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Souslin's Hypothesis

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Contents

1 Introduction

We begin with a characterisation of the real line.

1.1. Theorem. *Let L be a non-empty linearly ordered set. Suppose L satisfies the following properties:*

1. *L has no minimum or maximum.*
2. *L is densely ordered; that is, for all $p < q$ there is some $r \in L$ with $p < r < q$.*
3. *L is complete; that is, every non-empty and bounded $A \subseteq L$ has an infimum and a supremum.*
4. *The order topology of L is separable.*

Then L is order-isomorphic to $\mathbb{R}$.

Let L be a linearly ordered set. Then L is said to be a ccc linear order (or to satisfy the countable chain condition) if every pairwise disjoint family of open intervals is countable.

Clearly, if the order topology of L is separable then L is a ccc linear order: if Q is a countable dense subset of L and $\mathcal{F}$ is a pairwise disjoint family of open intervals then any map which sends each $A \in \mathcal{F}$ to some element of $A \cap Q$ is an injection into Q.

A natural question is whether a non-empty linear order L which satisfies properties (1)-(3) and the countable chain condition is isomorphic to $\mathbb{R}$. This is the content of Souslin's question, which was first posed by Mikhail Souslin and published in 1920 at the end of the first volume of Fundamenta Mathematicae and which is nowadays known as Souslin's Hypothesis.

1.2. Souslin's Hypothesis (SH). *If L is a non-empty complete dense linear order without a minimum or maximum which satisfies the countable chain condition, then L is isomorphic to $\mathbb{R}$.*

It turns out (SH) is independent of ZFC. If L is a non-empty order which satisfies (1)-(3) and the countable chain condition but is not isomorphic to $\mathbb{R}$, then L is said to be a Souslin line. Thus, (SH) is equivalent to the statement "no Souslin line exists".

In this thesis, we will not be able to directly prove the independence of (SH). Rather, we will derive both it and its negation from two statements which are known to be consistent with the axioms of set theory. Both of these statements belong to infinitary combinatorics and Souslin's Hypothesis is an appealing example of how infinitary combinatorics may see use in proving statements which do not look combinatorial in the least at first glance.

2 Trees

To reduce Souslin's Hypothesis to a combinatorial problem, we consider a special kind of partially ordered sets which play a central role in infinitary combinatorics.

2.1. Definition. A tree $(T, <)$ is a partially ordered set for which $\hat{t} = \{s \in T : s < t\}$ is well-ordered for each $t \in T$. For a given tree T we define the following notions:

- The height $\mathrm{ht}(t)$ of an element $t \in T$ is the order type of $\hat{t}$.
- The height $\mathrm{ht}(T)$ is defined to be equal to $\sup\{\mathrm{ht}(t) + 1 : t \in T\}$.
- The α-th level $T(\alpha) = \{t \in T : \mathrm{ht}(t) = \alpha\}$ of T consists of the elements at height α.
- We write $T_{(\alpha)}$ for $\bigcup_{\xi<\alpha} T(\xi)$ and $<_\alpha$ for $<$ restricted to $T_{(\alpha)}$.
- A branch is a maximal chain in T.
- The immediate successors of $t \in T$ are the $t < s \in T$ such that $\mathrm{ht}(s) = \mathrm{ht}(t) + 1$.

Note that any subset of a tree is itself a tree under the induced order.

The utility of trees lies in the fact that many problems in set theory (including Souslin's Hypothesis) can be restated as a problem about trees. Trees are pleasant to work with because they have the useful property of being well-founded: if A is a non-empty subset of a tree T, then every $a \in A$ at level $\min\{\mathrm{ht}(a) : a \in A\}$ is a minimal element of A. Consequently, one can perform induction and recursion along trees.

The following lemma gives an explicit description of how recursion along trees is handled. It tells us we can deal with minimal elements, elements of height a successor and elements of height a non-zero limit separately. One can derive it from the recursion principle for well-founded relations or prove it directly by constructing approximations. We will not go into details and instead leave the proof to the reader.

2.2. Lemma. *Let m denote the set of minimal elements of T, M the set of maximal elements, $L = \{T^{\hat{t}} : t \in T,\ ht(t) > 0 \text{ a limit}\}$ and for each $t \in T$, let $I(t)$ denote the set of immediate successors of t. Suppose we have functions $F_0 : m \to Y, F_1 : \bigcup L \to Y$ and $F_2 : T \setminus M \to \bigcup_{t \in T} Y^{I(t)}$ such that $F_2(t) \in Y^{I(t)}$ for all $t \in T \setminus M$. Then there is a unique function $f : T \to Y$ such that:*

- *If $t \in m$ then $f(t) = F_0(t)$.*
- *If $ht(t) > 0$ is a limit then $f(t) = F_1(f \restriction \hat{t})$.*
- *If s is an immediate successor of t then $f(s) = F_2(t)(s)$.* $\square$

The next definition gives a class of trees which are useful because of their generality: many trees arise naturally as subsets of these objects.

