup of } G_v \text{, for all } v \in \Sigma;\\
\Pi_v &:= \text{a complex smooth } G_v\text{-representation for all } v \text{, unramified for nearly all } v;\\
G &:= \prod\nolimits'_{v \in \Sigma} G_v;\\
G_S &:= \prod\nolimits'_{v \in \Sigma \setminus S} G_v \quad \text{for } S \subset \Sigma \text{ subset;}\\
G^S &:= \prod\nolimits'_{v \in \Sigma \setminus S} G_v \quad \text{for } S \subset \Sigma \text{ subset;}\\
K &:= \prod\nolimits_{v \in S} K_v \quad \text{for } S \subset \Sigma \text{ subset.}
\end{aligned}$$

((III.15))

Proposition 21. — *Let* $(\Sigma, G_v, K_v, \Pi_v)$ *be as in (*III.*15). The restricted tensor product* ${\bigotimes'}_v \Pi_v$ *is smooth.*

Proof. — Recall that, in a topological group, to show that a subgroup H of G is open, it suffices to show that it contains an open subgroup of G. Let $x \in \bigotimes'_v \Pi_v$. Write x as a finite sum of elementary tensors. The stabiliser of x in G contains the intersection of the stabilisers of these elementary tensors. Therefore, we may assume that $x = \bigotimes_v x_v$ is an elementary tensor. We have $\mathrm{stab}(x) = \prod'_v \mathrm{stab}(x_v) \subset G$. Because $\mathrm{stab}(x_v) \subset G_v$ is open and $\mathrm{stab}(x_v) \supset K_v$ for nearly all v, this subgroup $\mathrm{stab}(x) \subset G$ is open.

Lemma 22. — *Let* $(\Sigma, G_v, K_v, \Pi_v)$ *be as in (*III.*15). Let, moreover* $H_v \subset G_v$ *for each* v *be an open subgroup such that* $H_v = K_v$ *for nearly all* v*. Then* $\left(\bigotimes'_v \Pi_v\right)^H$ *is the inductive limit over all* $S \subset \Sigma$ *of the spaces* $\bigotimes_{v\in\Sigma\setminus S} \Pi_v^{K_v} \otimes \bigotimes_{v\in S} \Pi_v^{H_v}$.

Proof. — The functor $\square^H$ is left exact, so it commutes with inductive limits and we have $\left(\bigotimes'_v \Pi_v\right)^H = \varinjlim_S \left(\bigotimes_{v\in\Sigma\setminus S} \Pi_v^{K_v} \otimes \bigotimes_{v\in S} \Pi_v\right)^H$. If S contains all $v \in \Sigma$ such that $H_v \neq K_v$, then, by the Lemma 19

$$\left(\bigotimes_{v\in\Sigma\setminus S} \Pi_v^{K_v} \otimes \bigotimes_{v\in S} \Pi_v\right)^H = \bigotimes_{v\in\Sigma\setminus S} \Pi_v^{K_v} \otimes \left(\bigotimes_{v\in S} \Pi_v\right)^{\prod_{v\in S} H_v} = \bigotimes_{v\in\Sigma\setminus S} \Pi_v^{K_v} \otimes \bigotimes_{v\in S} \Pi_v^{H_v}.$$

This completes the proof.

Proposition 23. — *Let* $(\Sigma, G_v, K_v, \Pi_v)$ *be as in (*III.*15). Assume furthermore that* $\dim \Pi_v^{K_v} \leq 1$ *for nearly all* v*, and that* Π_v *is* G_v*-smooth-admissible for all* v*. Then* ${\bigotimes'}_v \Pi_v$ *is a smooth-admissible* G*-representation.*

Proof. — Let $H' \subset G$ be an open subgroup. There exists a subgroup $H \subset H'$ such that $H \subset G$ is of the form $\prod'_v H_v$ with $H_v \subset G_v$ open for every $v \in \Sigma$ and $H_v = K_v$ for nearly all $v \in \Sigma$. We have $\Pi^{H'} \subset \Pi^H$, so it suffices to show that Π^H is finite dimensional. By Lemma 22 we have $\Pi^H \cong \bigotimes'_v \Pi_v^{H_v}$, so $\dim \Pi^H = \prod_v \dim \Pi_v^{H_v}$ is finite.

Theorem 24. — *Let* $(\Sigma, G_v, K_v, \Pi_v)$ *be as in (*III.*15). Assume furthermore that* Π_v *is* G_v*-smooth-admissible for all* v*, and that* $\dim \Pi_v^{K_v} = 1$ *for nearly all* v*. The restricted tensor product* $\Pi = \bigotimes'_v \Pi_v$ *is* G*-irreducible if and only if* Π_v *is* G_v*-irreducible for all* v*.*

Proof. — Assume that Π_v has a non-trivial G_v-invariant subspace $V \subset \Pi_v$ for some $v \in \Sigma$. Then Π has admits $V \otimes {\bigotimes'}_{w\neq v} \Pi_w$ as non-trivial subspace, so it is reducible. This proves the implication "⇒".

The other implication follows from Proposition 20 in case $\#\Sigma = 2$. By induction, the statement follows follows for finite Σ. The remaining case, $\#\Sigma = \infty$, is now easy. Assume Π_v is G_v-irreducible for every $v \in \Sigma$. Let $x, y \in \bigotimes'_v \Pi_v$. We show that there exists a $\lambda \in \mathbf{C}[G]$ such that $\lambda x = y$. Let $S \subset \Pi$ be a finite set such that $x, y \in \bigotimes_{v\in\Sigma\setminus S} \Pi_v^{K_v} \otimes \bigotimes_{v\in S} \Pi_v =: T(S)$ and $\dim \Pi_v^{K_v} = 1$ for all $v \in \Sigma\setminus S$. The space $T(S)$ is an irreducible representation of the group $G_S = \prod_{v\in S} G_v$. We may now find a $\lambda' \in \mathbf{C}[G_S]$ such that $\lambda' x = y$. Recall $G(S) = G_S \times \prod_{v\in\Sigma\setminus S} K_v$. The canonical surjection $G(S) \to G_S$ induces a surjection on the group algebras $\mathbf{C}[G(S)] \to \mathbf{C}[G_S]$. Take any λ in $\mathbf{C}[G(S)] \subset \mathbf{C}[G]$ mapping to λ'. For this λ we have $\lambda x = y$.

3.4. Hecke algebras and restricted tensor products

In the previous section we took restricted tensor products of G_v-representations to obtain a G-representation. We will now do the inverse, i.e. we want to go from a representation Π of G to representations Π_v of the G_v such that $\Pi \cong \bigotimes'_{v\in\Sigma} \Pi_v$. In general this is not possible: We need to impose strong conditions on the G_v and on the Π_v. At the end we will have to verify that the groups $\mathrm{GL}_n(\mathbf{Q}_p)$ satisfy those properties.

There are two technical hypotheses that we will impose on our locally profinite groups G considered:

(**Hyp**) *For all (equivalently, one) open profinite subgroups $K \subset G$ the index $|G/K|$ is countable, and any left Haar measure on G is also a right Haar measure on G (unimodular, see* [**23**].*).*

For any affine algebraic group $\underline{G}$ over a non-Archimedean local field F the group $\underline{G}(F)$ has a countable index open profinite subgroup. To see this, pick a closed immersion $\underline{G} \hookrightarrow \mathrm{GL}_n$, then pull back the subgroup $\mathrm{GL}_n(\mathcal{O}_F) \subset \mathrm{GL}_n(F)$ in $\underline{G}(F)$. In general, the group $\underline{G}(F)$ is not unimodular (for example the F-points of the Borel group $\left(\begin{smallmatrix} * & * \\ & * \end{smallmatrix}\right) \subset \mathrm{GL}_2$). The group $\mathrm{GL}_n(F)$ is unimodular.

Assume G is locally profinite and satisfies (Hyp). Fix a Haar measure μ on G. Let $\mathcal{H}(G)$ be the set of locally constant complex valued functions $G \to \mathbf{C}$ with compact support. If $f, g \in \mathcal{H}(G)$ are two such functions, then their sum $f+g$ is defined by $x \mapsto f(x)+g(x)$, and their product is the convolution:

$$f * h := \left(g \mapsto \int_G f(x)h(x^{-1}g)\mathrm{d}\mu(x) \right). \qquad \text{((III.16))}$$

Together with the operations $+$, $*$, and scalar multiplication with elements $c \in \mathbf{C}$, the set $\mathcal{H}(G)$ is an associative $\mathbf{C}$-algebra. The algebra $\mathcal{H}(G)$ has no unit if G is not compact, usually $\mathcal{H}(G)$ is not commutative.

Let Π be a smooth G-representation, and $\pi\colon G \to \mathrm{GL}(\Pi)$ the structural morphism. Then we can let $\mathcal{H}(G)$ act on Π in the following manner. Let $x \in \Pi$ and $h \in \mathcal{H}(G)$. Pick a compact open subgroup $K \subset G$ which fixes v and h (under right translation). We define:

$$h \cdot v := \mu(K) \cdot \sum_{g \in G/K} h(g)\pi(g)x, \qquad \text{((III.17))}$$

(in Lemma 25 we check that this makes sense). The sum in equation (III.17) is usually written as an integral:

$$h \cdot v = \int_G h(g)gx\mathrm{d}\mu(g). \qquad \text{((III.18))}$$

Lemma 25. — *In equation (*III.*17), the number of $g \in G/K$ such that $h(g)\pi(g)x \neq 0$ is finite. Moreover, the sum $\mu(K)\sum_{g\in G/K} h(g)\pi(g)x$ does not depend on the choice of K.*

Proof. — The mapping

$$G \longrightarrow V, \quad g \mapsto h(g)\pi(g)x,$$

has compact support because h has compact support. Therefore, the sum in equation (III.17) is finite.

Let us prove the second statement. Assume K' is another compact open subgroup of G fixing x and f. Then $K' \cap K$ also fixes x and f, so we may assume $K' \subset K$. Let $g \in G/K$. If $g' \in G/K'$ equals to g modulo K, then $h(g')\pi(g')x = h(g)\pi(g)x$. For each g, there are exactly $n = [K : K']$ such g', so

$$\sum_{g' \in G/K'} h(g')\pi(g')x = n \sum_{g \in G/K} h(g)\pi(g)x.$$

Note $\mu(K) = n\mu(K')$, so we conclude

$$\mu(K) \cdot \sum_{g \in G/K} h(g)\pi(g)x = \mu(K') \cdot \sum_{g \in G/K'} h(g')\pi(g')x.$$

This completes the proof.

If $\Pi \to \Pi'$ is a morphism of G-representations, then it is also a morphism of $\mathcal{H}(G)$-modules with respect to the module structure on Π, Π' defined in equation (III.17). Thus Π is naturally an $\mathcal{H}(G)$-module. We call an $\mathcal{H}(G)$-module *smooth* if $\mathcal{H}(G) \cdot \Pi = \Pi$. Any $\mathcal{H}(G)$-module obtained from a smooth G-representation is a smooth $\mathcal{H}(G)$-module.

If $H \subset G$ is compact open, then we write $e_H \in \mathcal{H}(G)$ for the locally constant function taking value $\frac{1}{\mu(H)}$ on H, and 0 on $H^{\complement} \subset G$. We define $\mathcal{H}(G : H) := e_H * \mathcal{H}(G) * e_H$. This algebra is the endomorphism ring of the functor $\Pi \mapsto \Pi^H$ from the category of smooth G-representations to the category of complex vector spaces. More precisely, if $h \in \mathcal{H}(G : H)$, then h acts on Π^H for all smooth G-representations Π, and it does so functorially in Π: If $\Pi_1 \to \Pi_2$ is a morphism of smooth G-representations, then $\Pi_1^H \to \Pi_2^H$ is h-equivariant. Therefore we have a morphism $\mathcal{H}(G : H)$ to the endomorphism ring of the functor $\Pi \to \Pi^H$. This morphism is an isomorphism.

The algebra $\mathcal{H}(G : H)$ is also equal to the algebra of functions $f \colon H\backslash G/H \to \mathbf{C}$ with finite support, where the product of two such functions is defined by the convolution integral (see (III.16)).

Theorem 26. — *Let G be a locally profinite group satisfying (Hyp).*

1. *The category of smooth G-representations is isomorphic to the category of smooth $\mathcal{H}(G)$-modules.*
2. *Let $H \subset G$ be a compact open subgroup. Then $\Pi \mapsto \Pi^H$ induces a bijection between the set of equivalence classes of smooth irreducible G-representations with $\Pi^H \neq 0$, and the set of isomorphism classes of simple $\mathcal{H}(G : H)$-modules.*

Proof. — [**7**, section 1.4].

Lemma 27. — *Let G_1 and G_2 be two locally profinite groups satisfying (Hyp). Write $G = G_1 \times G_2$, and let Π be an irreducible smooth-admissible G-representation. There exist an irreducible smooth-admissible G_i-representation Π_i for $i = 1, 2$ such that $\Pi \cong \Pi_1 \otimes \Pi_2$. The Π_i are unique up to (non-canonical) isomorphism.*

Proof. — Let $K = K_1 \times K_2 \subset G$ a compact open subgroup, such that $\Pi^K \neq 0$. Then Π^K is finite dimensional over $\mathbf{C}$ and $\mathcal{H}(G : K)$-simple by Theorem 26. Note that $\mathcal{H}(G : K) = \mathcal{H}(G_1 : K_1) \otimes \mathcal{H}(G_2 : K_2)$, so we may apply Proposition 20, to find (up to isomorphism unique) simple $\mathcal{H}(G_i : K_i)$-modules $\Pi_i^{(K_i)}$ for $i = 1, 2$ and an isomorphism $a_K \colon \Pi^K \overset{\sim}{\to} \Pi_1^{(K_1)} \otimes \Pi_2^{(K_2)}$. Pick $K' \subset K$ another compact open in G, and let $b_{K,K'} \colon \Pi_1^{(K_1)} \otimes \Pi_2^{(K_2)} \to \Pi_1^{(K_1')} \otimes \Pi_2^{(K_2')}$ be the unique $\mathcal{H}(G : K)$-morphism making the diagram

((III.19))
$$\begin{array}{ccc} \Pi^K & \xrightarrow{a_K} & \Pi_1^{(K_1)} \otimes \Pi_2^{(K_2)} \\ \downarrow & & \downarrow b_{K,K'} \\ \Pi^{K'} & \xrightarrow{a_{K'}} & \Pi_1^{(K_1')} \otimes \Pi_2^{(K_2')} \end{array}$$

commute. We have $b_{K,K'} = b_1(K_1, K_1') \otimes b_2(K_2, K_2')$ for certain $\mathcal{H}(G_i : H_i)$-morphisms $b_i(K_i, K_i') \colon \Pi_i^{(K_i)} \to \Pi_i^{(K_i')}$. To see this, we have

$$\Pi_1^{(K_1)} \otimes \Pi_2^{(K_2)} \xrightarrow{\sim} \Pi^K = (\Pi^{K'})^K \xrightarrow{\sim} (\Pi_1^{(K_1')} \otimes \Pi_2^{(K_2')})^K = (\Pi_1^{(K_1')})^{K_1} \otimes (\Pi_2^{(K_2')})^{K_2}.$$

On comparing the left and right hand side, we get from Theorem 20 isomorphisms $\Pi_i^{(K_i)} \overset{\sim}{\to} (\Pi_i^{(K_i')})^{K_i}$ and by composing with the embedding $(\Pi_i^{(K_i')})^{K_i} \subset \Pi_i^{(K_i')}$ the sought for morphisms $b_i(K_i, K_i')$.

These $b_i(K_i, K_i')$ are unique once they exist, so the set consisting of the Π^{K_i} together with the $b_i(K_i, K_i')$ is an inductive system, and we may form the inductive limit $\Pi_i \overset{\text{def}}{=} \varinjlim \Pi_i^{(K_i)}$ with $K = K_1 \times K_2$ ranging over the compact open subgroups of G. Hence $\Pi_i^{K_i} = \Pi_i^{(K_i)}$, and by taking the inductive limit of the a_K, we get an isomorphism $\Pi \cong \Pi_1 \otimes \Pi_2$.

Theorem 28. — *Let $(\Sigma, G_v, K_v, G_S, \Pi_v)$ be defined as in (III.15). Assume that the G_v satisfy (Hyp) and that $\mathcal{H}(G_v : K_v)$ is commutative for nearly all $v \in \Sigma$. Then for any smooth-admissible irreducible G-representation Π there exist unique irreducible smooth-admissible G_v-representations Π_v, unramified for nearly all v, such that $\Pi \cong \bigotimes_{v \in \Sigma}' \Pi_v$ as G-representations. The Π_v are unique up to (non-canonical) isomorphism.*

Proof. — Let $S \subset \Sigma$ be the finite subset of $v \in \Sigma$ such that $\mathcal{H}(G_v : K_v)$ is not commutative. We have

$$G = G^S \times \prod_{v \in S} G_v.$$

By the Lemma 27, there exists a G^S-representation Π^S and for every $v \in S$, an G_v-representation Π_v such that

$$\Pi \cong \Pi^S \otimes \bigotimes_{v \in S} \Pi_v.$$

Therefore, we may (and will) assume that $\mathcal{H}(G_v : K_v)$ is commutative for all v. Let $K = \prod_v K_v$. The morphism

$$\bigotimes_{v \in \Sigma} \mathcal{H}(G_v : K_v) \longrightarrow \mathcal{H}(G : K), \quad \bigotimes_{v \in \Sigma} f_v \mapsto \prod_v f_v,$$

is surjective. Therefore, $\mathcal{H}(G:K)$ is commutative. The $\mathcal{H}(G:K)$-module Π^K is simple (Theorem 26), and thus one-dimensional over $\mathbf{C}$. Let $x \in \Pi^K$ be a basis of this one-dimensional space.

Let $S \subset \Sigma$ be a finite subset and $G_S := \prod_{v\in S} G_v$. Consider the morphism

$$\mathbf{C}[G_S] \longrightarrow \Pi, \quad \lambda \mapsto \lambda x.$$

We denote with Π_S the image of this morphism, and with I_S the kernel of this morphism. If $S = \{v\}$ is a singleton, then we will write $\Pi_v = \Pi_{\{v\}}$ and $I_v := I_{\{v\}}$ to relax notations.

The G_S-representation Π_S is irreducible. To see this, we prove it is isomorphic to an irreducible G_S-representation. By Lemma 27, there exists $\tilde{\Pi}_S$ (resp. $\tilde{\Pi}^S$) irreducible G_S (resp. G^S) representations such that $\varphi\colon \Pi \xrightarrow{\sim} \tilde{\Pi}_S \otimes \tilde{\Pi}^S$ as G-representations. Take K-invariants on both sides, to obtain an isomorphism $\Pi^K \cong \tilde{\Pi}_S^{K_S} \otimes (\tilde{\Pi}^S)^{K^S}$, where $K_S = \prod_{v\in S} K_v$ and $K^S = \prod_{v\in\Sigma\setminus S} K_v$. The space $(\tilde{\Pi}^S)^{K^S}$ is one-dimensional; we choose a basis $y \in (\tilde{\Pi}^S)^{K^S}$ for it, so that $\tilde{\Pi}_S \otimes (\tilde{\Pi}^S)^{K^S}$ is isomorphic to $\tilde{\Pi}_S$ as $\mathbf{C}[G_S]$-modules. The $\mathbf{C}[G_S]$-submodule of Π generated by Π^K is Π_S, and is, via the basis y and the isomorphism φ, identified with a non-trivial $\mathbf{C}[G_S]$-submodule of $\tilde{\Pi}_S$. Because $\tilde{\Pi}_S$ is irreducible as G_S-representation we obtain an $\mathbf{C}[G_S]$-isomorphism $\Pi_S \cong \tilde{\Pi}_S$. In particular Π_S is irreducible.

Fix $S \subset \Sigma$ a finite subset. Define the S-multilinear map ψ by

$$\psi\colon \prod_{v\in S} \mathbf{C}[G_v] \longrightarrow \Pi_S, \quad (\lambda_v)_{v\in S} \mapsto \left(\prod_v \lambda_v\right) x.$$

The ideal $\prod_v I_v \subset \prod_{v\in S} \mathbf{C}[G_v]$ is contained in the kernel of ψ. Therefore, ψ induces a S-multilinear surjection

$$\prod_{v\in S} \mathbf{C}[G_v]/I_v = \prod_{v\in S} \Pi_v \longrightarrow \Pi_S,$$

and hence a G_S-surjection

$$\bigotimes_{v\in S} \Pi_v \longrightarrow \Pi_S.$$

Because $\bigotimes_{v\in S} \Pi_v$ is G_S-irreducible, this surjection is an isomorphism. If $S' \subset \Sigma$ is a finite subset containing S, then we have a map $\iota_{S,S'}\colon \bigotimes_{v\in S} \Pi_v \to \bigotimes_{v\in S'} \Pi_v$ given by $\bigotimes_{v\in S} x_v \mapsto \bigotimes_{v\in S'\setminus S} x \otimes \bigotimes_{v\in S} x_v$. The map $\iota_{S,S'}$ fits in the following commutative diagram:

$$\begin{array}{ccc}
\bigotimes_{v\in S} \Pi_v & \rightarrowtail\!\!\!\twoheadrightarrow & \Pi_S \\
\downarrow{\scriptstyle \iota_{S,S'}} & & \downarrow \\
\bigotimes_{v\in S'} \Pi_v & \rightarrowtail\!\!\!\twoheadrightarrow & \Pi_{S'}
\end{array}$$

The inductive limit of the $\bigotimes_{v\in S} \Pi_v$ relative to the morphisms $\iota_{S,S'}$ is the restricted tensor product $\bigotimes'_{v\in\Sigma} \Pi_v$. The inductive limit of the Π_S, where S varies, relative to the inclusion morphisms, is the representation Π. Therefore Π is isomorphic to $\bigotimes'_{v\in\Sigma} \Pi_v$.

3.5. The groups $\mathbf{GL}_n(\mathbf{Q}_p) \supset \mathbf{GL}_n(\mathbf{Z}_p)$

Proposition 29. — *Let G be a locally profinite group satisfying (Hyp), and let $K \subset G$ a compact open subgroup. Then $\mathcal{H}(G : K)$ is abelian if there exists an isomorphism $\varphi \colon G \to G^{\mathrm{op}}$ (of locally profinite groups) such that $\varphi^2 = \mathrm{id}$ and φ acts trivially on the set $\{KgK | g \in G\}$.*

Proof. — This is a straightforward calculation. Let $f, h \in \mathcal{H}(G : K)$. We write f^φ for the function sending g to $f(\varphi(g))$ (and similarly for h^φ). For all $g \in G$

$$\begin{aligned}(f * h)(g) &= \int_G f^\varphi(\varphi(x)) h^\varphi(\varphi(g)\varphi(x)^{-1}) \mathrm{d}\mu(x) \\ &= \int_G h(\varphi(g)\varphi(x)^{-1}) f(\varphi(x)) \mathrm{d}\mu(x) \\ &= \int_G h(\varphi(g)x) f(x^{-1}) \mathrm{d}\mu(x) \\ &= (h * f)(\varphi(g))\end{aligned}$$

The morphism φ acts trivially on the characteristic functions $\mathbf{1}_{KgK}$ of the classes KgK, so φ also acts trivially on $f_2 * f_1$; hence $(f_2 * f_1)(\varphi g) = (f_2 * f_1)(g)$.

Corollary 30. — *The Hecke algebra $\mathcal{H}(\mathrm{GL}_n(\mathbf{Q}_p) : \mathrm{GL}_n(\mathbf{Z}_p))$ is commutative.*

Proof. — We take φ in the above theorem the morphism sending a matrix to its transpose. The classes $\mathrm{GL}_n(\mathbf{Z}_p) g \mathrm{GL}_n(\mathbf{Z}_p)$ for $g \in \mathrm{GL}_n(\mathbf{Q}_p)$ are represented by matrices g which are diagonal. Therefore, φ acts trivially on the set of these classes.

The *Satake isomorphism* gives in fact an isomorphism between $\mathcal{H}(\mathrm{GL}_n(\mathbf{Q}_p) : \mathrm{GL}_n(\mathbf{Z}_p))$ and the commutative algebra $\mathbf{C}[X_1, \ldots, X_n, X_1^{-1}, \ldots, X_n^{-1}]^{\mathfrak{S}_n}$, where $\mathfrak{S}_n = \mathrm{Aut}_{\mathfrak{Sets}}(\{1, \ldots, n\})$ acts on the algebra by permutation of the variables. For more details, see [**16**].

Corollary 31. — *Let for every prime number p, Π_p be an irreducible smooth $\mathrm{GL}_n(\mathbf{Q}_p)$-representation, unramified for nearly all p, and $\Pi := \bigotimes_p' \Pi_p$. Then:*

1. *The $\mathrm{GL}_n(\mathbf{A}^\infty)$-representation Π is irreducible and smooth-admissible.*
2. *For unramified p the dimension of $\Pi_p^{\mathrm{GL}_n(\mathbf{Z}_p)}$ is 1.*
3. *Assume that $\{\Pi'_p : p \text{ prime}\}$ is another set of representations like above, then $\bigotimes_p' \Pi'_p \cong \bigotimes_p' \Pi_p$ if and only if for all primes p we have $\Pi'_p \cong \Pi_p$.*

Inversely, if Π' is an irreducible and smooth-admissible $\mathrm{GL}_n(\mathbf{A}^\infty)$-representation, then for every p there exists an irreducible and smooth-admissible $\mathrm{GL}_n(\mathbf{Q}_p)$-representation Π'_p such that $\Pi' \cong \bigotimes_p' \Pi'_p$.

Proof. — We first prove the second point. By Theorem 26 the $\mathcal{H}(\mathrm{GL}_n(\mathbf{Q}_p) : \mathrm{GL}_n(\mathbf{Z}_p))$-module $\Pi_p^{\mathrm{GL}_n(\mathbf{Z}_p)}$ is simple, and by Corollary 30 the algebra $\mathcal{H}(\mathrm{GL}_n(\mathbf{Q}_p) : \mathrm{GL}_n(\mathbf{Z}_p))$ is commutative, hence the dimension of $\Pi_p^{\mathrm{GL}_n(\mathbf{Z}_p)}$ is one.

By the second point, the conditions of Theorem 23 are verified, so the restricted product of smooth-admissible is again smooth-admissible. By Theorem 24, the restricted tensor product is irreducible.

To prove the last statement, it suffices to note that by Corollary 30, Theorem 28 applies. The third point follows from uniqueness in Theorem 28.

3.6. The infinite part

Let $O(n) \subset \mathrm{GL}_n(\mathbf{R})$ be the orthogonal group, i.e. matrices g for which the transpose is the inverse. Let $\mathfrak{gl}_n = M_n(\mathbf{C})$ be the complexified Lie-algebra of $\mathrm{GL}_n(\mathbf{R})$, with Lie-brackets defined by $[X, Y] := XY - YX$, for all $X, Y \in M_n(\mathbf{C})$.

An $(O(n), \mathfrak{gl}_n)$*-module* is a complex vector space V, with an action of $\mathfrak{gl}_n$ and a locally finite and continuous action of $O(n)$. This last continuity condition of the action of $O(n)$ on V means that V is the union of the finite dimensional continuous $O(n)$-subrepresentations of V. The actions of $O(n)$ and $\mathfrak{gl}_n$ on V are subject to the following conditions. For all $v \in V$, all $k \in O(n)$, all $X \in \mathfrak{gl}_n$ we have:

1. $k(Xv) = (kXk^{-1})(kv)$;
2. $\dim \mathbf{C}[O(n)]v < \infty$;
3. $\forall Y \in \mathit{Lie}(O(n)) \otimes \mathbf{C} \subset \mathfrak{gl}_n : (Yv) = \frac{\mathrm{d}}{\mathrm{d}t} \left(e^{tY} v\right)\big|_{t=0}$.

The module V is *admissible* if for all irreducible $O(n)$-representations W the dimension of $\mathrm{Hom}_{O(n)}(W, V)$ is finite. A *morphism of* $(O(n), \mathfrak{gl}_n)$*-modules* is a morphism of vector spaces which is equivariant for the action of both factors.

An $(O(n), \mathfrak{gl}_n) \times \mathrm{GL}_n(\mathbf{A}^\infty)$-module is a complex vector space V with actions of $\mathrm{GL}_n(\mathbf{A}^\infty)$, $\mathfrak{gl}_n$ and $O(n)$, such that there exists a smooth-admissible $\mathrm{GL}_n(\mathbf{A}^\infty)$-representation V_1 and a $(O(n), \mathfrak{gl}_n)$-module V_2, such that there exists a $\mathbf{C}$-linear bijection $V \xrightarrow{\sim} V_1 \otimes V_2$ which is equivariant for the actions of $\mathrm{GL}_n(\mathbf{A}^\infty)$, $\mathfrak{gl}_n$ and $O(n)$.

We say that the $(O(n), \mathfrak{gl}_n) \times \mathrm{GL}_n(\mathbf{A}^\infty)$-module V is *smooth-admissible* if the module V_2 is admissible. We say that V is *irreducible*, if it has precisely two subspaces which are stable for the actions of $\mathrm{GL}_n(\mathbf{A}^\infty)$, $\mathfrak{gl}_n$ and $O(n)$. In this case, V_1 and V_2 are also irreducible.

3.7. L-factors

Let (π, V) be a smooth-admissible $\mathrm{GL}_n(\mathbf{Q}_p)$-representation. Let $(\pi^\vee, V^\vee)$ be the *contragredient representation* of π (the subrepresentation of the dual representation (π^*, V^*) consisting of all vectors with open stabiliser in G). We denote the paring $V^\vee \times V \to V$ by $\langle \cdot, \cdot \rangle$. The *space of coefficients* C_π of π is the $\mathbf{C}$-subspace of $\mathrm{Map}(\mathrm{GL}_n(\mathbf{Q}_p), \mathbf{C})$ generated by all functions of the type $\mathrm{GL}_n(\mathbf{Q}_p) \to \mathbf{C}$, $g \mapsto \langle v', \pi(g)v \rangle$, for $v \in V, v' \in V^\vee$. A *coefficient* f of π is an element of C_π. If f is a coefficient of π, then $f^\vee := (g \mapsto f(g^{-1})$ is a coefficient of $\pi^\vee$.

Let $M_n(\mathbf{Q}_p)$ be the space of $n \times n$-matrices with entries in $\mathbf{Q}_p$. Embed $\mathrm{GL}_n(\mathbf{Q}_p)$ in $M_n(\mathbf{Q}_p)$ in the usual way. Denote with $S_n(\mathbf{Q}_p)$ the space of locally constant compactly supported functions $M_n(\mathbf{Q}_p) \to \mathbf{C}$. Fix a non-trivial continuous morphism $\psi \colon \mathbf{Q}_p \to \mathbf{C}^\times$.

Let μ be the *self-dual Haar measure* on $M_n(\mathbf{Q}_p)$ with respect to $\psi \circ \mathrm{Tr}$. Let us recall what this means. First, the *Fourier transform* $\Phi^\wedge$ of a $\Phi \in \mathcal{S}_n(\mathbf{Q}_p)$ is defined by

$$\Phi^\wedge(x) = \int_{\mathrm{GL}_n(\mathbf{Q}_p)} \Phi(y)\psi[\mathrm{Tr}(yx)]\mathrm{d}\mu y.$$

A Haar-measure is *self-dual*, with respect to the fixed morphism $\psi \circ \mathrm{Tr}\colon M_n(\mathbf{Q}_p) \to \mathbf{C}^\times$, if $\Phi^{\wedge,\wedge}(x) = \Phi(-x)$ for all $\Phi \in \mathcal{S}_n(\mathbf{Q}_p)$ and all $x \in M_n(\mathbf{Q}_p)$. Given a continuous morphism $M_n(\mathbf{Q}_p) \to \mathbf{C}^\times$), there is precisely one self-dual Haar measure with respect to this morphism.

Let $f \in C_\pi, \Phi \in \mathcal{S}_n(\mathbf{Q}_p)$ and $s \in \mathbf{C}$, we write $Z(\Phi, s, f)$ for the integral

$$\int_{\mathrm{GL}_n(\mathbf{Q}_p)} \Phi(g)|\det g|^s f(g)\mathrm{d}\nu(g),$$

where ν is a Haar measure on $\mathrm{GL}_n(\mathbf{Q}_p)$.

***Theorem 32* (Jacquet, Godement).** — *Assume π is irreducible. The following holds:*

1. *There exists an $s_0 \in \mathbf{R}$ such that the integral $Z(\Phi, s, f)$ converges absolutely on the half-plane $\mathbf{C}_{\geq s_0} := \{s \in \mathbf{C} |\, \Re(s) \geq s_0\}$.*
2. *For all $\Phi \in \mathcal{S}_n(\mathbf{Q}_p)$ and all $f \in C_\pi$, there exists a unique rational function $Z_{\Phi,f} \in \mathbf{C}(X)$ such that $Z_{\Phi,f}(p^{-s}) = Z(\Phi, s, f)$ for all $s \in \mathbf{C}_{\geq s_0}$.*
3. *Write $Z_{\Phi,f} = T_{\Phi,f}/N_{\Phi,f}$ with $T_{\Phi,f}, N_{\Phi,f} \in \mathbf{C}[X]$ coprime and $N_{\Phi,f}$ monic. The elements $N_{\Phi,f} \in \mathbf{C}[X]$ with Φ resp. f ranging over $\mathcal{S}_n(\mathbf{Q}_p)$ resp. C_π have a non-zero least common multiple.*
4. *Let $\psi \neq 1$ be a continuous morphism $\mathbf{Q}_p \to \mathbf{C}^\times$. There is a rational function $G(x, \pi, \psi) \in \mathbf{C}(x)$ such that for all coefficients f of π and $\Phi \in \mathcal{S}_n(\mathbf{Q}_p)$ we have*

$$Z(\Phi^\wedge, 1 - s + (n-1)/2, f^\vee) = G(p^s, \pi, \psi)Z(\Phi, s, f),$$

for all $s \in \mathbf{C}_{\geq s_0}$.

Proof. — [**19**].

In the next section we give a brief outline of the proof of this theorem, let us first indicate the important consequence of this theorem.