2.3. Definition. Let α be an ordinal, X some non-empty set. The full X-ary tree of height α, denoted by $X^{<\alpha}$, is defined to be the set $\bigcup_{\xi<\alpha} X^\xi$, ordered by extension; that is, $f \leq g$ if $f \subseteq g$. The predecessors of f are just the restrictions of f to a proper initial segment of $\operatorname{dom} f$, so $\hat{f} \cong \operatorname{dom} f$ is well-ordered, which implies $X^{<\alpha}$ is in fact a tree. We use the following notation:

- If s is not maximal, then $\operatorname{dom} s = \beta$ is not a maximum of α. Define for each $x \in X$ the function $s \cdot x$ with domain $\beta + 1$ which agrees with s on β and for which $(s \cdot x)(\beta) = x$ holds. It is easy to see the immediate successors of s are precisely the functions of the form $s \cdot x$ with $x \in X$.

- We write elements of $X^{<\alpha}$ using the standard notation for ordinal-indexed sequences. For example, $\langle\rangle$ denotes the empty function.

In this section, we will prove the existence of a Souslin line is equivalent to the existence of a certain kind of tree, namely to the existence of a Souslin tree. In the next two sections, we will consider proofs of the (non)-existence of both Souslin lines and Souslin trees. The proofs for Souslin lines will contain topological arguments, while the proofs for Souslin trees will be of a purely combinatorial nature.

2.4. Definition. A Souslin tree is a tree $(T, <)$ with the following properties:

1. $\operatorname{ht}(T) = \omega_1$.

2. Every branch of T is countable.

3. Every antichain of T is countable.

An Aronszajn tree is a tree which satisfies the conditions mentioned above, except "antichain" is replaced with "level". Thus, Souslin trees are a special kind of Aronszajn trees. Unlike Souslin trees, one can prove the existence of Aronszajn trees in ZFC. Generalizing to arbitrary cardinal numbers κ, a κ-Aronszajn tree is a tree of height κ such that each level and each branch is of cardinality $< \kappa$ and a κ-Souslin tree is a κ-Aronszajn tree in which every antichain is of cardinality $< \kappa$.

2.5. Proposition. *An $\aleph_1$-Aronszajn tree exists.*

Proof. The tree will be a subset of $\omega^{<\omega_1}$. Let $S = \{s \in \omega^{<\omega_1} : s \text{ is an injection}\}$. This tree has no branches of length ω_1 since such a chain would give rise to an injection $\omega_1 \to \omega$, but its levels are uncountable. Define $s \approx t$ if and only if $|\{\alpha : f(\alpha) \neq g(\alpha)\}| < \omega$. We will construct a sequence $\{s_\alpha\}_{\alpha \in \omega_1}$ such that $\operatorname{dom} s_\alpha = \alpha$ for all $\alpha \in \omega_1$ and such that $s_\beta \restriction \alpha \approx s_\alpha$ for $\alpha < \beta$; then, $T = \{s \in S : (\exists \alpha)(s \approx s_\alpha \wedge s \in \omega^\alpha)\}$ is as desired.

To make sure we can continue the construction at limit steps, we demand the complement of the range of s_α is infinite for all $\alpha < \omega_1$. s_0 is simply the empty function and if s_α is defined, let $s_{\alpha+1}$ be the function $s_\alpha \cdot a$, where a is some element in the complement of the range of s_α.

For limits $\gamma \in \omega_1$, suppose s_α is given for $\alpha < \gamma$. Let $(\gamma_n)_{n \in \omega}$ be an increasing sequence with $\lim_{n \in \omega} \gamma_n = \gamma$ and define recursively $s'_{\gamma_{n+1}}$ to be the function which agrees with s'_{γ_n} on γ_n and which agrees with $s_{\gamma_{n+1}}$ on $\gamma_{n+1} \setminus \gamma_n$. Then $s'_{\gamma_n} \approx s_{\gamma_n}$ for all $n \in \omega$ by the properties of the s_{γ_n} and $s'_{\gamma_m} \restriction s'_{\gamma_n} = s_{\gamma_n}$ for all $n < m$ by construction. Let $t = \bigcup_{n \in \omega} s'_{\gamma_n}$ and let s_γ be the function which agrees with t for all $\xi \notin \{\gamma_n : n \in \omega\}$ and which satisfies $s_\gamma(\gamma_n) = t(\gamma_{2n})$. Then $\omega \setminus \operatorname{ran} s_\gamma \supseteq \{t(\gamma_{2n+1}) : n \in \omega\}$ is infinite and the other properties are easily checked. $\square$

It is easy to see there are no $\aleph_0$-Aronzajn trees: one defines recursively a sequence $(a_n)_{n\in\omega}$ such that a_0 is minimal and such that each a_{n+1} is an immediate successor of a_n which has an infinite number of successors; the set $\{a_n : n \in \omega\}$ is then a branch of length ω. This result is known as König's lemma.

For many cardinal numbers κ, the existence of κ-Aronszajn trees is related to the existence of certain large cardinals. An exposition of these large cardinals and the (non)-existence of κ-Aronszajn trees is outside of the scope of this thesis.

Before we prove the equivalence of the existence of Souslin lines and Souslin trees we first reduce the problem to normal Souslin trees.

2.6. Lemma. *If there exists a Souslin tree T, then there exists a normal Souslin tree; that is, a Souslin tree T' which satisfies the following properties:*

1. *For every $x \in T'$ and $\alpha > \mathrm{ht}(x)$ there is an $y \in T'$ with $\mathrm{ht}(y) = \alpha$ and $y > x$.*
2. *If $\mathrm{ht}(x) = \mathrm{ht}(y) = \gamma$, where γ is a limit ordinal, then $\hat{x} = \hat{y}$ implies $x = y$.*
3. *If $x \in T'$ is not a maximal element then it has an infinite number of immediate successors.*
4. *T' has a least element.*

Proof. We add and remove elements of the tree in such a way that the new tree satisfies all the necessary properties, making sure we do not lose properties along the way. Since this is easily checked at each step of the construction, the verification will be omitted.

Let $T_x = \{y \in T : y \geq x\}$. Let T_1 be the tree which consists of just the $x \in T$ with $|T_x| = \aleph_1$. If $x \in T_1$ and $\alpha > \mathrm{ht}(x)$ then there must be an $y \in T$ such that $\mathrm{ht}(y) = \alpha$ and $|T_y| = \aleph_1$: otherwise $|T_y| = \omega$ for all y at level α, so $T_x = \bigcup_{\{y:ht(y)=\alpha\}}([x, y] \cup T_y)$ (where $[x, y] = \{a \in T : x \leq a \leq y\}$) is countable as a countable union of countable sets. Thus, T_1 satisfies the first property.

Define T_2 by adding a new point a_C for every chain C of the form $\hat{y}$ with $\mathrm{ht}(y)$ a limit ordinal and defining $a_C < x$ for all x with $\hat{y} \subseteq \hat{x}$ and $a_C > x$ for all $x \in C$. Then T_2 satisfies property 2.

Define a branching point to be a point which has at least two successors. Since T_2 satisfies (1), each point has at least one successor and since T_2 has no uncountable chains (such a chain would give rise to a branch), there must be uncountably many branching points. Thus T_3, the tree consisting of all the branching points of T_2, is still of height ω_1. Let T_4 be the three consisting of all the points of T_3 at limit height. Let $s \in T$ be at limit height γ. Since each point of T_3 has at least two successors, there is an infinite number of $t \in T$ at height $\gamma + \omega$ above s, so T_4 satisfies (3).

Finally, add a point a to T_4 which is smaller than all $t \in T_4$ to satisfy (4); it is then not hard to show the resulting tree T' has all the desired properties. □

We now prove the main result of this section.

2.7. Theorem. *There is a Souslin tree if and only if there is a Souslin line.*

Proof. ($\Rightarrow$): If there is a Souslin tree, then there is a normal Souslin tree by Lemma 2.6, so let T be a normal Souslin tree.

The line L will consist of the set of all branches of T with a suitable linear order. Let $\mathcal{N}(t)$ denote the set of immediate successors of t. Since T is normal, each $\mathcal{N}(t)$ is countably infinite; pick a linear order $<_t$ so that $\mathcal{N}(t)$ with this order is isomorphic to $\mathbb{Q}$. Let a, b be branches. If $a \neq b$ then the first ordinal for which they differ is a successor ordinal $\alpha + 1$ by (2), so let $t \in a \cap b$ be at level α and let $a_\alpha \in a$, $b_\alpha \in b$ be its immediate successors in a respectively b; we define $a < b$ if and only if $a_\alpha <_t b_\alpha$. This defines a linear order.

For $a < b$, let $a_\alpha <_t b_\alpha$ be as above and let c_α be some immediate successor such that $a_\alpha <_t c_\alpha <_t b_\alpha$. Then any branch c which extends $(a \cap b) \cup \{c_\alpha\}$ satisfies $a < c < b$; thus, L is densely ordered.

If $\{(a_i, b_i) : i \in I\}$ is a family of intervals, pick s_i for each $i \in I$ as we picked c_α in the previous paragraph. For each $x \in T$, define $I_x = \{c \in L : x \in c\}$; each I_x is an interval. If $I_{s_i} \cap I_{s_j} = \emptyset$ then s_i and s_j are incomparable. Thus, $\{s_i\}_{i \in I}$ is an antichain, so I is countable. We conclude L satisfies the countable chain condition.

Finally, if C is a countable set of branches, let $s(C)$ be the set of order types of elements of C and let $\xi = \sup s(C) + 2$. Then $c < \xi$ for all $c \in C$ and we have $\xi \in \omega_1$. Let $x \in T$ be an element at the ξ-th level. Then I_x contains no elements of C, so C is not dense in L. Thus, L is not separable. L obviously has no endpoints, so we conclude its Dedekind completion $\hat{L}$ is a Souslin line. □

($\Leftarrow$): Let L be a Souslin line. Write $L' = L \cup \{-\infty, \infty\}$ and define the ordering on L' in the obvious manner. We recursively define a map $2^{<\omega_1} \to (L')^2$ as follows:

- The empty map gets mapped to the pair $(-\infty, \infty)$.
- If s gets mapped to (a_s, b_s), let c_s be some element between a_s and b_s if $a_s \neq b_s$ and let $b_s = c_s = a_s$ if $b_s = a_s$. Define $a_{s\cdot 0} = a_s, a_{s\cdot 1} = c_s, b_{s\cdot 0} = c_s, b_{s\cdot 1} = b_s$ and map $s \cdot 0$ to $(a_{s\cdot 0}, b_{s\cdot 0})$ and $s \cdot 1$ to $(a_{s\cdot 1}, b_{s\cdot 1})$.
- If $\operatorname{dom} s$ is a limit and (a_t, b_t) is given for all $t < s$, take a_s, b_s such that $\bigcap_{t<s}[a_t, b_t] = [a_s, b_s]$.