Proposition 33. — *Let $I(\pi)$ be the $\mathbf{C}$-subspace of $\mathbf{C}(x)$ spanned by the $Z_{\Phi,f}$ where $\Phi \in \mathcal{S}_n(\mathbf{Q}_p)$ and $f \in C_\pi$ vary. Then*

1. *The space $I(\pi) \subset \mathbf{C}(x)$ is a $\mathbf{C}[x, x^{-1}]$-fractional ideal;*
2. *One has $1 \in I(\pi)$;*
3. *There exists a generator $P \in \mathbf{C}[x]$ such that $I(\pi) = (1/P)$ and $P(0) = 1$.*

Definition 34. — Let (π, V) be a smooth irreducible $\mathrm{GL}_n(\mathbf{Q}_p)$-representation. The *L-factor* of (π, V) is defined by $L(s, \pi) := P(p^{-s})^{-1}$. The *ε-factor* of (π, V) with respect to a continuous morphism $\psi\colon \mathbf{Q}_p \to \mathbf{C}^\times$ is defined by $\varepsilon(s, \pi, \psi) = G(s, \pi, \psi)L(s, \pi)/L(1 - s, \pi^\vee)$.

Proof of Proposition 33. — (1) We prove $\mathbf{C}[x, x^{-1}] \cdot I(\pi) \subset I(\pi)$. To show this, it suffices that $x \cdot I(\pi) = I(\pi)$ holds, which is true if $x \cdot Z(\Phi, s, f) = c \cdot Z(\Phi_1, s, f_1)$, for a certain $c \in \mathbf{C}, \Phi_1 \in \mathcal{S}_n(\mathbf{Q}_p)$ and coefficient f_1. Let $h \in \mathrm{GL}_n(\mathbf{Q}_p)$ be a matrix with $\det(h) = p$, and define $\Phi_1 := (g \mapsto \Phi_1(gh))$, $f_1 := (g \mapsto f(gh))$. Then f_1 is again a coefficient of π and $\Phi_1 \in \mathcal{S}_n(\mathbf{Q}_p)$. Two rational functions on $\mathbf{C}$ are equal if and only if they agree on an infinite subset of $\mathbf{C}$, hence, by Proposition 33 it suffices to show $Z(\Phi_1, s, f_1) = c \cdot Z(\Phi, s, f)$ for all $s \in \mathbf{C}_{\geq s_0}$ where $s_0 \in \mathbf{R}$ is large enough. We compute:

$$Z(\Phi_1, s, f_1) = \int_{\mathrm{GL}_n(\mathbf{Q}_p)} \Phi(gh) |\det g|^s f(gh) \mathrm{d}\mu(g) = \int_{\mathrm{GL}_n(\mathbf{Q}_p)} \Phi(g) |\det gh^{-1}|^s f(g) |\det h| \mathrm{d}\mu(g)$$
$$= |\det h|^{-s} |\det h| \int_{\mathrm{GL}_n(\mathbf{Q}_p)} \Phi(g) |\det g|^s f(g) \mathrm{d}\mu(g) = q^{-s} |\det h| Z(\Phi, s, f).$$

Hence $I(\pi)$ is an $\mathbf{C}[x, x^{-1}]$-submodule of $\mathbf{C}(x)$. By 33.(3), the module $I(\pi)$ is a fractional ideal.

(2) Notice that $h = (g \mapsto \Phi(g) |\det g|^s f(g))$ is a locally constant function on $\mathrm{GL}_n(\mathbf{Q}_p)$ for all Φ and f. Pick an open and closed neighborhood $U \subset M_n(\mathbf{Q}_p)$ of $e \in \mathrm{GL}_n(\mathbf{Q}_p)$ such that $h|_U$ is constant. Let $\chi_U \colon M_n(\mathbf{Q}_p) \to \mathbf{C}$ be the characteristic function U and take $\Phi' = \Phi \cdot \chi_U$. If Φ and f are such that $\Phi(e) |\det e|^s f(e) \neq 0$, then $Z(\Phi', s, f) \in I(\pi)$ is constant and non-zero.

(3) The ring $\mathbf{C}[x, x^{-1}]$ is a principal ideal domain.

3.8. Remarks on the proof of Theorem 32

The proof of Theorem 32 is long. At the moment I do not have much interest in studying those calculations, but I do want to make the L-factors more explicit. Therefore, I have to say some words on the proof.

We introduce the notion "cuspidal representation". Assume $\alpha = (n_1, \ldots, n_r) \in \mathbf{Z}_{\geq 1}^r$ with $r \in \mathbf{Z}_{\geq 1}$ is a partition of n, i.e. $\sum n_i = n$. Set

$$P_\alpha = \begin{pmatrix} \mathrm{GL}_{n_1} & & * \\ & \ddots & \\ & & \mathrm{GL}_{n_r} \end{pmatrix} \subset \mathrm{GL}_n, \qquad U_\alpha = \begin{pmatrix} I_{n_1} & & * \\ & \ddots & \\ & & I_{n_r} \end{pmatrix} \subset P,$$

where I_{n_i} is the $n_i \times n_i$-identity matrix. Identify P_α / U_α with $\prod_{i=1}^r \mathrm{GL}_{n_i}$ via the obvious isomorphism. Now assume (σ, V) is a representation of $\mathrm{GL}_n(\mathbf{Q}_p)$; we may restrict it to obtain a representation of $P_\alpha(\mathbf{Q}_p)$. The largest quotient on which $U_\alpha(\mathbf{Q}_p)$ acts trivially, $V_\alpha / \langle \sigma(u) \cdot v - v : \sigma \in U_\alpha(\mathbf{Q}_p) \rangle$, is a $P_\alpha(\mathbf{Q}_p)/U_\alpha(\mathbf{Q}_p)$-representation, denoted $r_\alpha(\sigma)$. The representation σ is called *cuspidal* if for all non-trivial partitions α (so with $r > 1$) the representation $r_\alpha(\sigma)$ is 0.

From Frobenius duality, it follows that any non-cuspidal irreducible $\mathrm{GL}_n(\mathbf{Q}_p)$-representation is obtained in the following way. Let σ_i be a smooth $\mathrm{GL}_{n_i}(\mathbf{Q}_p)$-representation, where $i \in \{1, \ldots, r\}$. Let $\sigma = \bigotimes_{i=1}^r \sigma_i$ be the representation of $P_\alpha / U_\alpha(\mathbf{Q}_p)$ induced by the σ_i, and define:

$$I(\sigma_1, \ldots, \sigma_r) := \iota_{P_\alpha(\mathbf{Q}_p)}^{\mathrm{GL}_n(\mathbf{Q}_p)} (P_\alpha(\mathbf{Q}_p) \longrightarrow P_\alpha / U_\alpha(\mathbf{Q}_p) \xrightarrow{\sigma} \mathrm{GL}(\Pi)), \quad \text{((III.20))}$$

where Π is the space of σ. The $\iota_{P_\alpha(\mathbf{Q}_p)}^{\mathrm{GL}_n(\mathbf{Q}_p)}$ is the unitary induction. Let us very briefly explain this construction, for more details see [**7**]. Let G be locally profinite and $H \subset G$ a closed subgroup.

The forgetful functor from the category of smooth complex G-representations to the category of smooth complex H-representations admits a right adjoint functor Ind_H^G. For V a smooth H-representation, $\mathrm{Ind}_H^G V$ may be defined as the space of locally constant functions $f\colon G \to V$ such that for all $h \in H$, all $g \in G$ we have $f(hg) = \rho(h)f(g)$ and there is a compact open subgroup K_f such that f factors over G/K_f.

Let δ_H be the *modulus* of H, this means the following. Let μ be a left Haar measure on H, then any other left Haar measure ν differs by a scalar from μ, we denote this scalar by $\frac{\mu}{\nu}$. For any automorphism $\sigma \in \mathrm{Aut}(H)$ we see that $\mu \circ \sigma$ is a Haar measure; we put $\Delta(\sigma) := \frac{\mu\circ\sigma}{\mu}$, which provides a morphism $\mathrm{Aut}(H) \to \mathbf{R}_{>0}^\times$. We can compose this morphism with $H \to \mathrm{Aut}(H)$ ($h \in H$ maps to conjugation by h), to obtain a continuous morphism $\delta_H\colon H \to \mathbf{R}_{>0}^\times \subset \mathbf{C}^\times$; this δ_H is the modulus of H.

We are now in place to define the *unitary induction*. By definition $\iota_H^G := \mathrm{Ind}_H^G \otimes \delta_H^{1/2}$. The unitary induction has some technical advantages over usual induction, most notably (1) it preserves unitary representations, (2) it is compatible with direct sums and local Langlands (suitably normalized).

Theorem 35. — *Let Π be an irreducible smooth-admissible $\mathrm{GL}_n(\mathbf{Q}_p)$-representation. There exists a partition $n = n_1 + \cdots + n_r$, with $n_i, r \in \mathbf{Z}_{\geq 1}$, smooth-admissible cuspidal $\mathrm{GL}_{n_i}(\mathbf{Q}_p)$-representations σ_i for $i = 1, \ldots, r$, such that Π is an irreducible subquotient of $I(\sigma_{n_1}, \ldots, \sigma_{n_r})$.*

Proof. — This follows from the discussion above.

Using the notion of cuspidal, the proof of Theorem **??** reduces to the following 4 steps.

1. If Theorem 32 is true for $\sigma_1, \ldots, \sigma_r$, then Theorem 32 is also true for $I(\sigma_1, \ldots, \sigma_r)$. (Proved in paragraphs (2.4)—(2.6) in [**24**].)
2. Theorem 32 is true in case $n = 1$. (Proved in Tate's thesis, [**9**].)
3. Theorem 32 is true in case $n > 1$ and the representation is cuspidal. (Proved in [**19**].)
4. If the conclusions of Theorem 32 hold for a (not necessarily irreducible) smooth-admissible $\mathrm{GL}_n(\mathbf{Q}_p)$-representation π, then the theorem is also true for the irreducible subrepresentations of π. (Proved in (2.7) in [**24**].)

3.9. Unramified L-factors

If a smooth-admissible irreducible $\mathrm{GL}_n(\mathbf{Q}_p)$-representation is unramified, then there exist morphisms $\sigma_1, \ldots, \sigma_n\colon \mathbf{Q}_p^\times \to \mathbf{C}^\times$ (with open kernel) such that $\Pi = I(\sigma_1, \ldots, \sigma_n)$. In particular, the L-factor of Π is just the product of the L-factors of the σ_i, namely:

$$L(\Pi, s) = \prod_{i=1}^{n} (1 - \sigma_i(p)p^{-s})^{-1}.$$

Because the morphisms σ_i are unramified, the unramified representation is completely determined by its L-factor.

We now understand, before even having stated it properly, that there is a Langlands correspondence for local, unramified representations. Let $\mathrm{Art}_{\mathbf{Q}_p}\colon \mathrm{GL}_1(\mathbf{Q}_p) \xrightarrow{\sim} \mathcal{G}_{\mathbf{Q}_p}^{\mathrm{ab}}$ be the reciprocity morphism (geometrically normalised). The map

$$\left\{\begin{matrix}\text{unramified, semi-simple, } n\text{-dimensional} \\ \ell\text{-adic } \mathcal{G}_{\mathbf{Q}_p}\text{-representations}\end{matrix}\right\}_{/\cong} \longrightarrow \left\{\begin{matrix}\text{unramified, irreducible, smooth-admissible} \\ \mathrm{GL}_n(\mathbf{Q}_p)\text{-representations over } \overline{\mathbf{Q}}_\ell\end{matrix}\right\}_{/\cong}$$

$$\rho \cong \bigoplus_{i=1}^{n} \sigma_i \longmapsto I(\sigma_1 \circ \mathrm{Art}_{\mathbf{Q}_p}^{-1}, \ldots, \sigma_n \circ \mathrm{Art}_{\mathbf{Q}_p}^{-1}),$$

is a bijection preserving the L-factors. The crucial point is that ρ decomposes as a direct sum of one dimensional representations.

Recall that any irreducible $(O(n), \mathfrak{gl}_n) \times \mathrm{GL}_n(\mathbf{A}^\infty)$-module is a restricted tensor product of local representations Π_v, unramified for nearly all v. In particular $\Pi_v = I(\sigma_{v,1}, \ldots, \sigma_{v,n})$ for nearly all v. I have not treated properly the prime at infinity, so we regard only $v = p$ a finite place, and we assume that it is different from ℓ. The idea of the global Langlands conjecture is that for "good" $(O(n), \mathfrak{gl}_n) \times \mathrm{GL}_n(\mathbf{A}^\infty)$-modules, the representations $\bigoplus_{i=1}^n \sigma_{v,i} \circ \mathrm{Art}_{\mathbf{Q}_p}^{-1}$ should all be restrictions of *one* global Galois representation $\rho\colon \mathcal{G}_{\mathbf{Q}} \to \mathrm{GL}_n(\overline{\mathbf{Q}}_\ell)$ to the decomposition group at p.

3.10. The L-function associated to an $(O(n), \mathfrak{gl}_n) \times \mathbf{GL}_n(\mathbf{A}^\infty)$-module

Let Π be an irreducible smooth-admissible $(O(n), \mathfrak{gl}_n) \times \mathrm{GL}_n(\mathbf{A}^\infty)$-module. We define

$$L(\Pi, s) = \prod_{p \text{ prime}} L(\Pi_p, s), \qquad \text{((III.21))}$$

where the Π_p are admissible $\mathrm{GL}_n(\mathbf{Q}_p)$-representations such that $\Pi \cong \bigotimes_p' \Pi_p$.

Similar to what we saw in chapter 1 on Galois representations, the L-function has no reason to converge in some right half plane. It will converge only for certain special representations.

3.11. The space of smooth functions on $\mathbf{GL}_n(\mathbf{A})$

A complex valued function $f\colon \mathrm{GL}_n(\mathbf{A}) \to \mathbf{C}$ is called *smooth* if for all $x \in \mathrm{GL}_n(\mathbf{A})$ there exist open subsets $U \subset \mathrm{GL}_n(\mathbf{R})$ and $V \subset \mathrm{GL}_n(\mathbf{A}^\infty)$ and a $\mathcal{C}^\infty$-smooth function $\tilde{f}\colon U \to \mathbf{C}$ such that $x \in U \times V$ and $f|_{U \times V}$ equals to the composition $U \times V \to U \xrightarrow{\tilde{f}} \mathbf{C}$. We denote with C^∞ the $\mathbf{C}$-vector space of all smooth functions $\mathrm{GL}_n(\mathbf{A}) \to \mathbf{C}$.

Several groups/algebras act on the space $\mathcal{C}^\infty$. The group $\mathrm{GL}_n(\mathbf{A})$ acts on C^∞ via right translations, $\mathrm{GL}_n(\mathbf{A}) \times C^\infty \to C^\infty$, $(g, f) \longmapsto (h \mapsto f(hg))$. The $\mathrm{GL}_n(\mathbf{A})$-action on C^∞ induces an action of $\mathfrak{gl}_n$ on C^∞ given by:

$$\mathfrak{gl}_n \times C^\infty \longrightarrow C^\infty, \quad (X, f) \mapsto \left(h \mapsto \left[\frac{\mathrm{d}}{\mathrm{d}t} f\left(he^{tX}\right)\right]_{t=0}\right).$$

Finally, let $\mathfrak{z}_n$ be the endomorphism ring of the identity functor, $\mathrm{id}_{\mathfrak{gl}_n\text{-rep}}$, on the category of $\mathfrak{gl}_n$-representations (representations as Lie-algebra). Then $\mathfrak{z}_n$ acts on C^∞, because C^∞ is $\mathfrak{gl}_n$-representation.

3.12. The algebra $\mathfrak{z}_n$

We have to know more of the structure of the algebra $\mathfrak{z}_n$.

Lemma 36. — *Let A be an associative unitary* **C***-algebra. Denote with $Z(A)$ the center of A. The canonical morphism*

$$\Phi\colon Z(A) \longrightarrow \mathrm{End}(\mathrm{id}_{A\text{-mod}}), \quad a \mapsto \text{multiplication by } a,$$

is an isomorphism.

Proof. — Define the map

$$\Psi\colon \mathrm{End}(\mathrm{id}_{A\text{-mod}}) \longrightarrow Z(A), \quad F \mapsto F(A)(1).$$

Here with $F(A)(1)$ we mean the following. Note that A is a module over itself, so $F(A)$ defines an endomorphism of A as A-module. With $F(A)(1)$ we mean the image of $1 \in A$ under this endomorphism. For any $a \in A$ we have $aF(A)(1) = F(A)(a) = F(A)(1)a$, so $F(A)(1) \in Z(A)$, so the image of Ψ lies in the center of A.

Let $a \in Z(A)$, then $(\Psi \circ \Phi)(a) = a \cdot 1 = a$, so $\Psi \circ \Phi = \mathrm{id}_{Z(A)}$. Inversely, let $F \in \mathrm{End}(\mathrm{id}_{A\text{-mod}})$. Let M be an A-module and let $m \in M$. We have a commutative square

$$\begin{array}{ccc} A & \xrightarrow{F(M)} & M \\ \uparrow & & \uparrow \\ A & \xrightarrow[F(A)]{} & A \end{array}$$

where the vertical morphisms send 1 to m. The bottom map is multiplication by $F(A)(1)$, and therefore, the top map sends m to $F(A)(1) \cdot m$. This proves that $(\Phi \circ \Psi)(F) = F$, so it proves lemma.