Let T be the subset of $2^{<\omega_1}$ consisting of all the s for which $[a_s, b_s]$ is a non-degenerate interval; that is, the s for which $a_s \neq b_s$. Each antichain $A \subseteq T$ corresponds to a family of disjoint intervals (namely $\{(a_s, b_s) : s \in A\}$) and each branch B corresponds to a sequence of strictly nested open intervals (namely $\{(a_s, b_s) : s \in B\}$). Thus, since L is a Souslin line, each antichain and each chain is countable.

For each $x \in L$, let $f_x : \omega_1 \to 2$ be such that x is contained in $[a_{f_x\restriction\alpha}, b_{f_x\restriction\alpha}]$ for all $\alpha \in \omega_1$. Then, since each branch is countable, there is a limit ordinal γ such that $a_{f_x\restriction\gamma} = b_{f_x\restriction\gamma}$ and such that $f_{x\restriction\alpha} \in T$ for all $\alpha \in \gamma$. Thus, we can fit a non-degenerate interval with endpoints of the form $a_t, b_t (t \in T)$ in each neighborhood of x for any given x, which implies the set $A = \{a_t : t \in T\} \cup \{b_t : t \in T\}$ is dense in L. Since L is a Souslin line, A must be uncountable, so T is uncountable and therefore of cardinality $\aleph_1$ as a subset of $2^{<\omega_1}$. Any tree is the union of its levels, so since each level is countable and since T is of cardinality $\aleph_1$, T must be of height ω_1. We conclude T is a Souslin tree. $\square$

3 The $\diamondsuit$-Principle

In order to formulate the $\diamondsuit$-principle we first need some notions of size for subsets of cardinal numbers.

3.1. Definition. Let $\kappa > \omega$ be a cardinal number. Let $A \subseteq \kappa$.

- If $\sup A = \kappa$, in other words, if for all $\beta \in \kappa$ there is some $\gamma \in A$ with $\gamma > \beta$, then A is unbounded (or cofinal) in κ.
- The cofinality of κ is the smallest possible size of a cofinal set: $\text{cf } \kappa = \min\{|A| : A \text{ cofinal in } \kappa\}$. The cardinal numbers with $\text{cf } \kappa = \kappa$ are the regular cardinals, the cardinal numbers with $\text{cf } \kappa \neq \kappa$ are the singular cardinals.
- If for all limit ordinals $\beta \in \kappa$ such that $\sup(A \cap \beta)$ is unbounded in β we have $\beta \in A$, then A is closed in κ.
- The subsets of A which are both closed and unbounded in κ are the closed unbounded (cub) sets of κ.

A cardinal number κ is regular if and only if it cannot be written as a union of fewer than κ subsets of cardinality less than κ. Since the countable union of countable sets is countable, $\aleph_1$ is a regular cardinal.

Note that the closed sets as defined above coincide with the closed sets of the order topology on κ. Thus, they are closed under arbitrary intersections.

For regular κ, the cub sets of κ are the "large" subsets of κ in the sense that they form the basis of a filter. In fact, even more is true.

3.2. Theorem. *Let $\kappa > \omega$ be a regular cardinal. If $\lambda < \kappa$ and $\{C_\alpha\}_{\alpha\in\lambda}$ is a family of closed unbounded sets, then $\bigcap_{\alpha<\lambda} C_\alpha$ is closed unbounded as well.*

Proof. $\bigcap_{\alpha<\lambda} C_\alpha$ is closed as an intersection of closed sets in the topological sense, so it is closed in the sense of Definition 3.1 as well.

For every $\alpha < \lambda$, define $f_\alpha : \kappa \to \kappa, \gamma \mapsto \min\{\beta \in C_\alpha : \beta > \gamma\}$. Define $g(\gamma) = \sup_{\alpha<\lambda} f_\alpha(\gamma)$. Since $\lambda < \kappa$ and since κ is regular, this map is well-defined. Now let $\gamma \in \kappa$ be an arbitrary ordinal number. Define $h(\gamma, n)$ recursively as follows: $h(\gamma, 0) = \gamma$ and $h(\gamma, n+1) = g(h(\gamma, n))$. Let $\gamma^* = \sup_{n\in\omega} h(\gamma, n)$. Since κ is regular, we have $\gamma < \gamma^* \in \kappa$. Note $h(\gamma, n) < f_\alpha(h(\gamma, n)) \leq h(\gamma, n+1)$ for all $\alpha < \lambda$, so $\lim_{n\to\omega} f_\alpha(h(\gamma, n)) = \gamma^*$. Thus, since each C_α is closed, we have $\gamma^* \in \bigcap_{\alpha<\lambda} C_\alpha$, so $\bigcap_{\alpha<\lambda} C_\alpha$ is unbounded. □

This proposition would be false for ω if we defined cub sets of ω in the obvious way: the order topology on ω is trivial, so the sets of odd respectively even numbers would be cub, but their intersection is empty. Similarly, if κ is an uncountable cardinal number of countable cofinality and $\{\alpha_n\}_{n\in\omega}$ is an increasing cofinal sequence, then the subsequences consisting of the α_n for which n is odd respectively even is a pair of cub sets with empty intersection.