The *universal enveloping algebra* $U(\mathfrak{gl}_n)$ of $\mathfrak{gl}_n$ is defined by the equality of functors from **C**-algebras to sets

$$\mathrm{Hom}_{\mathbf{C}\text{-algebras}}(U(\mathfrak{gl}_n), \square) = \mathrm{Hom}_{\text{Lie-algebras}}(\mathfrak{gl}_n, \square),$$

(any **C**-algebra is also a Lie-algebra). In particular the category of $\mathfrak{gl}_n$-representations is isomorphic to the category of $U(\mathfrak{gl}_n)$-modules. Therefore $\mathfrak{z}_n$ is also the endomorphism ring of the functor $\mathrm{id}_{U(\mathfrak{gl}_n)\text{-mod}}$, hence we get an identification $\mathfrak{z}_n = Z(U(\mathfrak{gl}_n))$, where $Z(U(\mathfrak{gl}_n))$ is the center of $U(\mathfrak{gl}_n)$. The algebra $U(\mathfrak{gl}_n)$ may be constructed as the quotient of the tensor algebra, $\bigoplus_{r \in \mathbf{N}} \mathfrak{gl}_n^{\otimes r}$ modulo the ideal $\langle [x,y] - xy + yx | x, y \in \mathfrak{gl}_n \rangle$.

Let $\mathfrak{S}_n = \mathrm{Aut}_{\mathfrak{Sets}}(\{1, \dots, n\})$ act on $\mathbf{C}[X_1, \dots, X_n]$ by permutation of the variables. Let $H = (x_1, \dots, x_n) \in \mathbf{C}^n$ be an n-tuple of complex numbers. We define $\theta_H\colon \mathbf{C}[X_1, \dots, X_n]^{\mathfrak{S}_n} \to \mathbf{C}$, $f \mapsto f(x_1, \dots, x_n)$. There exists a canonical isomorphism

$$\gamma_{\mathrm{HC}}\colon Z(U(\mathfrak{gl}_n)) \xrightarrow{\sim} \mathbf{C}[X_1, \dots, X_n]^{\mathfrak{S}_n},$$

called the *Harish-Chandra isomorphism*. See [**6**] or [**16**]. The isomorphism is characterised as follows. Assume $a_1 \geq a_2 \geq \cdots \geq a_n$ are integers and (ρ, V) is the irreducible representation of

$\mathfrak{gl}_n$ with $(a_1, \ldots, a_n) \in \mathbf{Z}^n$, as highest weight (for the usual choice of maximal torus and Borel subgroup). Let $H(\rho)$ be the multiset

$$\{a_1 + (n-1)/2, a_2 + (n-3)/2, \ldots, a_n + (1-n)/2\} \in \mathfrak{S}_n \backslash \mathbf{C}^n.$$

Let $U(\rho)\colon U(\mathfrak{gl}) \to \mathrm{End}(V)$ be the $U(\mathfrak{gl})$-representation induced by ρ. The morphism γ_{HC} should satisfy $U(\rho)(z) = \theta_{H(\rho)}(\gamma_{HC}(z))$ for all $z \in Z(U(\mathfrak{gl}_n))$, which determines γ_{HC} uniquely.

Example. — In case $n = 1$ it is easy to see that $\mathfrak{z}_1 = \mathbf{C}[X_1]$. In case $n = 2$ it turns out that $\mathfrak{z}_2 = \mathbf{C}[c, C]$, where $c = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $h = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, $a_+ = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, $a_- = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) \in \mathfrak{gl}_2$ and $C = \frac{1}{2}h^2 + a_+a_- + a_-a_+$.

Corollary 37. — *The mapping* $\mathfrak{S}_n \backslash \mathbf{C}^n \to \mathrm{Hom}(\mathfrak{z}_n, \mathbf{C})$, $H \mapsto \theta_H \circ \gamma_{\mathrm{HC}}$ *is a bijection.*

Proof. — It suffices to note that $\mathfrak{S}_n \backslash \mathbf{C}^n \to \mathrm{Hom}(\mathbf{C}[X_1, \ldots, X_n]^{\mathfrak{S}_n}, \mathbf{C})$, $H \mapsto \theta_H$ is a bijection (here $\mathfrak{S}_n$ acts on $\mathbf{C}^n$ by permuting the axises).

This result will become important when we state the global Langlands conjecture: The conjecture compares the infinitesimal characters with Hodge-Tate numbers, which are now both multisets of complex numbers.

3.13. The space of cusp forms

For a partition $n = n_1 + n_2$ with $n_1, n_2 \in \mathbf{Z}_{\geq 1}$ define $N_{n_1,n_2} := \left(\begin{smallmatrix} I_{n_1} & * \\ & I_{n_2} \end{smallmatrix}\right) \subset \mathrm{GL}_n$, where I_{n_i} is the $n_i \times n_i$-identity matrix ($i = 1, 2$). Fix a morphism $\chi\colon \mathfrak{z}_n \to \mathbf{C}$, or, equivalently by Corollary 37, a multiset of n complex numbers.

Definition 38. — The *space of cuspidal automorphic forms with infinitesimal character* χ, notation $\mathcal{A}^\circ_\chi \subset C^\infty$, is the subspace of C^∞ formed by smooth functions $f\colon \mathrm{GL}_n(\mathbf{A}) \to \mathbf{C}$ satisfying:

($\mathcal{A}1$) *Left* $\mathrm{GL}_n(\mathbf{Q})$*-invariant*. The function f is invariant under the action of $\mathrm{GL}_n(\mathbf{Q})$ by left translations and thus factors to a function $\overline{f}\colon \mathrm{GL}_n(\mathbf{Q}) \backslash \mathrm{GL}_n(\mathbf{A}) \to \mathbf{C}$.

($\mathcal{A}2$) $O(n) \times \mathrm{GL}_n(\hat{\mathbf{Z}})$*-finite*. The translates of f under the subgroup $O(n) \times \mathrm{GL}_n(\hat{\mathbf{Z}}) \subset \mathrm{GL}_n(\mathbf{A})$ span a finite dimensional vector space.

($\mathcal{A}3$) *Infinitesimal character* χ. For all $z \in \mathfrak{z}_n$ one has $z \cdot f = \chi(z) \cdot f$.

($\mathcal{A}4$) *Cuspidal*. For each partition $n = n_1 + n_2$ with $n_1, n_2 \in \mathbf{Z}_{\geq 1}$ and all $g \in \mathrm{GL}_n(\mathbf{A})$, the integral $\int_{N_{n_1,n_2}(\mathbf{Q}) \backslash N_{n_1,n_2}(\mathbf{A})} f(ug)\mathrm{d}\mu(u)$ vanishes.

($\mathcal{A}5$) *Growth condition*. The function f is bounded on $\mathrm{GL}_n(\mathbf{A})^1 = \{g \in \mathrm{GL}_n(\mathbf{A}) || \det g| = 1\}$.

Note that condition $\mathcal{A}_5$ in the definition of cuspidal automorphic form above, is different from the one given by Taylor in [**39**]. Without this modification, the function $\mathrm{GL}_n(\mathbf{A}) \to \mathbf{C}, x \mapsto |\det x|$ will not be an automorphic form. This is problematic, because it should correspond to the cyclotomic character under the global Langlands conjecture (for $n = 1$). So without this modification, the conjecture is false.

Inside the space of smooth functions C^∞, property ($\mathcal{A}2$) is not $\mathrm{GL}_n(\mathbf{A})$-stable but obviously $(O(n) \times \mathrm{GL}_n(\hat{\mathbf{Z}}), \mathfrak{gl}_n)$-stable. Because any two compact open subgroups of $\mathrm{GL}_n(\mathbf{A}^\infty)$ are *commensurable*[(1)], it follows that property ($\mathcal{A}2$) is $\mathrm{GL}_n(\mathbf{A}^\infty)$-stable. The other properties ($\mathcal{A}1$), ($\mathcal{A}3$), ($\mathcal{A}4$) and ($\mathcal{A}5$) are stable under the $\mathrm{GL}_n(\mathbf{A})$ action, and thus in particular under the actions of $\mathrm{GL}_n(\mathbf{A}^\infty), O(n) \times \mathrm{GL}_n(\hat{\mathbf{Z}})$ and $\mathfrak{gl}_n$.

The space $\mathcal{A}_\chi^\circ$ is an $(\mathfrak{gl}_n, O(n)) \times \mathrm{GL}_n(\mathbf{A}^\infty)$-module [**4**], and decomposes as a direct sum of irreducible representations, $\mathcal{A}_\chi^\circ = \bigoplus_{i \in I} M_i$, each occuring with multiplicity one: $i \neq j \implies M_i \ncong M_j$ (multiplicity one theorem, [**4**]). An $\mathrm{GL}_n(\mathbf{A}^\infty), O(n) \times \mathrm{GL}_n(\hat{\mathbf{Z}})$-module Π is a *cuspidal automorphic representation of* $\mathrm{GL}_n(\mathbf{A})$ *with infinitesimal character* χ if it is isomorphic to an irreducible subspace of $\mathcal{A}_\chi^\circ$. Let M, N be two cuspidal automorphic representations. Decompose $M \cong \bigotimes_v' M_v$ and $N \cong \bigotimes_v' N_v$, then $M \cong N$ if and only for nearly all v, $N_v \cong M_v$ [**4**].

(1) Two subgroups $H_1, H_2 \subset G$ are commensurable if $H_1 \cap H_2$ is a finite index subgroup of H_1 and of H_2.

Chapter 4: The global Langlands Conjecture

Conjecture 39 **(Langlands, Fontaine-Mazur, ...)**. — *Let $\iota\colon \overline{\mathbf{Q}}_\ell \to \mathbf{C}$ be an embedding, H a multiset of n integers, and ℓ a prime number. There exists a bijection between*

$$\left\{ \begin{array}{c} \text{Irreducible cuspidal automorphic } \mathrm{GL}_n(\mathbf{A})\text{-representations} \\ \text{with infinitesimal character } H \end{array} \right\} / \cong,$$

and

$$\left\{ \begin{array}{c} \text{Irreducible geometric } \ell\text{-adic } \mathcal{G}_{\mathbf{Q}}\text{-representations over } \overline{\mathbf{Q}}_\ell \\ \text{with Hodge-Tate numbers } H \end{array} \right\} / \cong,$$

such that if Π is an automorphic representation corresponding to an ℓ-adic $\mathcal{G}_{\mathbf{Q}}$-representation V under the above bijection, then $L_p(\Pi, p^{-s}) = L_p^{(\iota)}(V, p^{-s})$, for nearly all p.

We give some remarks to help the reader understand the content of this conjecture, and some of its implications.

We refer to the first set in Conjecture 39 as "the automorphic side", and the second set above as "the Galois side". Moreover, we say that a geometric representation "corresponds" to an automorphic representation if the above bijection maps the geometric representation to the automorphic representation, where we assume that ι is fixed once and for all.

We say a Galois representation V on the Galois side is *automorphic* if it correspond to an automorphic representation on the automorphic side.

The bijection in 39, if it exists, is automatically unique. Actually, something stronger is true: Assume a map $\varphi_{\ell,\iota,H}$ from a *subset of* the automorphic side to the Galois side exists, which preserves the L-factors in the manner described in the conjecture above. Then $\varphi_{\ell,\iota,H}$ is automatically unique and injective. Also inversely, any map from a subset of the Galois side to the automorphic side, which preserves L-factors, is automatically unique and injective.

The notion of ramification on both sides coincides. Let p be a prime number, different from ℓ. If Π corresponds to V in Conjecture 39, then Π_p is ramified if and only if V_p is ramified. This follows directly from the fact that (on both sides) "being ramified at a prime" can be read off from the L-factors. More generally, to both a geometric and automorphic representation one may associate a conductor. These conductors are equal if the two representations correspond.

Sometimes cuspidal automorphic forms are assumed to be bounded on $\mathrm{GL}_n(\mathbf{A})$. For the conjecture in the above form, it is really important to take cuspidal automorphic forms to be bounded only on $\mathrm{GL}_n(\mathbf{A})^1$, like we defined in $\mathcal{A}_5$. Otherwise, Conjecture 39 is wrong: there will be no irreducible cuspidal automorphic representation on $\mathrm{GL}_1(\mathbf{A})$ for the cyclotomic character.

The article [**13**] explains the relation between automorphic forms and modular forms for $n = 2$. Briefly, to any newform, or cuspidal eigenform for Hecke operators, one may associate an irreducible cuspidal automorphic representation of $\mathrm{GL}_2(\mathbf{A})$, such that L-factors correspond to L-factors. This article also describes which automorphic $\mathrm{GL}_2(\mathbf{A})$-representations come from a modular form. Therefore, modularity of odd 2-dimensional Galois representations (up to twist by an integral power of the cyclotomic character) is equivalent to automorphy.

(In this paragraph we assume Conjecture 39 is true). The Galois side depends on the prime number ℓ, but the automorphic side does not. Therefore, one may start with a representation V on the Galois side, take the corresponding Π on the automorphic side, change ℓ to a different prime number λ, and go back to the Galois side (with the bijection of the conjecture corresponding to λ), to find an λ-adic Galois representation $V^{(\lambda)}$. Doing this for all prime numbers λ, we find V sitting in a family of geometric representations $\{V^{(\lambda)} | \lambda \text{ prime}\}$. This family of representations has the property that $L_p^{(\iota_\lambda)}(V^{(\lambda)}, p^{-s}) = L_p^{(\iota_{\lambda'})}(V^{(\lambda')}, p^{-s})$, and $HT(V^{(\lambda)}) = HT(V^{(\lambda')})$, for all triples of prime numbers (p, λ, λ') (the $\iota_\lambda \colon \overline{\mathbf{Q}}_\lambda \to \overline{\mathbf{C}}$ are fixed embeddings).

If you go back to chapter 1, where we introduced étale cohomology, the cyclotomic character and discussed Tate modules of elliptic curves, then you see that for all those cases the choice of the prime ℓ was arbitrary, so those representations we already expected to sit in some kind of family. However, a geometric representation can also be a *strict* subquotient V of the étale cohomology of some smooth proper $\mathbf{Q}$-scheme X.

4.1. Another conjecture

Note that in Conjecture 39 we actually put Langlands and Fontaine-Mazur. This is to say that the conjecture we gave above is actually a combination of the Langlands conjecture with the Fontaine-Mazur conjecture. The Langlands conjecture on its own states that the set of automorphic representations as above is in bijection with the irreducible subquotients of the étale cohomology of varieties. In a precise form it is formulated as:

Conjecture 40 **(Langlands)**. — *Let $\iota \colon \overline{\mathbf{Q}}_\ell \to \mathbf{C}$ be an embedding, $H \subset \mathbf{Z}$ a multiset of n integers, and ℓ a prime number. There exists a bijection between*

$$\left\{ \begin{matrix} \text{Irreducible cuspidal automorphic } \mathrm{GL}_n(\mathbf{A})\text{-representations} \\ \text{with infinitesimal character } H \end{matrix} \right\} / \cong,$$

and

$$\left\{ \begin{matrix} \text{Irreducible } \ell\text{-adic } \mathcal{G}_{\mathbf{Q}}\text{-representations } V \text{ such that} \\ \exists X \text{ proper smooth } \mathbf{Q}\text{-scheme, } i \in \mathbf{Z}_{\geq 0},\ j \in \mathbf{Z}\text{:} \\ V \text{ is isomorphic to a subquotient of } H^i(X_{\overline{\mathbf{Q}},\text{ét}}, \overline{\mathbf{Q}}_\ell(r)), \text{ and} \\ \text{the Hodge-Tate numbers of } V \text{ equal to } H \end{matrix} \right\} / \cong,$$

such that if Π is an automorphic representation corresponding to an ℓ-adic $\mathcal{G}_{\mathbf{Q}}$-representation V under the above bijection, then $L_p(\Pi, p^{-s}) = L_p^{(\iota)}(V, p^{-s})$, for nearly all p.

Note that Conjecture 39 and 40 should be equivalent, but they are so *only conjecturally.* To prove their equivalence, one needs the Fontaine-Mazur conjecture.

Chapter 5: Class Field Theory

In this chapter we prove the one dimensional version of the conjectures 39 and 40 for the field $\mathbf{Q}$.

5.1. The automorphic forms on $\mathbf{GL}_1$

Let $r \in \mathbf{Z}$ be an integer. Let $f\colon \mathbf{A}^\times \to \mathbf{C}$ be a smooth function. The conditions $\mathcal{A}_4$ and $\mathcal{A}_5$ on f (recall definitions 38) are empty in case $n = 1$. To see this for $\mathcal{A}_5$, notice hat $(\mathbf{A}^\times)^1 = \{g \in \mathbf{A}^\times | |g| = 1\}$ is compact modulo $\mathbf{Q}^\times$. Moreover, the subgroup $O(1) \times \hat{\mathbf{Z}}^\times \subset \mathbf{A}^\times$ is the only maximal compact subgroup of $\mathbf{A}^\times$, so $\mathcal{A}^\circ_{\{r\}}$ is a $\mathbf{A}^\times$-representation.