For cardinals κ of uncountable cofinality, one can show the intersection of less than $\text{cf } \kappa$ cub sets is again cub. Since we only encounter regular cardinals in proving the consistency of $\neg$(SH), we will not expand any further on cub sets of singular cardinals.

The following definition gives a notion of a non-neglibible subset.

3.3. Definition. Let κ be a regular cardinal number. A subset $S \subseteq \kappa$ which has non-empty intersection with each closed unbounded set of κ is called stationary.

If C is a cub set and S is stationary, then $C \cap S$ is stationary, for if D is a cub set then $(S \cap C) \cap D = S \cap (C \cap D)$ is non-empty since $C \cap D$ is cub by Theorem 3.2. As another corollary of Theorem 3.2, note that any cub set is stationary. Note that stationary sets are unbounded since they have non-empty intersection with the closed unbounded interval (α, κ) for all $\alpha < \kappa$.

If $\{I_\alpha : \alpha < \lambda\}$ is a family of $\lambda < \kappa$ non-stationary sets, let $\{C_\alpha : \alpha < \lambda\}$ be a family of closed unbounded sets such that $I_\alpha \cap C_\alpha = \emptyset$ for all $\alpha < \lambda$. Then $\bigcap_{\alpha<\lambda} C_\alpha$ is closed unbounded by Theorem 2.2, so since $(\bigcup_{\alpha<\lambda} I_\alpha) \cap (\bigcap_{\alpha<\lambda} C_\alpha) = \emptyset$, we conclude the union of less than κ non-stationary subsets of κ is non-stationary.

We now have enough material to formulate the $\diamondsuit$-principle. A $\diamondsuit$-sequence is a sequence $\{A_\alpha\}_{\alpha\in\omega_1}$ such that $A_\alpha \subseteq \alpha$ for all $\alpha \in \omega_1$ and such that $\{\alpha \in \omega_1 : A_\alpha = A \cap \alpha\}$ is stationary for all $A \subseteq \omega_1$. Thus, such a sequence contains information about all subsets of ω_1. In the proof of the consistency of $\neg$(SH), we will see how to use this information.

3.4. The $\diamondsuit$-principle ($\diamondsuit$). *A $\diamondsuit$-sequence exists.*

Note that in the definition of a $\diamondsuit$-sequence we can not replace "stationary" with "closed unbounded": if A, B are distinct subsets of ω_1 then

$$\{\alpha \in \omega_1 : A \cap \alpha = A_\alpha\} \cap \{\alpha \in \omega_1 : B \cap \alpha = A_\alpha\}$$

must necessarily be countable, but by Theorem 2.2 it would have to be closed unbounded as well. The existence of ω_1 stationary subsets with countable pairwise intersection is provable from ZFC. In fact:

3.5. Theorem. *If $S \subseteq \omega_1$ is stationary then there is a pairwise disjoint family $\mathcal{S}$ of stationary subsets of ω_1 such that $|\mathcal{S}| = \omega_1$ and $\bigcup \mathcal{S} = S$.*

Proof. For each $\alpha \in \omega_1$, let $f_\alpha : \omega \to \alpha$ be a surjection. Define $A^n_\xi = S \cap \{\alpha : f_\alpha(n) = \xi\}$. Note that for fixed ξ, the A^n_ξ are pairwise disjoint. Since the f_α are surjective, we have $\bigcup_{n\in\omega} A^n_\xi = S \cap (\xi, \omega_1)$. This set is stationary, so since no union of less than ω_1 stationary subsets is stationary, there must be some $n_\xi \in \omega$ such that $A^{n_\xi}_\xi$ is stationary. Pick such an n_ξ for every $\xi \in \omega_1$ and define $f : \omega_1 \to \omega, \xi \mapsto n_\xi$. Since ω_1 is not the countable union of countable sets, one of the $f^{-1}\{n\}$ must be uncountable (this is a special instance of the Pigeonhole principle; for more, see chapter 4). Now $\{A^n_\xi : \xi \in f^{-1}\{n\}\}$ is a family of ω_1 disjoint stationary subsets of S. Since supersets of stationary sets are stationary, one can make this family into a partition by adding the remaining elements of S to one of the A^n_ξ. $\square$

In order to make use of the $\diamondsuit$-principle, we need to know it is consistent. This is the content of the following theorem.