Let $\psi\colon \mathbf{A}^\times \to \mathbf{C}^\times$ is a continuous morphism. We let $\mathbf{C}(\psi)$ be the $\mathbf{A}^\times$-representation with space $\mathbf{C}$ and $\mathbf{A}^\times$-action defined by $\mathbf{A}^\times \times \mathbf{C}(\psi) \to \mathbf{C}(\psi)$, $(x,z) \longmapsto \psi(x) \cdot z$. The vector space $\mathbf{C}(|\cdot|^{-r}) \otimes \mathcal{A}^\circ_{\{r\}}$ equals $\mathcal{A}^\circ_{\{0\}}$ as $\mathbf{A}^\times$-representations. In case $r = 0$, it follows from $\mathcal{A}_3$ that any $f \in \mathcal{A}^\circ_{\{0\}}$ is locally constant, so we conclude

$$\mathcal{A}^\circ_{\{0\}} = \{f\colon \mathbf{Q}^\times \mathbf{R}^\times_{>0} \backslash \mathbf{A}^\times \longrightarrow \mathbf{C} | f \text{ locally constant}\} = \bigoplus_{\psi \in \mathrm{Hom}_{\mathrm{cts}}(\mathbf{Q}^\times \mathbf{R}^\times_{>0} \backslash \mathbf{A}^\times, \mathbf{C}^\times)} \mathbf{C}(\psi).$$

Twisting back the r, we get

$$\mathcal{A}^\circ_{\{r\}} = \bigoplus_{\psi \in \mathrm{Hom}_{\mathrm{cts}}(\mathbf{Q}^\times \backslash \mathbf{A}^\times, \mathbf{C}^\times),\ \psi(\mathbf{R}^\times_{>0}) = \{1\}} \mathbf{C}(|\cdot|^r \cdot \psi), \tag{(V.22)}$$

as $\mathbf{A}^\times$-representations.

Let $\mathbf{C}(|\cdot|^r \cdot \psi)$ be one of the irreducible components of $\mathcal{A}^\circ_{\{r\}}$ and p a prime number which is unramified in $\pi := |\cdot|^r \cdot \psi$. The L-factor of π at p is given by

$$L_p(\pi, s) = \frac{1}{1 - \pi(\hat{p}) p^{-s}} = \frac{1}{1 - \psi(\hat{p}) p^{-r-s}} \in \mathbf{C}(p^{-s}), \tag{(V.23)}$$

where $\hat{p} \in \mathbf{A}^\times = \mathbf{A}^{p,\times} \times \mathbf{Q}_p^\times$ equals $(1, p)$.

5.2. Hodge-Tate characters

Theorem 41. — *Let $\rho\colon \mathcal{G}_\mathbf{Q} \to \overline{\mathbf{Q}}_\ell^\times$ be a continuous morphism and $r \in \mathbf{Z}$. The following are equivalent:*

1. *ρ is de Rham at ℓ with $HT(\rho) = \{r\}$;*
2. *ρ is Hodge-Tate at ℓ with $HT(\rho) = \{r\}$;*
3. *there exists a continuous morphism $\phi\colon \mathcal{G}_\mathbf{Q} \to \overline{\mathbf{Q}}_\ell^\times$ with finite image such that $\rho = \phi \otimes \chi_\ell^r$.*

Before we prove this theorem, we first give a remark and state and prove a lemma.

Remark. — The local Galois group $\mathcal{G}_{\mathbf{Q}_\ell}$ has a one-dimensional ℓ-adic Hodge-Tate representation $\rho\colon \mathcal{G}_{\mathbf{Q}_\ell} \to \overline{\mathbf{Q}}_\ell^\times$ with $HT(\rho) = 0$ and infinite image. In fact, any continuous morphism $\rho\colon \mathcal{G}_{\mathbf{Q}_\ell} \to \overline{\mathbf{Q}}_\ell^\times$ which is unramified is Hodge-Tate. We have $\mathcal{G}_{\mathbf{Q}_\ell}/\mathcal{I}_{\mathbf{Q}_\ell} = \hat{\mathbf{Z}}$, and $\hat{\mathbf{Z}}$ admits plenty of continuous morphisms $\rho\colon \hat{\mathbf{Z}} \to \overline{\mathbf{Q}}_\ell^\times$. Such a morphism extends to a global continuous morphism $\mathcal{G}_{\mathbf{Q}} \to \overline{\mathbf{Q}}_\ell^\times$ if and only if the image of ρ is finite.

Lemma 42. — *Let $\rho\colon \mathcal{G}_{\mathbf{Q}_\ell} \to \mathrm{GL}(V)$ be a $\mathcal{G}_{\mathbf{Q}_\ell}$-representation over $\overline{\mathbf{Q}}_\ell$ with finite image. Then ρ is de Rham with $HT(\rho)$ the multiset in which 0 occurs with multiplicity* $\dim V$.

Proof. — Let $L := (\overline{\mathbf{Q}}_\ell)^{\ker(\rho)}$, $G := \mathcal{G}_{\mathbf{Q}_\ell}/\mathcal{G}_L = \mathrm{Gal}(L/\mathbf{Q}_\ell)$ and $V_L := L \otimes_{\mathbf{Q}_\ell} V$. Then

$$\mathbf{D}_{\mathrm{dR}}(V) = (\mathrm{B}_{\mathrm{dR}}^{\mathcal{G}_L} \otimes_{\mathbf{Q}_\ell} V)^G = (V_L)^G.$$

We have $L \cong \mathbf{Q}_\ell[G]$ as $\mathbf{Q}_\ell[G]$-modules, so

$$(V_L)^G \cong (\mathbf{Q}_\ell[G] \otimes_{\mathbf{Q}_\ell} V)^G \cong V,$$

as $\overline{\mathbf{Q}}_\ell$-vector spaces. Therefore, $\dim \mathbf{D}_{\mathrm{dR}}(V) = \dim V$ and V is de Rham. The second statement is proved in the same way by replacing B_{dR} in the above calculations with $\mathbf{C}_\ell$ and using that $\mathbf{C}_\ell^{\mathcal{G}_L} = L$.

Proof of Theorem 41. — The implication (1)⇒(2) is true for ℓ-adic Galois representations of any dimension, see Proposition 13, and the implication (3)⇒(1) follows from Lemma 42. The only difficult implication is (2)⇒(3), the proof runs until the end of this section. We first need several preparations.

We introduce the *Tate trace* for $K/\mathbf{Q}_\ell$ a finite extension, contained in $\overline{\mathbf{Q}}_\ell$. For $n \in \mathbf{Z}_{\geq 0}$ define $K_n := K(\boldsymbol{\mu}_{\ell^n})$ and $K_\infty := K(\boldsymbol{\mu}_{\ell^\infty})$. The Tate trace is for each $n \in \mathbf{Z}_{\geq 0}$ defined by:

$$R_n\colon K_\infty \longrightarrow K_n, x \mapsto \frac{1}{[K_n(x) : K_n]} \mathrm{Tr}_{K_n(x)/K_n}(x).$$

There are several facts of the Tate trace that we will use. (T1) R_n is uniformly continuous, so it extends uniquely to a continuous, additive map $R_n\colon \widehat{K_\infty} \to K_n$. And (T2) $\lim_{n\to\infty} R_n(x) = x$ for all $x \in \widehat{K_\infty}$. Next (T3), R_n is K_n-linear, and (T4) for all $\sigma \in \mathcal{G}_K$ and $x \in \widehat{K_\infty}$ we have $R_n(\sigma x) = \sigma R_n(x)$.

The Tate trace is introduced in the article [**38**]. Proposition 6 in this article proves (T1). The properties (T2), (T3) and (T4) are easily seen to be true by checking them on the dense subspace $K_\infty \subset \widehat{K_\infty}$.

Proposition 43. — *Let $\rho\colon \mathcal{G}_K \to K^\times$ be a continuous morphism with infinite image, such that* $\ker(\chi_\ell|_{\mathcal{G}_K}) \subset \mathrm{Ker}(\rho)$. *Then*

$$(\mathbf{C}_\ell \otimes_K K(\rho))^{\mathcal{G}_K} = \{0\},$$

where $\mathcal{G}_K$ acts on $\mathbf{C}_\ell$ via the Galois action, and $K(\rho)$ is the space K with $\mathcal{G}_K$-action via ρ. The tensor product $\mathbf{C}_\ell \otimes_K K(\rho)$ is equipped with the $\mathcal{G}_K$-action on both factors.

Proof. — Via the bijection

$$\mathbf{C}_\ell \otimes_K K(\rho) \xrightarrow{\sim} \mathbf{C}_\ell, \quad a \otimes b \mapsto ab,$$

we get an action of $\mathcal{G}_K$ on $\mathbf{C}_\ell$, which is different from the Galois action. Let us denote this action by $\bullet$ to distinguish it clearly from the Galois action. If $x \in \mathbf{C}_\ell$ and $\sigma \in \mathcal{G}_{\mathbf{Q}_\ell}$ then $\sigma(x)$ is the result of σ acting on x via the Galois action of $\mathcal{G}_{\mathbf{Q}_\ell}$ on $\mathbf{C}_\ell$.

Let now $x \in \mathbf{C}_\ell^{\bullet,\mathcal{G}_K}$ be a vector, invariant for the action "$\bullet$". Then for all $\sigma \in \mathrm{Ker}(\chi_\ell|_{\mathcal{G}_K})$ we have $x = \sigma \bullet x = \sigma(x)$. Therefore

$$x \in \mathbf{C}_\ell^{\mathrm{Ker}(\chi_\ell|_{\mathcal{G}_K})} = \widehat{K_\infty},$$

(use Ax-Sen-Tate, [**10**, 2.6]). We have

$$x = \lim_{n\to\infty} R_n(x).$$

So if we can prove that $R_n(x) = 0$ for all n, then clearly we have $x = 0$ and the theorem follows.

For all $\sigma \in \mathcal{G}_K$ we have

$$x = \sigma \bullet x = \rho(\sigma)\sigma(x),$$

hence, for all $\sigma \in \mathcal{G}_K$, $\sigma(x) = \frac{x}{\rho(\sigma)}$. By properties (T2) and (T3) of the Tate trace, we get

$$\sigma(R_n(x)) = R_n(\sigma(x)) = R_n\left(\frac{x}{\rho(\sigma)}\right) = \frac{1}{\rho(\sigma)} R_n(x).$$

Assume for a contradiction that $R_n(x) \neq 0$. Because $\#\mathrm{Im}(\rho) = \infty$ the element $R_n(x) \in K_n$ has infinitely many Galois conjugates over K. But $R_n(x) \in K_n$ is algebraic over K, so that it is impossible. We conclude $R_n(x) = 0$ for all n. This completes the proof of Proposition 43.

Continuation of the proof of Theorem 41. — Let $\rho\colon \mathcal{G}_\mathbf{Q} \to \overline{\mathbf{Q}}_\ell^\times$ be a continuous morphism which is Hodge-Tate at ℓ. Then, by Proposition 7, ρ is of the form $\rho = \phi_1\phi_2$, with $\phi_1 : \mathcal{G}_\mathbf{Q} \to \overline{\mathbf{Q}}_\ell^\times$ with finite image, and $\phi_2 \colon \mathcal{G}_\mathbf{Q} \to \overline{\mathbf{Q}}_\ell^\times$ unramified outside ℓ. Because ϕ_1 has finite image, we may twist it away and asume that $\rho = \phi_2$, i.e. that ρ is unramified outside ℓ. The maximal abelian extension, unramified outside ℓ of $\mathbf{Q}$ is $\mathbf{Q}(\boldsymbol{\mu}_{\ell^\infty})$. Therefore, the kernel of ρ contains the kernel of the cyclotomic character. We may replace ρ by $\rho\chi_\ell^{-r}$ where $\{r\} = HT(\rho)$, so we may assume that ρ is $\mathbf{C}_\ell$-admissible. By Proposition 4, there exists a finite extension K of $\mathbf{Q}_\ell$, contained in $\overline{\mathbf{Q}}_\ell$ such that $\mathrm{Im}(\rho) \subset K^\times \subset \overline{\mathbf{Q}}_\ell^\times$. The property "$\mathbf{C}_\ell$-admissible", is stable for finite extensions of the base field. Therefore, $\rho|_{\mathcal{G}_K}$ is $\mathbf{C}_\ell$-admissible, and Proposition 43 applies to the morphism $\rho|_{\mathcal{G}_K}$. Therefore $\mathrm{Im}(\rho|_{\mathcal{G}_K})$ is finite. Clearly, the image of ρ must then also be finite. This completes the proof of Theorem 41.

5.3. Proof of Conjecture 39 for $n = 1$

Using the results of the previous section it is now easy to deduce Conjecture 39 for $n = 1$. Let $\rho\colon \mathcal{G}_\mathbf{Q} \to \overline{\mathbf{Q}}_\ell^\times$ be a continuous morphism. By Theorem 41 is ρ of the form $\rho = \phi\chi_\ell^r$, where $r \in \mathbf{Z}$ and $\phi\colon \mathcal{G}_\mathbf{Q} \to \overline{\mathbf{Q}}_\ell^\times$ is a continuous morphism with finite image.

Let p be a prime number different from ℓ and such that ρ is not ramified at p. Then $L_p^{(\iota)}(\rho, s) = \frac{1}{1-\iota\phi(p)p^{-s+r}} \in \mathbf{C}(p^{-s})$. Let ψ be the product of the morphism $|\cdot|^r : \mathbf{Q}^\times\backslash\mathbf{A}^\times \to \mathbf{C}^\times$ with the morphism

$$\mathbf{Q}^\times\backslash\mathbf{A}^\times \longrightarrow \mathbf{R}^\times\mathbf{Q}^\times\backslash\mathbf{A}^\times = \hat{\mathbf{Z}}^\times \xrightarrow{\sim} \mathcal{G}_\mathbf{Q}^{\mathrm{ab}} \xrightarrow{\phi} \overline{\mathbf{Q}}_\ell^\times \xrightarrow{\iota} \mathbf{C}^\times.$$

If v is a finite place p different from ℓ where ψ is not ramified, then

$$L_p(\psi_p, s) = \frac{1}{1-\psi_p(p)p^{-s}} = \frac{1}{1-\iota(\rho(\mathrm{Frob}_p))p^{-s}} \in \mathbf{C}(p^{-s}).$$

Hence ψ corresponds to ρ.

The Conjecture 40 for $n = 1$ is now also true, because we have verified the Fontaine-Mazur conjecture for continuous morphisms $\mathcal{G}_{\mathbf{Q}} \to \overline{\mathbf{Q}}_\ell^\times$ in chapter 1.

5.4. Class field theory

We show how the Artin reciprocity map for $\mathbf{Q}$ may be obtained from the bijection in the Langlands conjecture (Conjecture 39). Note that class field theory for $\mathbf{Q}$ is very classical; it follows directly from the Kronecker-Weber theorem (which actually gives more). For this reason, the reader should not take this section too seriously: we have only put it here to further explain the relation between class field theory and the one dimensional Langlands conjecture (the case $n = 1$ in Conjecture 39). On the other hand, the argument that we give here will probably extend to general number fields, if one writes down a proper statement of the global Langlands conjecture for general number fields (or even global fields).

Take the infinitesimal character H equal to $\{0\}$. By formula (V.22), the set A of (isomorphism classes of) cuspidal automorphic $\mathrm{GL}_1(\mathbf{A})$-representations with infinitesimal character 0 is given by $A = \mathrm{Hom}_{\mathrm{cts}}(\hat{\mathbf{Z}}^\times, \mathbf{C}^\times)$.

We denote D the set of (isomorphism classes of) one-dimensional ℓ-adic de Rham representations which are (1) unramified at nearly all primes and (2) have Hodge-Tate weight 0. The sets A and D have the structure of a group via tensor products and duals.

For every embedding $\iota \colon \overline{\mathbf{Q}}_\ell \to \mathbf{C}$, the Langlands conjecture gives a bijection $\mathrm{Art}_{\mathbf{Q}}^{\vee,\ell,\iota} \colon D \to A$ such that L-factors correspond to L-factors. By comparing L-factors we see that $\mathrm{Art}_{\mathbf{Q}}^{\vee,\ell,\iota}$ is a morphism of groups.

Let $\rho \in D$; we show that ρ has finite image. The morphism $\psi = \mathrm{Art}_{\mathbf{Q}}^{\vee,\ell,\iota,-1}(\rho) \in A$ has finite image, and so ρ has finite image on the set $\{\mathrm{Frob}_p | p \text{ unramified prime}\} \subset \mathcal{G}_{\mathbf{Q}}^{\mathrm{ab}} / \ker(\rho)$. We claim that the subgroup C generated by this set is dense. To see this, let $\vartheta \colon \mathcal{G}_{\mathbf{Q}}^{\mathrm{ab}}/\overline{C} \to \overline{\mathbf{Q}}_\ell^\times$ be with finite image, then it corresponds to a $\phi \in A$. For nearly all p we have $\phi(\hat{p}) = 1$, where $\hat{p} = (1, p^{-1}) \in \mathbf{Z}_p^\times \times \mathbf{Z}^{\{p\},\times}$, so ϕ is trivial and thus ϑ is trivial, hence $(\mathcal{G}_{\mathbf{Q}}^{\mathrm{ab}}/\overline{C})^\vee = 1$, and $\overline{C} = \mathcal{G}_{\mathbf{Q}}^{\mathrm{ab}}$. Because C is dense, ρ has finite image. Inversely, we know that if ρ has finite image, then it is de Rham (see Lemma 42 of the previous section).

We conclude $D = \mathrm{Hom}_{\text{grp, cts, fin. img}}(\mathcal{G}_{\mathbf{Q}}^{\mathrm{ab}}, \overline{\mathbf{Q}}_\ell^\times)$, so we have an isomorphism

$$\mathrm{Art}_{\mathbf{Q}}^{\vee,\ell,\iota} \colon \mathrm{Hom}_{\text{grp, cts, fin. img}}(\mathcal{G}_{\mathbf{Q}}^{\mathrm{ab}}, \overline{\mathbf{Q}}_\ell^\times) \xrightarrow{\sim} \mathrm{Hom}_{\text{grp, cts}}(\hat{\mathbf{Z}}^\times, \mathbf{C}^\times).$$

On applying ι we find an isomorphism

$$\mathrm{Art}_{\mathbf{Q}}^{\vee} \colon \mathrm{Hom}_{\text{grp, cts}}(\mathcal{G}_{\mathbf{Q}}^{\mathrm{ab}}, \mathbf{C}^\times) \xrightarrow{\sim} \mathrm{Hom}_{\text{grp, cts}}(\hat{\mathbf{Z}}^\times, \mathbf{C}^\times),$$

which does not depend on ι. Apply Pontryagin duality to obtain the Artin map $\mathrm{Art}_{\mathbf{Q}} \colon \hat{\mathbf{Z}}^\times \xrightarrow{\sim} \mathcal{G}_{\mathbf{Q}}^{\mathrm{ab}}$.

Chapter 6: Elliptic curves with complex multiplication

In this chapter we will compute for an explicit elliptic curve with complex multiplication the automorphic representation corresponding to the Tate Module of this curve.

6.1. Hecke characters

Let K be a number field, with ring of integers $\mathcal{O}_K$. The *ring of K-adèles*, notation $\mathbf{A}_K$, is the restricted product of the completions K_v, where v runs over all places of K, and with respect to the valuation rings $\mathcal{O}_v \subset K_v$. As topological K-algebras one has $\mathbf{A}_K = \mathbf{A}_\mathbf{Q} \otimes_\mathbf{Q} K$. We write $\mathbf{A}_K^\infty$ for the finite adèles of K, and $K_\infty = K \otimes_\mathbf{Q} \mathbf{R}$ for the infinite adèles of K. We denote with $K_\infty^{\times,\circ}$ the connected component of $1 \in K_\infty^\times$, and $K_v^{\times,\circ}$ the connected comonent of $1 \in K_v^\times$.

A continuous morphism $\chi\colon \mathbf{A}_K^\times / K^\times \to \mathbf{C}^\times$ is a *Hecke character* if for each Archimedian prime v of K there exist $a, b \in \mathbf{Z}$ such that for all $z \in K_v^{\times,\circ}$ we have $\chi|_{K_v^{\times,\circ}}(z) = z^a \bar{z}^b$. This condition is equivalent to the existence of a morphism of algebraic groups $T(\Box)\colon (\Box \otimes_\mathbf{R} K_\infty)^\times \to (\Box \otimes_\mathbf{R} \mathbf{C})^\times$, such that $T(\mathbf{R})|_{K_\infty^{\times,\circ}} = \chi|_{K_\infty^{\times,\circ}}$. Such a morphism $T(\Box)$ is automatically unique once it exists.

Note that we have to make the restriction to the connected component of identity, $K_\infty^{\times,\circ} \subset K_\infty^\times$, because otherwise, in case $K_\infty^\times$ is non-connected, the adelic norm $|\cdot|\colon \mathbf{A}_K^\times \to \mathbf{C}^\times$ would not be a Hecke character. For example if $K = \mathbf{Q}$, then the composition $\mathbf{R}^\times \to \mathbf{A}_\mathbf{Q}^\times \xrightarrow{|\cdot|} \mathbf{C}^\times$, is the usual absolute value on $\mathbf{R}^\times$, which does not come from a morphism of algebraic groups.

If v is a place of K, then χ_v is defined as the composition

$$K_v^\times \longrightarrow \mathbf{A}_K^\times / K^\times \longrightarrow \mathbf{C}^\times.$$

Assume $v = \mathfrak{p}$ is a finite place. We say that χ is *unramified* at $\mathfrak{p}$ if $\chi_\mathfrak{p}$ is trivial on $\mathcal{O}_{K_\mathfrak{p}}^\times$. In this case $\chi_\mathfrak{p}(\pi_\mathfrak{p})$ does not depend on the choice of a local uniformizer $\pi_\mathfrak{p} \in K_\mathfrak{p}$. If χ is unramified at $\mathfrak{p}$, then the L-factor of χ at $\mathfrak{p}$ is defined by

$$L_\mathfrak{p}(\chi, s) \stackrel{\text{def}}{=} (1 - \chi_\mathfrak{p}(\pi_\mathfrak{p}) N(\mathfrak{p})^{-1})^{-1} \in \mathbf{C}(p^{-s}).$$

If χ is ramified at $\mathfrak{p}$, then the L-factor is defined to be 1. The Hecke character χ has also L-factors for the primes at infinity, in this thesis we do not bother about these factors.

The L-function of χ is the product of all the L-factors, and it is known to converge in some right half plane, have meromorphic continuation and satisfy a functional equation (Tate's thesis, see [**9**]).

Let $\chi\colon \mathbf{A}_K^\times \to \mathbf{C}^\times$ be a Hecke character. Let us define the *conductor* of χ. Recall that the open subgroups $1 + \mathfrak{p}^n \subset K_\mathfrak{p}^\times$, where $n \in \mathbf{Z}_{\geq 0}$ varies, form a fundamental system of open neighborhoods of identity. Therefore, there exists a minimal integer $n_\mathfrak{p} \in \mathbf{Z}_{\geq 0}$ such that $\chi_\mathfrak{p}$ is trivial on $1 + \mathfrak{p}^{n_\mathfrak{p}}$. The conductor of χ is the $\mathcal{O}_K$-ideal

$$\prod_{\mathfrak{p} \subset \mathcal{O}_K \text{ prime}} \mathfrak{p}^{n_\mathfrak{p}}.$$

6.2. Some remarks on the Tate module

Proposition 44. — *Let $E/\mathbf{Q}$ be an elliptic curve and ℓ a prime number. Then*

1. *the Hodge Tate multiset $HT(V_\ell(E))$ is equal to $\{0,1\}$;*
2. *the $\mathcal{G}_{\mathbf{Q}}$-representation $V_\ell(E)$ is irreducible;*
3. *if E has CM over the field $K/\mathbf{Q}$, then the $\mathcal{G}_K$-representation $V_\ell(E)$ is reducible.*

Proof. — (**1**) This is proved in [**38**, p. 180, cor. 2]. We can give a direct proof, but we have to use the (highly non-trivial) result of Faltings stating that the Hodge-Tate multiset of $V_\ell(E)$ is independent of ℓ, see Remark 2.3 for the precise statement. Recall also that $V_\ell(E)$ is isomorphic to the dual of $H^1(E_{\overline{\mathbf{Q}}_\ell,\text{ét}}, \overline{\mathbf{Q}}_\ell)$ as $\overline{\mathbf{Q}}_\ell[\mathcal{G}_{\mathbf{Q}_\ell}]$-modules [**27**, 15.1].

Let $\{a,b\} \in \mathbf{N}^{(\mathbf{Z})}$ be the Hodge-Tate multiset of E. We accept that the numbers a, b do not depend on ℓ.

We claim that there exists a prime number ℓ such that E has ordinary reduction at ℓ. There are several ways to prove this. One way is to use modularity to get the following 'optimal' results: We have $a_p \neq 0$ for a density 1 set of primes p if E has no CM, and if E has CM, then this density equals to $\frac{1}{2}$. So certainly, a prime ℓ where E has ordinary reduction exists.

Without using modularity, we may prove that such a prime exists in the following manner. Let S be the finite set consisting of ℓ, together with the primes p where E has bad reduction. The representation $\rho\colon \mathcal{G}_{\mathbf{Q}} \to \mathrm{GL}(V_\ell(E))$ factors to a representation $\rho'\colon \mathcal{G}_{\mathbf{Q},S} \to \mathrm{GL}(V_\ell(E))$, where $\mathcal{G}_{\mathbf{Q},S}$ is the Galois group of the maximal unramified-outside-S-extension of $\mathbf{Q}$. Consider the closed subset Z of $\mathrm{GL}(V_\ell(E))$ consisting of all matrices with trace 0. The image of $1 \in \mathcal{G}_{\mathbf{Q},S}$ in $\mathrm{GL}(V_\ell(E))$ does not lie in Z, because the identity matrix has trace $2 \neq 0$. Therefore, the inverse image of the complement $Z^{\complement} \subset \mathrm{GL}(V_\ell(E))$ in $\mathcal{G}_{\mathbf{Q},S}$ is a non-empty open subset. By the Chebotariev density theorem, there exists a prime $p \notin S$ such that Frob_p lies in this inverse image. By construction, the trace of Frob_p acting on $V_\ell(E)$ is non-zero and so E has ordinary reduction at p.

(As an exercise, the reader may try to adapt the above argument to prove that there is a density $\frac{1}{48}$ set of primes (within the set of all primes) where E has ordinary reduction.)

Pick ℓ a prime where E has ordinary reduction. Let $\tilde{E}/\mathbf{Z}[1/N]$ be an elliptic curve with generic fiber E, where N is the product of the primes where E has bad reduction. Observe that $\ell \nmid N$.

Consider the exact sequence

$$0 \longrightarrow A \longrightarrow T_\ell(E) \longrightarrow T_\ell(\tilde{E} \times \mathbf{F}_\ell) \longrightarrow 0, \tag{(VI.24)}$$

of $\mathbf{Z}_\ell[\mathcal{G}_{\mathbf{Q}_\ell}]$-modules, where A is the kernel of $T_\ell(E) \to T_\ell(\tilde{E} \times \mathbf{F}_\ell)$. The Galois action on $T_\ell(\tilde{E} \times \mathbf{F}_\ell) \cong \mathbf{Z}_\ell$ is trivial and $\mathbf{C}_\ell \otimes_{\mathbf{Z}_\ell} T_\ell(E) \cong \mathbf{C}_\ell(a) \oplus \mathbf{C}_\ell(b)$. Therefore, one of the Hodge-Tate numbers of $V_\ell(E)$ is 0.

Existence of the Weil paring $T_\ell(E) \times T_\ell(E) \to \mathbf{Z}_\ell(1)$ implies that $\bigwedge^2 T_\ell(E) = \mathbf{Z}_\ell(1)$, so $a + b = 1$, hence $HT(V_\ell(E)) = \{0,1\}$. This completes the proof of point (1).

(**2**) This is proved in [**38**]. We may give a direct proof, but again we have to admit the (non-trivial) result which states that $V_\ell(E)$ is a Hodge-Tate representation (in fact it is de Rham).

Assume for a contradiction that $V_\ell(E)$ is reducible. Then $V_\ell(E)^{\mathrm{ss}} = U \oplus W$ where $(u,U),(w,W)$ are two one dimensional subrepresentations of $V_\ell(E)^{\mathrm{ss}}$. Then U and W are Hodge-Tate, because the property "Hodge-Tate" is stable under subquotients.

Assume $HT(u) = \{0\}$ and $HT(w) = \{1\}$. By Theorem 41 we know that u and $v\chi_\ell^{-1}$ have finite image. Let S be the set consisting of ℓ and the primes where E has bad reduction. The group $\mathcal{G}_{\mathbf{Q},S}$ is the absolute-unramified-outside-S-quotient of $\mathcal{G}_{\mathbf{Q}}$. The representation ρ factors to a representation $\mathcal{G}_{\mathbf{Q},S} \to \mathrm{GL}(V_\ell(E))$, which we also denote by ρ. Let $H = \ker(u) \cap \ker(w\chi_\ell^{-1}) \subset \mathcal{G}_{\mathbf{Q}}$. Then H is an open subgroup of $\mathcal{G}_{\mathbf{Q},S}$. By the Chebotariev density theorem, there exists a prime $p \neq \ell$ where E has good reduction, such that Frob_p lies in H. We have $u(\mathrm{Frob}_p) = 1$ and $w(\mathrm{Frob}_p) = p$. But $a_p(E) = u(p) + w(p) = p + 1$ so $\#\tilde{E}(\mathbf{F}_p) = 0$ (Theorem 8), which is impossible.

(**3**) Assume that E has complex multiplication over K. Because $T_\ell(\square)$ is a functor we have a map $\mathcal{O}_K \to \mathrm{End}(T_\ell(E))$. The $\mathcal{G}_K$-action on $T_\ell(E)$ is $\mathcal{O}_K$-linear, and $T_\ell(E)$ is a free $\mathbf{Z}_\ell \otimes \mathcal{O}_K$-module of rank 1, [**37**, II.1.4]. This implies that $V_\ell(E)$ is abelian as $\mathcal{G}_K$-representation and in particular it is reducible.

6.3. Elliptic curves with complex multiplication

Let $E/\mathbf{Q}$ be an elliptic curve with complex multiplication by the ring of integers $\mathcal{O}_K$ of a quadratic imaginary field $K \subset \mathbf{C}$. Let N be the conductor of E, and $\tilde{E}/\mathbf{Z}[1/N]$ an elliptic curve equipped with an isomorphism $\tilde{E} \times \mathbf{Q} \cong E$.

Let $\mathfrak{p}$ be a prime of K where $E_K = E \times K$ has good reduction, let $\tilde{E}_{K,\mathfrak{p}}$ be the reduction of E_K modulo $\mathfrak{p}$, and let $\phi_\mathfrak{p} \colon \tilde{E}_{K,\mathfrak{p}} \to \tilde{E}_{K,\mathfrak{p}}$ be the Frobenius, uniquely determined by

$$\tilde{E}_{K,\mathfrak{p}}(\overline{\kappa(\mathfrak{p})}) \longrightarrow \tilde{E}_{K,\mathfrak{p}}(\overline{\kappa(\mathfrak{p})}), \quad (x,y,z) \mapsto (x^{1/\#\kappa(\mathfrak{p})}, y^{1/\#\kappa(\mathfrak{p})}, z^{1/\#\kappa(\mathfrak{p})}),$$

in projective coordinates, where $\overline{\kappa(\mathfrak{p})}$ is an algebraic closure of the residue field $\kappa(\mathfrak{p})$ at $\mathfrak{p}$. Pick for every K-prime $\mathfrak{p}$ a uniformizer $\pi_\mathfrak{p} \in K_\mathfrak{p}$, and write $\hat{\pi}_\mathfrak{p} := (1, \pi_\mathfrak{p}) \in \mathbf{A}_K^{\mathfrak{p},\times} \times K_\mathfrak{p}^\times$. By [**37**, 10.4], there exists a unique Hecke character $\chi \colon \mathbf{A}_K^\times / K^\times \to \mathbf{C}^\times$ such that for all primes $\mathfrak{p}$ where E has good reduction, we have $\chi_E(\hat{\pi}_\mathfrak{p}) \in \mathcal{O}_K$ and the endomorphism $[\chi_E(\hat{\pi}_\mathfrak{p})] \in \mathrm{End}(\tilde{E}_{\mathcal{O}_K[1/N]})$ reduces modulo $\mathfrak{p}$ to the Frobenius $\phi_\mathfrak{p}$.

We have

$$L_\mathfrak{p}(\overline{\chi_E}, s) L_\mathfrak{p}(\chi_E, s) = L_\mathfrak{p}^{(\iota)}(V_\ell(E_K), s), \qquad \text{((VI.25))}$$

for all $\mathfrak{p}$ where E has good reduction, and where $\iota \colon \overline{\mathbf{Q}}_\ell \to \mathbf{C}$ is an isomorphism of fields [**37**, 10.4.1]. The Hecke character χ_E induces a cuspidal automorphic representation $\Pi_{E,K}$ of $\mathrm{GL}_1(\mathbf{A}_K)$. The $\mathcal{O}_K$-module $T_\ell(E_K)$ is free of rank 1, and $\mathcal{G}_K$ acts on it linearly. We get a morphism $\mathcal{G}_K \to \mathcal{O}_{K,\ell}^\times$. By equation (VI.25), the representation $V_\ell(E_K)$ corresponds to $\Pi_{E,K}$ under the global Langlands correspondence (for the number field K).

Let $\mathfrak{a} \subset \mathcal{O}_K$ be an ideal which is coprime with N. We define $\chi_E(\mathfrak{a}) := \chi_E(\prod_\mathfrak{p} \pi_\mathfrak{p}^{v_\mathfrak{p}(\mathfrak{a})})$. Let $L(\chi_E, s)$ be the L-function of χ_E. Then $L(\chi_E, s)$ is of the form $\sum_{n \in \mathbf{Z}_{\geq 1}} a_n n^{-s}$. The function

$\sum_{n\in\mathbf{Z}_{\geq 1}} a_n q^n$ is the q-expansion of a modular form $f\colon \mathfrak{h} \to \mathbf{C}$, see [**30**, 4.8.2, 4.5.16]. This modular form is a cupsidal eigenform, has weight 2 and has conductor N.

The L-factor of f at a prime p not dividing N, is given by $(1 - a_p(f)p^{-s} + pp^{-2s})^{-1}$, where $a_p(f) = 0$ if p is inert and $a_p(f) = \chi_E(\mathfrak{p}) + \chi_E(\overline{\mathfrak{p}})$ if $p = \mathfrak{p}\overline{\mathfrak{p}}$ in $\mathcal{O}_K$ splits. By equation (VI.25), f is the modular form corresponding to E.