3.6. Theorem. *If ZFC is consistent, then so is ZFC+$\diamondsuit$.*

One can prove the $\diamondsuit$-principle in the constructible universe, thus proving its consistency. A treatment of the Axiom of Constructibility and how it implies the $\diamondsuit$-principle can be found in chapter 13 of [2].

By the following lemma, $\diamondsuit$ is a stronger version of the continuum hypothesis.

3.7. Lemma. $\diamondsuit \Rightarrow$(CH).

Proof. Let $C = \{A_\alpha\}_{\alpha\in\omega_1}$ be a $\diamondsuit$-sequence. Let $A \subseteq \omega$. Since C is a $\diamondsuit$-sequence, the set of α such that $A \cap \alpha = A_\alpha$ is stationary, hence unbounded. Therefore, there is an $\alpha > \omega$ such that $A \cap \alpha = A_\alpha$. Define $f : \mathcal{P}(\omega) \to \omega_1, A \mapsto \min\{\alpha > \omega : A \cap \alpha = A_\alpha\}$. We have $A \subseteq f(A)$, so $A_{f(A)} = A \cap f(A) = A$. Therefore, if $f(A) = f(B)$ then $A = A_{f(A)} = A_{f(B)} = B$, so f is injective. We conclude $|\mathcal{P}(\omega)| \leq |\omega_1|$, so $2^{\aleph_0} = \aleph_1$. □

We now describe how $\diamondsuit$ can be useful in constructions. Suppose we can recursively construct a certain structure S which has ω_1 as its underlying set and suppose ϕ is an unwanted property of certain subsets of S we wish to eliminate. Then, given that we can fix this problem for an arbitrary $A \subseteq \omega_1$ by modifying the construction at stage α and given this can be done for a closed unbounded number of α, $\diamondsuit$ postulates there is a stationary number of ordinals for which we can safely do so. Thus, since the intersection of a closed unbounded set and a stationary subset is non-empty, we will be able to fix up every subset of A at some point in the construction.

The following lemma is crucial in our proof of the consistency of $\neg$(SH) because it provides us with exactly that closed unbounded set of opportunities. We call a function $\kappa^n \to \kappa$, $n \in \omega$, a finitary function.

3.8. Lemma. *Let $\kappa > \omega$ be a regular cardinal. Let $\{f_n : \kappa^{k_n} \to \kappa : n \in \omega\}$ be a countable family of finitary functions. Then the set C of α such that α is closed under f_n (i.e. the α such that dom $f_n \upharpoonright \alpha^n \subseteq \alpha$) is closed unbounded.*

Proof. Suppose $C \cap \alpha$ is unbounded in α. We will prove α is closed under each of the f_n. For arbitrary $n \in \omega$, let $(x_1, ..., x_{k_n}) \in \alpha^{k_n}$. Let $\beta \in C$ be such that $\beta \geq \max\{x_1, ..., x_{k_n}\}$. By definition of C, β is closed under f_n, so $f_n(x_1, ..., x_{k_n}) \in \beta \subseteq \alpha$. Thus, $\alpha \in C$, so C is closed.

Let $\gamma \in \kappa$. Define γ_n recursively as follows: $\gamma_0 = \gamma$ and γ_{m+1} is the maximum of $\gamma_m + 1$ and $\sup_{n\in\omega}\{f_m(s) : s \in (\gamma_m)^{k_n}\}$. Then $\gamma^* = \sup_{n\in\omega} \gamma_n > \gamma$ and we will show it is closed under the f_n. So let $n \in \omega$ and let $(x_1, ..., x_{k_n}) \in (\gamma^*)^{k_n}$. Then each of the x_i is in some γ_{n_i}. Let $n' \in \omega$ be such that $\gamma_{n'} = \max\{\gamma_{n_i} : i \leq k_n\}$. Then $f_n(x_1, ..., x_{k_n}) \in \{f_n(s) : s \in (\gamma_{n'})^{k_n}\}$, so $f_n(x_1, ..., x_{k_n}) < \gamma_n < \gamma^*$. □

The rest of this section is devoted to proving the existence of Souslin lines and Souslin trees directly from the $\diamondsuit$-principle. As we will see, the construction of a Souslin line is a bit more involved and less transparent than the construction of a Souslin tree. Let us start with the Souslin line.

3.9. Theorem ($\diamondsuit$) *There is a Souslin line.*

We first reduce the problem to a slightly easier to construct order.

3.10. Lemma. *Suppose there is a linearly ordered set L such that*

- *L is densely ordered and has no minimum or maximum.*
- *Each closed nowhere dense subset of L with respect to the order topology is countable.*
- *L is not separable.*

Then there is also a Souslin line.

Proof. First of all, note L is ccc: if $\mathcal{I} = \{(a_i, b_i) : i \in I\}$ is a family of disjoint open intervals, then we can find (using Zorn's lemma) a maximal family of disjoint open intervals $\mathcal{I}'$ with respect to inclusion containing $\mathcal{I}$. Thus, $L \setminus \mathcal{I}'$ is a closed nowhere dense set by maximality of $\mathcal{I}'$, which implies I is countable since all the a_i, b_i are in $L \setminus \mathcal{I}'$.

The Dedekind completion $\hat{L}$ of L is again a linearly, densely ordered set which has no minimum or maximum and satisfies the countable chain condition (see the appendix). $\hat{L}$ is not separable because if it were, it would be isomorphic to $\mathbb{R}$, so L would be isomorphic to a subspace of $\mathbb{R}$ and therefore be separable itself. We conclude $\hat{L}$ is as desired. □

Next, we describe a construction which almost works, except that the resulting set may have closed nowhere dense subsets which are not countable. Let $\mathcal{C}$ be the set of countable limit ordinals. Since we are dealing with limits, if $\alpha \in \mathcal{C}$ then $\alpha + \omega$ is the successor of α according to the induced well-order on $\mathcal{C}$. For every $\alpha \in \mathcal{C}$, $\lhd_\alpha$ will be a dense linear order on $[0, \alpha)$ which has no minimum or maximum. If $\alpha \leq \beta$, then $\lhd_\beta$ will be an extension of $\lhd_\alpha$. The final order $\lhd$ will be an order on ω_1 and is defined from the previous orders by $\lhd = \bigcup_{\beta \in \mathcal{C}} \lhd_\beta$.

As is often the case with recursive constructions, the limit steps are easy: if γ is a limit in $\mathcal{C}$, so a limit of limits, simply take $\lhd_\gamma = \bigcup_{\beta<\gamma} \lhd_\beta$. Then the conditions so far already ensure $\lhd = \bigcup_{\beta \in \mathcal{C}} \lhd_\beta$ is a dense linear order without a minimum or maximum.

To make sure the final order is not separable, note each $[0, \beta)$, $\beta \in \mathcal{C}$, is order isomorphic to $\mathbb{Q}$. For each $\beta \in \mathcal{C}$, let D_β be some Dedekind cut of $[0, \beta)$ and define $\lhd_{\beta+\omega}$ on $[0, \beta + \omega)$ as follows: the elements of $[\beta, \beta + \omega)$ are ordered as $\mathbb{Q}$ and come after the elements of D_β and before the elements of $\beta \setminus D_\beta$. Thus, if $A \subseteq \omega_1$ is countable, then A is not dense in $\sup A + \omega$, so ω_1 under the new order is not separable.

Given a linearly ordered set $(X, <)$ and a closed nowhere dense subset $A \subseteq X$, we say a Dedekind cut D avoids A if there are $a, b \in X$ such that $a < b, [a, b] \cap A = \emptyset, a \in D$ and $b \notin D$. The following lemma specifies, under certain circumstances, how we can make sure a closed nowhere dense subset A of the final order stays countable.

3.11. Lemma. *Let $A \subseteq \omega_1$ be a closed nowhere dense subset. Suppose $\alpha \in \mathcal{C}$ is a limit ordinal such that $A \cap \alpha$ is closed nowhere in α with respect to $\lhd_\alpha$ and such that, for each interval (γ, δ) with $\gamma, \delta \in \alpha$, $(\gamma, \delta) \cap A \neq \emptyset$ implies $(\gamma, \delta) \cap A \cap \alpha \neq \emptyset$. Suppose each D_β with $\beta \geq \alpha$ avoids $A \cap \alpha$. Then for all $\alpha \leq \beta \in \mathcal{C}$ we have*

- *$A \cap \alpha = A \cap \beta$.*
- *For all $\gamma \in \beta \setminus A$, there are $\gamma', \delta \in \alpha$ such that $\gamma' \lhd \gamma \lhd \delta$ and $(\gamma, \delta) \cap A = \emptyset$.*

Proof. We proceed by induction on β. If $\beta = \alpha$, then the second point is immediate since $\alpha \setminus A$ is a dense open set. Assume the statement holds for some $\beta \geq \alpha$. Then

$A \cap \beta = A \cap \alpha$ is closed nowhere dense in β, so D_β avoids $A \cap \beta$, so there are $\eta, \eta' \in \beta$ such that $[\eta, \eta'] \cap A \cap \beta = \emptyset$ and $[\beta, \beta+\omega) \subseteq (\eta, \eta')$. By the induction hypothesis, pick $\gamma, \gamma', \delta, \delta' \in \alpha$ so that they satisfy the properties mentioned in the second point for η respectively η'. Then we have $(\gamma, \delta') \cap A \cap \alpha = \emptyset$, so by assumption, $(\gamma, \delta') \cap A = \emptyset$. Therefore, since $[\beta, \beta + \omega)$ is contained in (γ, δ'), we have $A \cap \alpha = A \cap \beta = A \cap (\beta + \omega)$ and γ and δ' suffice for the second point. The limit step is straightforward. □

As a consequence of this lemma, $A \subseteq \alpha$, so it is countable. We have proven we can make sure a closed nowhere dense subset A stays countable given there is some $\alpha \in \mathcal{C}$ which satisfies certain properties, so the next step in the proof is to show there is a closed unbounded number of such α. We will make use of Lemma 3.8.