Let $\Pi_E = \Pi_{E,\infty} \otimes \bigotimes'_p \Pi_{E,p}$ be the automorphic representation corresponding to E, under the global Langlands correspondence for K. For the primes p where E has good reduction, one has $\Pi_{E,p} \cong \iota_{B(\mathbf{Q}_p)}^{\mathrm{GL}_2(\mathbf{Q}_p)}(\mu_{p,1}, \mu_{p,2})$, where $\mu_{p,1}, \mu_{p,2}\colon \mathbf{Q}_p^\times \to \mathbf{C}^\times$ are two morphisms which are trivial on $\mathbf{Z}_p^\times$. The representation $(\mu_{p,1}, \mu_{p,2})$ is the morphism from $B(\mathbf{Q}_p) = \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \subset \mathrm{GL}_2(\mathbf{Q}_p)$ to $\mathbf{C}^\times$, sending a matrix $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$ to $\mu_{p,1}(a)\mu_{p,2}(d) \in \mathbf{C}^\times$. By comparing L-factors we have

$$\{\mu_{1,p}(p), \mu_{2,p}(p)\} = \begin{cases} \{(-p)^{-\frac{1}{2}}, -(-p)^{-\frac{1}{2}}\} & p \text{ is inert in } K \\ \{\chi_E(\mathfrak{p}), \chi_E(\overline{\mathfrak{p}})\} & p \text{ splits into } \mathfrak{p}\cdot\overline{\mathfrak{p}} \text{ in } K, \end{cases}$$

which determines $\Pi_{E,p}$ up to isomorphism.

The module $\Pi_{E,\infty}$ is (up to isomorphism) the same for all elliptic curves E. This follows from the fact that the infinitesimal character of E is given by $H = \{0, 1\}$, and the classification of the $(\mathfrak{gl}_n, O(n))$-modules occurring in a cuspidal automorphic representation, see [**39**, p. 20].

6.4. Example: The elliptic curve $y^2 = x^3 - 4x$

Let E be the elliptic curve over $\mathbf{Q}$ given by the Weierstrass equation $y^2 = x^3 - 4x$. The curve E has complex multiplication over the field $K = \mathbf{Q}(i)$, where $i \in \mathbf{C}$ acts by $E(\overline{\mathbf{Q}}) \to E(\overline{\mathbf{Q}}), (x, y) \mapsto (-x, iy)$. The polynomial $x^3 - 4x$ is separable modulo all primes p different from 2, so the model $\tilde{E} \subset \mathbf{P}^2_{\mathbf{Z}[1/2]}$ given by the same equation ($y^2 = x^3 - 4x$) is smooth and defines an elliptic curve over $\mathbf{Z}[1/2]$ with generic fiber E. The representation $V_\ell(E)$ is unramified outside 2ℓ. For primes p congruent to 3 mod 4, we have $L_p^{(\iota)}(V_\ell(E), s) = (1 + pp^{-2s}) \in \mathbf{C}(p^{-s})$. The conductor of E over $\mathbf{Q}$ is 2^6; over $\mathbf{Q}(i)$ it equals $(1 + i)^4$. The Hecke character χ_E corresponding to E has conductor $(1 + i)^4$. To determine χ_E, we will calculate all Hecke characters of $\mathbf{A}^\times_{\mathbf{Q}(i)}$ with conductor $\leq (1 + i)^4$.

Define:

$$\begin{aligned} U^\infty &:= \{x \in \widehat{\mathbf{Z}[i]}^\times | v_{(1+i)}(1 - x) \geq 4\} \subset \mathbf{A}^{\infty,\times}_{\mathbf{Q}(i)} \\ U &:= \{1\} \times U^\infty \subset \mathbf{A}^\times_{\mathbf{Q}(i)} \\ V &:= \{x \in \mathbf{Z}_2[i]^\times | v_{(1+i)}(1 - x) \geq 4\} = 1 + (1 + i)^4\mathbf{Z}_2[i] \\ \varphi &\colon \mathbf{C}^\times \longrightarrow (\mathbf{C}^\times \times \mathbf{A}^{\infty,\times})/\mathbf{Q}(i)^\times U, \quad z \mapsto (z, 1). \end{aligned}$$

We have $\ker(\varphi) = (1 + i)^{\mathbf{Z}} i^{\mathbf{Z}} \cap V = 1$.

The cokernel of φ is given by:

$$\begin{aligned} \operatorname{coker}(\varphi) &= \mathbf{C}^\times\mathbf{Q}(i)^\times U \backslash \mathbf{A}^\times_{\mathbf{Q}(i)} = \mathbf{Q}(i)^\times U^\infty \backslash \mathbf{A}^{\infty,\times}_{\mathbf{Q}(i)} = \mathbf{Q}(i)^\times (V \times \widehat{\mathbf{Z}[1/2, i]}^\times) \backslash \mathbf{A}^{\infty,\times}_{\mathbf{Q}(i)} \\ &= (\mathbf{Q}(i)^\times \cap \widehat{\mathbf{Z}[1/2, i]}^\times) V \backslash \mathbf{Q}_2(i)^\times = (i + 1)^{\mathbf{Z}} i^{\mathbf{Z}} V \backslash \mathbf{Q}_2(i)^\times = i^{\mathbf{Z}} V \backslash \mathbf{Z}_2[i]^\times. \end{aligned}$$

One has $(\mathbf{Z}/4\mathbf{Z})[i]^\times \cong \langle i\rangle \times \langle 3+2i\rangle$, so $\mathbf{Z}_2[i]/i^{\mathbf{Z}}V = \langle 3+2i\rangle$, and the sequence

$$1 \longrightarrow \mathbf{C}^\times \xrightarrow{\varphi} \mathbf{Q}(i)^\times U\backslash \mathbf{A}_{\mathbf{Q}(i)}^\times \longrightarrow \operatorname{coker}(\varphi) \longrightarrow 1,$$

splits.

We conclude that there is an isomorphism

$$\mathbf{Z}^2 \times \operatorname{Hom}(\langle 3+2i\rangle, \{\pm 1\}) \xrightarrow{\sim} \{\text{Hecke characters with conductor} \leq (1+i)^4\}$$

$$((a,b),\varepsilon) \longmapsto \begin{cases} \mathbf{Q}(i)^\times U\backslash \mathbf{A}_{\mathbf{Q}(i)}^\times \cong & \mathbf{C}^\times \times \langle 3+2i\rangle \to \mathbf{C}^\times \\ & (z,\eta) \longmapsto z^a \overline{z}^b \varepsilon(\eta). \end{cases}$$

Let π be prime of $\mathbf{Q}(i)$ different from (π), and χ a Hecke character with conductor $\leq (1+i)^4$. We calculate the local L-factor $L_\pi(\chi,s)$. There exist an unique $x \in \mathbf{Z}/4\mathbf{Z}$ and $y \in \{0,1\}$ such that $\pi \cdot i^x \cdot (3+2i)^y \in V$. Let $\hat{\pi}$ be the element of $\mathbf{A}_{\mathbf{Q}(i)}^{(1+i),\infty,\times}$ with π on the coordinate corresponding to (π) and with 1 on all other coordinates.

We need some notation: With a tuple $(*,*,*) \in \mathbf{A}_{\mathbf{Q}(i)}^\times$ above, we mean the idèle viewed as element of $\mathbf{C}^\times \times \mathbf{Q}_2(i)^\times \times \mathbf{A}_{\mathbf{Q}(i)}^{\infty,(1+i),\times} = \mathbf{A}_{\mathbf{Q}(i)}^\times$. We may now calculate:

$$\begin{aligned}\chi(1,1,\hat{\pi}) &= \chi(\pi^{-1},\pi^{-1},\hat{\pi}/\pi) = \chi(\pi^{-1}, i^x(3+2i)^y, 1) \\ &= \chi(i^{-x}\pi^{-1},(3+2i)^y,1) = i^{by-ax}\pi^{-a}\overline{\pi}^{-b}\varepsilon((3+2i)^y),\end{aligned}$$

Therefore:

$$L_\pi(\chi,s) = \frac{1}{1 - i^{by-ax}\pi^{-a}\overline{\pi}^{-b}\varepsilon((3+2i)^y) N_{\mathbf{Q}(i)/\mathbf{Q}}(\pi)^{-s}}. \tag{(VI.26)}$$

We can now determine χ_E. The Hodgte-Tate multiset of an elliptic curve is given by $\{0,1\}$, so $a=1$ and $b=0$ in the above, and there are only two Hecke characters χ_ε left, the Hecke character with $\varepsilon \neq 1$ and the one with $\varepsilon = 1$. Some calculations yield the following table for the values $a_p(\chi_\varepsilon)$:

p:	5	13	17	29	37	41	53	61	73
$\varepsilon = 1$:	-2	6	2	-10	-2	10	14	-10	-6
$\varepsilon \neq 1$:	2	-6	2	10	2	10	-14	10	-6,

We have $a_p(E) = 1 - \#\tilde{E}(\mathbf{F}_p) + p$ if $p \neq \ell$, and by counting solutions to to the equation $y^2 = x^3 - 4x$ over $\mathbf{F}_5$ we conclude that the Hecke character χ_E corresponding to E is the Hecke character with $\varepsilon \neq 1$.

Recall that any Hecke character gives a cuspidal eigenform of weight 2, and that cuspidal eigenforms correspond to elliptic curves over $\mathbf{Q}$ modulo $\mathbf{Q}$-isogeny. Therefore, there should be an elliptic curve corresponding to $\varepsilon = 1$. Indeed, the curve $y^2 = x^3 - x$ has conductor $(1+i)^3 \leq (1+i)^4$ has complex multiplication over $\mathbf{Q}(i)$, and is not isogeneous to E, so its Hecke character is the one with $\varepsilon = 1$.

Chapter 7: Weil-Deligne representations

We introduce the Weil group and the notion of Weil-Deligne representations so that we can state the local Langlands theorem. Most of this chapter is copied from the book [**7**, chap. 7].

7.1. The Weil group of a local field

Let F be a non-Archimedian local field of residue characteristic p. Fix a separable closure F^s of F. The *Weil-group* $\mathcal{W}_F$ of F^s/F is the (abstract) subgroup of $\mathcal{G}_F = \mathrm{Gal}(F^s/F)$ generated by all the Frobenius elements. The topology on $\mathcal{W}_F$ is the sparsest topology for which the inertia subgroup $\mathcal{I}_F \subset \mathcal{G}_F$ is an open subgroup of $\mathcal{W}_F$ with its profinite topology.

Denote by k the residue field of F and with k^s the residue field of F. Then $\mathcal{W}_F$ is the inverse image of $\mathbf{Z} \subset \hat{\mathbf{Z}} = \mathrm{Gal}(k^s/k)$ under the morphism of reduction $\mathcal{G}_F \to \mathcal{W}_F$. Local class field theory yields an isomorphism $\mathrm{Art}_F \colon \mathcal{W}_F^{\mathrm{ab}} \to F^\times$.

Let E be a field of characteristic 0 and V an E-vector space. The group $\mathcal{W}_F$ is a locally profinite group. A $\mathcal{W}_F$-representation $\rho\colon \mathcal{W}_F \to \mathrm{GL}(V)$ is called *smooth* if the stabiliser of each vector $v \in V$ is an open subgroup of $\mathcal{W}_F$, equivalently, the map $\mathcal{W}_F \times V \to V$ is continuous for the discrete topology on V. The representation ρ is called *smooth-admissible* if it is smooth and the space of invariants V^H is finite dimensional for each compact open subgroup $H \subset \mathcal{W}_F$.

7.2. The monodromy theorem

Let F be a non-Archimedian local field, and F^s/F a separable closure of F. We denote with k^s the residue field of F^s and with k the residue field of k. Let ℓ be a prime number which is different from the residue characteristic p of F. Let F^{ur} resp. F^{tr} be the maximal unramified (resp. tamely ramified) extension field of F contained in F^s; it is the field corresponding to $\mathcal{I}_F$ resp. $\mathcal{I}_K^{\mathrm{wild}}$. Fix a prime $\pi \in F$.

Identify $\hat{\mathbf{Z}}$ with the Galois group of k^s/k by letting 1 correspond to the geometric Frobenius. We get a surjection $\mathcal{G}_F \to \hat{\mathbf{Z}}$ and thus a surjection $\mathcal{W}_F \to \mathbf{Z}$. We denote this surjection by v_F. Via the isomorphism $\mathrm{Art}_F \colon F^\times \xrightarrow{\sim} \mathcal{W}_F^{\mathrm{ab}}$, the surjection v_F corresponds to the valuation map on F.

We define the map

$$|\cdot|\colon \mathcal{W}_F \longrightarrow q^{\mathbf{Z}}, \quad x \mapsto q^{-v_F(x)},$$

where $q = \#k$.

Lemma 45. — *For all $\sigma \in \mathcal{I}_F$ and $w \in \mathcal{W}_F$ one has $w\sigma w^{-1} = \sigma^{|w|} \in \mathcal{W}_F/\mathcal{I}_F^{\mathrm{wild}}$.*

Proof. — By Kummer theory, the extension F^{tr} over F^{ur} is generated by elements $a \in F^{\mathrm{tr}}$ such that $a^n = \pi \in F$ for some $n \in \mathbf{Z}_{\geq 1}$ with $p \not| n$. Let $a \in F^{\mathrm{tr}}$ be such an element, and $\zeta, \eta \in F^{\mathrm{ur}}$ be the n-th roots of unity such that $\sigma(a) = \eta a$ and $w^{-1}(a) = \zeta a$. The map of reduction $\mathcal{O}_{F^{\rm tr}} \to \kappa(F^{\rm tr})$ is injective when restricted to $\mu(F^{\rm tr})$. Because $\overline{w}(\overline{\zeta}) = \overline{\zeta}^{|w|} \in \kappa(F^{\rm tr})$, we see $w(\zeta) = \zeta^{|w|}$ and $\sigma(\zeta) = \zeta$. We compute:

$$(w\sigma w^{-1})(a) = (w\sigma)(\zeta \cdot a) = w(\zeta\sigma(a)) = w(\zeta\eta a) = \eta^{|w|}w(\zeta a) = \eta^{|w|}a = \sigma^{|w|}(a),$$

This holds for all a and the assertion follows.

Any finite subextension $F^{\rm tr}/E/F^{\rm ur}$ is generated by an element $a \in E$ such that $a^n = \pi$, where $n = [E : F^{\rm ur}]$ is prime to p. For such an a the map $\mathrm{Gal}(E/F^{\rm ur}) \to \boldsymbol{\mu}_n$, $\sigma \mapsto \frac{\sigma(a)}{a}$ is an isomorphism, and does not depend on a. Passing to the projective limit over all E, we get

$$\mathrm{Gal}(F^{\rm tr}/F^{\rm ur}) = \mathcal{I}_F/\mathcal{I}_K^{\rm wild} \xrightarrow{\sim} \varprojlim_{n\in\mathbf{N},(n,p)=1} \boldsymbol{\mu}_n = \prod_{\lambda\neq p} \mathbf{Z}_\lambda(1). \qquad \text{((VII.27))}$$

Fix a surjection $t_\ell \colon \mathcal{I}_F \to \mathbf{Z}_\ell$.

Theorem 46. — *(Recall $\ell \neq p$). Let (ρ, V) be an ℓ-adic $\mathcal{W}_F$-representation over a closed subfield E of $\overline{\mathbf{Q}}_\ell$. There exists a unique nilpotent $N_\rho \in \mathrm{End}(V)$ and a open subgroup $H \subset \mathcal{I}_F$ such that for all $\sigma \in H$ one has $\rho(\sigma) = \exp(t_\ell(\sigma)N_\rho)$.*

Proof. — Proposition 4, is also true for ℓ-adic representations of $\mathcal{W}_F$, although this group is not profinite. To see this, let V be an ℓ-adic $\mathcal{W}_F$-representation with coefficients over $\overline{\mathbf{Q}}_\ell$, say. Then it is also an $\mathcal{I}_F$-representation, so we may find a finite extension $E_1/\mathbf{Q}_\ell$ over which V is defined as $\mathcal{I}_F$-representation. Now pick a geometric Frobenius Φ in $\mathcal{W}_F$, and enlarge this finite extension E_1 with the coefficients of the matrix of $\rho(\Phi)$ to find a finite extension E_2 with the property that the $\mathcal{W}_F$-representation V is obtained from extending scalars from a $E_2[\mathcal{W}_F]$-submodule of V.

Therefore, we may assume $E = \mathbf{Q}_\ell$. Assume an $N_\rho \in \mathrm{End}(V)$ exists such that $\rho(\sigma) = \exp(t_\ell(\sigma)N_\rho)$ holds for all σ in some open subgroup H of $\mathcal{I}_F$. Pick a $\sigma \in H$ such that $t_\ell(\sigma) \neq 0$, then $N_\rho = t_\ell(\sigma)^{-1}\log\rho(\sigma)$, so N_ρ is unique. Moreover, for all $w \in \mathcal{W}_F$ and $\sigma \in \mathcal{I}_F$ we have

$$\exp(t_\ell(\sigma)\rho(w)N_\rho\rho(w)^{-1}) = \rho(w\sigma w^{-1}) = \exp(t_\ell(\sigma)|w|N_\rho).$$

By uniqueness

$$\rho(w)N_\rho\rho(w)^{-1} = |w|N_\rho, \qquad \text{((VII.28))}$$

for all $w \in \mathcal{W}_F$. In particular $q \cdot N_\rho$ is conjugate to N_ρ and the set of eigenvalues of N_ρ is stable under multiplication by q. Therefore N_ρ cannot have any non-zero eigenvalues, and thus N_ρ is automatically nilpotent once it exists.

The image of $\mathcal{I}_F$ in $\mathrm{GL}(V)$ is compact hence contained in $\mathrm{GL}(L)$, where $L \subset V$ is some $\mathbf{Z}_\ell$-lattice in V. Let H' be the (open) kernel of the composition $\mathcal{I}_F \to \mathrm{GL}(L) \to \mathrm{GL}(L/\ell L)$. The image $\rho[H'] \subset \mathrm{GL}(L)$ is a pro-ℓ-group, so the kernel of $\rho|_{H'}$ has to contain $\mathcal{I}_F^{\rm wild} \cap H'$, which is pro-$p$. By equation (VII.27) the morphism $\rho|_{H'}$ factors to a map $\phi \colon \mathbf{Z}_\ell \to \mathrm{GL}(L)$, such that $\rho|_{H'} = \phi \circ t_\ell$. We may now apply the following theorem to ϕ:

Theorem 47. — *Let $F = \mathbf{R}$ or $F = \mathbf{Q}_\ell$ for some prime number ℓ. Let $\phi \colon G_1 \to G_2$ be a continuous morphism where G_1 and G_2 are two analytic groups over F. Then ϕ is locally analytical.*

Proof. — [**35**, p. 155].

Therefore $\phi\colon \mathbf{Z}_\ell \to \mathrm{GL}(L)$ is locally analytical. Hence there exist $r_k \in \mathrm{End}(V)$ for $k \in \mathbf{N}$ and an open $U \subset \mathbf{Z}_\ell$ with $0 \in U$ such that $\phi(x) = \sum_{k\geq 0} r_k \cdot x^k$ for all $x \in U$. Basic computation shows $\phi(x) = \exp(N_\rho x)$ for all $x \in U$, where $N_\rho := r_1$. Let us carry out this computation. Notice $r_0 = 1$. For all $x, y \in U$ we have

$$\sum_{k\geq 0} r_k(x+y)^k = \phi(xy) = \phi(x)\phi(y) = \left(\sum_{k\geq 0} r_k \cdot x^k\right)\left(\sum_{k\geq 0} r_k \cdot y^k\right) = \sum_{k\geq 0}\sum_{d=0}^{k} r_{k-d} r_d x^{k-d} y^d.$$

Assume that $r_{k-1} = \frac{N_\rho^{k-1}}{(k-1)!}$. We prove $r_k = \frac{N_\rho^k}{k!}$. We compute the coefficient of $x^{k-1}y$ on the left hand side and on the right hand side of the equation to get $kr_k = r_{k-1}r_1 = \frac{N_\rho^k}{(k-1)!}$, by induction on k, $\phi(x) = \exp(N_\rho x)$.

By making U smaller we may assume that it is an open subgroup of $\mathbf{Z}_\ell$. For all $\sigma \in H := (\rho|_{H'})^{-1}(U)$ we have $\rho(\sigma) = \exp(t_\ell(\sigma)N_\rho)$. This proves Theorem 46.

Corollary 48. — *One has $\rho(w)N_\rho\rho(w)^{-1} = |w|N_\rho$ for all $w \in \mathcal{W}_F$.*

Proof. — This follows from the proof of 46, see equation (VII.28).

7.3. Weil-Deligne representations

Let E be a field of characteristic 0, F a non-Archimedian local field of residue characteristic p. A *Weil-Deligne representation* of $\mathcal{W}_F$ over E is a triple (ρ, V, N), where (ρ, V) is a finite dimensional smooth representation of $\mathcal{W}_F$ and $N \in \mathrm{End}(V)$ a nilpotent endomorphism such that $\rho(w)N\rho(w)^{-1} = |w|N$ for all $w \in \mathcal{W}_F$. A *morphism of Weil-Deligne-representations* $(\rho, V, N) \to (\rho', V', N)$ is a morphism of representations $f\colon (\rho, V) \to (\rho', V')$ such that $f \circ N = N \circ f$.

Let ℓ be a prime number different from p. We will now transform ℓ-adic representations into Weil-Deligne representations. Let (ρ, V) be an ℓ-adic representation of $\mathcal{W}_F$ over $\overline{\mathbf{Q}}_\ell$. Fix a Frobenius element $\Phi \in \mathcal{W}_F$. We define

$$\rho_\Phi\colon \mathcal{W}_F \longrightarrow \mathrm{GL}(V), \quad w \mapsto \rho(w)\exp(-t_\ell(\Phi^{-v_F(w)}w)N_\rho). \tag{(VII.29)}$$

Lemma 49. — *Fix a surjection $t_\ell\colon \mathcal{I}_F/\mathcal{I}_K^{\mathrm{wild}} \to \mathbf{Z}_\ell$, and a Frobenius element $\Phi \in \mathcal{W}_F$ as above. The association*

$$D_{t_\ell,\Phi}\colon \begin{cases} (\rho, V) \mapsto (\rho_\Phi, V, N_\rho) \\ (V \to V') \mapsto (V \to V'), \end{cases}$$

is a functor from the category of finite-dimensional, continuous $\mathcal{W}_F$-representations in $\overline{\mathbf{Q}}_\ell$-vector spaces to the category of Weil-Deligne representations.

Proof. — Let us first verify that ρ_Φ is indeed a representation for all ρ. Let $w, v \in \mathcal{W}_F$. Then

$$\Phi^{-v_F(wv)}wv = \Phi^{-v_F(v)}vv^{-1}\left(\Phi^{-v_F(w)}w\right)v = \Phi^{-v_F(v)}v\left(\Phi^{-v_F(w)}w\right)^{|v|^{-1}},$$

(recall Lemma 45) so,

$$\begin{aligned}\rho_\Phi(wv) &= \rho(w)\rho(v)\exp(-t_\ell(\Phi^{-v_F(wv)}wv)N_\rho)\\ &= \rho(w)\rho(v)\exp(-t_\ell(\Phi^{-v_F(w)}w)\cdot|v|^{-1}\cdot N_\rho)\exp(-t_\ell(\Phi^{-v_F(v)}v)N_\rho).\end{aligned}$$

By Corollary 48, one has $\rho(v)N_\rho\rho(v)^{-1} = |v|N_\rho$, and thus

$$\rho(v)\exp(-t_\ell(\Phi^{-v_F(w)}w)\cdot|v|^{-1}\cdot N_\rho) = \exp(-t_\ell(\Phi^{-v_F(w)}w)N_\rho)\rho(v).$$

Combining the above two conclusions gives $\rho_\Phi(wv) = \rho_\Phi(w)\rho_\Phi(v)$.

Let $\phi\colon \rho \to \rho'$ is a morphism of finite-dimensional, continuous $\mathcal{W}_F$-representations over $\overline{\mathbf{Q}}_\ell$. By uniqueness, $N_\rho\circ\phi = \phi\circ N_\rho$, hence ϕ is also a morphism of the Weil-Deligne representations.

Theorem 50. — *Fix a surjection $t_\ell\colon \mathcal{I}_F/\mathcal{I}_K^{\mathrm{wild}} \to \mathbf{Z}_\ell$, and a Frobenius element $\Phi \in \mathcal{W}_F$. The functor $D_{t_\ell,\Phi}$ is an isomorphism between the category of finite-dimensional continuous representations of $\mathcal{W}_F$ over $\overline{\mathbf{Q}}_\ell$ and the category of Weil-Deligne representations of finite dimension over $\overline{\mathbf{Q}}_\ell$.*

Proof. — Let (ρ, V, N) be a Weil-Deligne representation. A similar computation like we did in Lemma 49 shows that $\rho^\Phi\colon \mathcal{W}_F \to \mathrm{GL}(V)$ defined by $\rho^\Phi(w) = \rho(w)\exp(t_\ell(\Phi^{-v_F(w)}w))$ is a representation. This construction is inverse to $(\rho, V) \mapsto (\rho_\Phi, V, N_\rho)$.

Fix an isomorphism $\iota\colon \overline{\mathbf{Q}}_\ell \xrightarrow{\sim} \mathbf{C}$. Define the functor $D_{t_\ell,\Phi,\iota}$ as the composition

$$(\square\otimes_{\overline{\mathbf{Q}}_\ell,\iota}\mathbf{C})\circ D_{t_\ell,\Phi}.$$

Proposition 51. — *Let $t'_\ell\colon \mathcal{I}_F \to \mathbf{Z}_\ell$, $\Phi' \in \mathcal{W}_F$ a geometric Frobenius and $\iota'\colon \overline{\mathbf{Q}}_\ell \xrightarrow{\sim} \mathbf{C}$ an isomorphism. The functors $D_{t_\ell,\Phi,\iota}$ and $D_{t'_\ell,\Phi',\iota'}$ are isomorphic.*

Sketch. — We give only a sketch, for more details see [**7**, p. 207]. The idependence of the functor on ι is clear. Let $\Phi' = \Phi x$, $x \in \mathcal{I}_F$ be another Frobenius. Define $A = \exp((q-1)^{-1}t_\ell(x)N_\rho)$; one verifies $A\circ\rho_\Phi\circ A^{-1} = \rho_{\Phi'}(g)$ for all $g \in \mathcal{W}_F$. Similarly, assume $t'_\ell = \alpha t_\ell$ with $\alpha \in \mathbf{Z}_\ell^\times$ is another surjection $\mathcal{I}_F/\mathcal{I}_F^{\mathrm{wild}} \to \mathbf{Z}_\ell$. Define $V_\lambda = \ker(\rho(\Phi^a) - \lambda)^{\dim V}$, where $a \in \mathbf{Z}_{\geq 1}$ is sufficiently big such that $\rho(\Phi^a)$ is central in $\rho(\mathcal{W}_F)$. Define B by $Bv = \mu_v v$, $v \in V_\lambda$, for a family of elements $\mu_\lambda \in \mathbf{Z}_\ell^\times$ satisfying $\alpha\mu_{\lambda q^n} = \mu_\lambda$, then B satisfies $BN_\rho B^{-1} = \alpha N_\rho$.

7.4. The local Langlands theorem

Also in this section we assume that $p \neq \ell$. Let q be the cardinality of the residue field k of F. Let (ρ', V', N) be a complex Weil-Deligne representation. We define

$$L(\rho, q^{-s}) = \det(1 - q^{-s}\rho|_{V^{I_F}\cap\ker(N)}(\mathrm{Frob}_p))^{-1} \in \mathbf{C}(q^{-s}).$$

Fix an isomorphism $\iota\colon \overline{\mathbf{Q}}_\ell \to \mathbf{C}$. With the above definition, the L-factor of an ℓ-adic $\mathcal{G}_F$-representation ρ equals to the L-factor of the Weil-Deligne representation ρ_Φ associated to ρ.

Consider the set A of isomorphism classes of quintuplets $(F^s/F, \psi, \mu, V, \rho)$, where F is a local field, F^s/F a separable closure, ψ a non-trivial continuous morphism $F \to \mathbf{C}^*$, a Haar-measure

μ on F, a finite dimensional complex vector space V and a smooth-admissible representation $\rho\colon W_F \to \mathrm{GL}(V)$. An isomorphism $(F^s/F, \psi, \mu, V, \rho) \xrightarrow{\sim} (F'^s/F', \psi', \mu', V', \rho')$, is a commutative diagram

$$\begin{array}{ccc} F & \longrightarrow & F^s \\ \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} \\ F' & \longrightarrow & F'^s, \end{array}$$

and an isomorphism of representations $r\colon f^*(\rho, V) \to (\rho', V')$, where f^* is the composition $W(F'^s/F) \to W(F^s/F) \xrightarrow{\rho} \mathrm{GL}(V)$. The morphism ψ and measure μ on F equal to the pull-back of ψ' and μ' to F.

In Section 3.7 we have defined the L-factors and ε-factors only for smooth-admissible irreducible representations of $\mathrm{GL}_n(\mathbf{Q}_p)$. These definitions also work for non-Archimedean local fields in general. See [**24**] for the definitions for general local fields.

***Theorem 52* (Langlands (and Dwork, Deligne)).** — *There exists an unique function $\varepsilon\colon A \to \mathbf{C}^\times$ such that*

1. *For all separable closures $(F \to F^s)$ of all local fields and exact sequences $0 \to \rho' \to \rho \to \rho'' \to 0$ of $W(F^s/F)$-representations we have $\varepsilon(\rho, \psi, \mu) = \varepsilon(\rho', \psi, \mu)\varepsilon(\rho'', \psi, \mu)$.*
2. *For all $a \in \mathbf{R}$ we have $\varepsilon(V, \psi, a\mu) = a^{\dim V}\varepsilon(V, \psi, \mu)$.*
3. *By (1), the function ε extends to virtual representations. If $E \subset F^s$ is finite over F and ρ_F is a virtual representation of $W(F^s/F)$ induced by a 0-dimensional virtual representation ρ_E of $W(F^s/E)$ we have $\varepsilon(\rho_F, \psi) = \varepsilon(\rho_E, \psi \circ \mathrm{Tr}_{L/K})$.*
4. *If $\dim \rho = 1$, then $\varepsilon(\rho, \psi, \mu)$ is the ε-factor defined by Tate in* [**9**].

Proof. — [**14**, p. 535].

We need to introduce the notion of Frobenius semi-simple Weil-Deligne representations. A Weil-Deligne representation (ρ, V, N) is Frobenius semi-simple if ρ is semi-simple. Any Weil Deligne-representation (ρ, V, N) has a canonical Frobenius semi-simplification $(\rho, V, N)^{\mathrm{ss}}$, which may be defined as follows. Pick a lift Φ of Frob_p to $\mathcal{W}_{\mathbf{Q}_\ell}$ and write $\rho(\Phi) = \Phi_s\Phi_u$, where Φ_s is semi-simple and Φ_u is unipotent. The semi-simplification is obtained by keeping N and $\rho|_{\mathcal{I}_p}$ unchanged and replacing $\rho(\Phi)$ by Φ_s.

Let $n \in \mathbf{N}$. We let $\mathcal{A}_n^0(F)$ be the set of equivalence classes of cuspidal smooth-admissible irreducible representations of $\mathrm{GL}_n(F)$, and $\mathcal{G}_n^0(F)$ be the set of equivalence classes of Frobenius semi-simple Weil Deligne-representations.

***Theorem 53* (Harris, Taylor, Henniart, Laumon, Stuhler,. . .)**

There exists a unique sequence of bijections $\{\sigma_n\colon \mathcal{A}_n^0(F) \to \mathcal{G}_n^0(F) | n \in \mathbf{N}\}$ such that for all $n \in \mathbf{N}, n' \in \mathbf{N}, \pi \in \mathcal{A}_n^0(F), \pi' \in \mathcal{A}_{n'}^0(F), \chi \in \mathcal{A}_1^0(F)$:

1. *The determinant of $\sigma_n(\pi)$ corresponds via Art_F to the central character of π.*
2. *Twists: $\sigma_n(\pi \otimes \chi) = \sigma_n(\pi) \otimes \sigma_1(\chi)$;*
3. *L-functions: $L(\pi \times \pi', s) = L(\sigma_n(\pi) \otimes \sigma_{n'}(\pi'), s)$;*

4. *ε-factors:* $\varepsilon(\pi \times \pi', s) = \varepsilon(\sigma_n(\pi) \otimes \sigma_{n'}(\pi'), s)$;
5. *Duals:* $\sigma_n(\pi^\vee) = \sigma_n(\pi)^\vee$.

There are several important remarks to make. The conditions (3) and (4) make the bijection unique, more precisely, the objects on both sides are uniquely (up to isomorphism) determined by the L-factors and ε-factors of the twists [**22**]. The map σ preserves the conductors of the representations [**8**]. As we will see, this result also passes on to the global Langlands correspondence.

We explain the compatibility between the global Langlands conjecture and the local Langlands theorem. Suppose that $\Pi = \bigotimes'_v \Pi_v$ is an automorphic $\mathrm{GL}_n(\mathbf{A})$-representation corresponding to a geometric (de Rham, and unramified almost everywhere) ℓ-adic $\mathcal{G}_{\mathbf{Q}}$-representation ρ (see Conjecture 39). For any prime number p, we consider $\rho|_{\mathcal{W}_{\mathbf{Q}_p}}$, where the injection $\mathcal{W}_{\mathbf{Q}_p} \to \mathcal{G}_{\mathbf{Q}}$ is induced from an injection of fields $\overline{\mathbf{Q}} \to \overline{\mathbf{Q}}_p$. Fix an isomorphism $\iota\colon \overline{\mathbf{Q}}_\ell \to \mathbf{C}$. Assume $p \neq \ell$. The representation $D_\iota(\rho|_{\mathcal{W}_{\mathbf{Q}_p}})$ is usually not irreducible (e.g. when ρ is unramified and $n > 1$), but we may write its Frobenius semi-simplification $D_\iota(\rho|_{\mathcal{W}_{\mathbf{Q}_p}})^{\mathrm{ss}}$ as a sum of irreducible representations $\sigma_1, \ldots, \sigma_k$. The local Langlands theorem associates to each σ_i a $\mathrm{GL}_n(\mathbf{Q}_p)$-representation π_i. The automorphic factor Π_p of the automorphic representation Π is isomorphic to $I(\pi_1, \ldots, \pi_n)$ (see equation (III.20) for the notation $I(*, \ldots, *)$).

At the prime $p = \ell$ the correspondence is different, and also not yet fully understood; one can go only from the Galois side to the automorphic side, but not in the other direction. To describe this correspondence, one needs (ϕ, γ)-modules to replace the Weil-Deligne representations. It goes beyond the scope of this thesis to discuss this case.

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