3.12. Lemma. *If $A \subseteq \omega_1$ is closed nowhere dense, then the set of $\alpha \in \omega_1$ which satisfies the following conditions is closed unbounded:*

- *$A \cap \alpha$ is closed nowhere dense in α.*
- *If $(\gamma, \delta) \cap A \neq \emptyset$ and $\gamma, \delta \in \alpha$, then $(\gamma, \delta) \cap A \cap \alpha \neq \emptyset$.*

Proof. Let $\phi : \omega_1 \to \omega_1^2$ be the map which sends ξ to $(0, 0)$ if $\xi \in A$ and to $\min\{(\gamma, \delta) : \gamma \lhd \xi \lhd \delta$ and $(\gamma, \delta) \cap A = \emptyset\}$ if $\xi \in \omega_1 \setminus A$. Here, we take the minimum with respect to some arbitary well-order on ω_1^2. This function is well-defined since A is closed. Write $\phi = (f, g)$ for some $f, g : \omega_1 \to \omega_1$. Then closure of α under f, g just means $A \cap \alpha$ is closed in α.

Similarly, let $h : \omega_1^2 \to \omega_1$ be the map which sends (γ, δ) to 0 if $\gamma \unrhd \delta$ and to $\min\{\xi \in (\gamma, \delta) : \xi \notin A\}$ otherwise. Since A is nowhere dense, this map is well-defined. Here, closure of α under h simply means $A \cap \alpha$ is nowhere dense in α.

Finally, let $h' : \omega_1^2 \to \omega_1$ be the map which sends (γ, δ) to 0 if $(\gamma, \delta) \cap A = \emptyset$ and $\min(\gamma, \delta) \cap A$ otherwise. Then closure of α under h simply means $A \cap (\gamma, \delta) \neq \emptyset$ implies $A \cap (\gamma, \delta) \cap \alpha \neq \emptyset$ for all $\gamma, \delta \in \alpha$.

Now, by Lemma 3.8 the $\alpha \in \omega_1$ which are closed under f, g, h and h' form a closed unbounded set. Since the α which are closed under f, g, h and h' are precisely the α which satisfy the necessary properties, we are done. □

Now let $\{A_\alpha : \alpha \in \omega_1\}$ be a $\Diamond$-sequence. Since $\{\alpha \in \omega_1 : A \cap \alpha = A_\alpha\}$ is stationary, it suffices to show we can pick D_β at each stage such that it avoids each A_α which is closed nowhere dense in β. Since β with the new order is isomorphic to $\mathbb{Q}$, the following lemma completes the proof.

3.13. Lemma. *If $\{A_n\}_{n\in\omega}$ is a family of closed nowhere dense subsets of $\mathbb{Q}$ then there is a Dedekind cut which avoids each A_n.*

Proof. The family of closures $\{\overline{A_n}\}_{n\in\omega}$ is a family of closed nowhere dense subsets of $\mathbb{R}$. By Baire's category theorem, $\bigcup_{n\in\omega} \overline{A_n}$ is not equal to $\mathbb{R}$, so there is some irrational $x \in \mathbb{R}$ which is contained in none of the $\overline{A_n}$. The Dedekind cut D_x corresponding to x is as desired. □

This concludes the construction of the Souslin line. □

Next up is the Souslin tree.

3.14. Theorem. ($\Diamond$) *There is a Souslin tree.*

As before, we begin by giving a construction which almost works, except this time there may be uncountable antichains in the constructed tree. The underlying set T will be ω_1 (so $T = \omega_1$) and the order will be denoted by $\lhd$. During the proof, we will use the usual notation: $T_{(\alpha)}$ consists of the elements at level $< \alpha$, $T(\alpha)$ consists of the elements at level α and $\lhd_\alpha$ is the order $\lhd$ restricted to $T_{(\alpha)}$. For each α, $T(\alpha)$ will consist of the elements in $[\omega \cdot \alpha, \omega \cdot \alpha + \omega)$.

To make matters easier, we will construct a tree for which every point has two successors; such a tree is called splitting. By the following lemma, in doing so, we only need to worry about making a high enough tree which has no antichains:

3.15. Lemma. *Suppose $(T, \lhd)$ is a splitting tree of height ω_1. Then, if T has a branch of length ω_1 it has an antichain of width ω_1 as well.*

Proof. Suppose B is a branch of length ω_1. For each α, let $t_\alpha \in B$ be at level α. Since t has precisely two immediate successors, we can pick s_α at level $\alpha + 1$ such that $s_\alpha \notin B$. Now $\{s_\alpha : \alpha \in \omega_1\}$ is an antichain, for if we would have $s_\alpha \rhd s_\beta$, then by construction we would have $s_\alpha \rhd t_{\beta+1}$, which contradicts the fact $\hat{s_\alpha}$ is well-ordered since $t_{\beta+1}$ and s_β are incomparable. $\square$

Thus, if each antichain is countable, then each branch is countable as well.

To make sure we get a high enough tree, we will actually construct the tree in such a manner that for each t at level β and each $\alpha \rhd \beta$ there is an s at level α above t.

If γ is a limit ordinal, then we simply let $\lhd_\gamma$ be $\bigcup_{\alpha<\gamma} \lhd_\alpha$. If α is any ordinal number and $\lhd_{\alpha+1}$ has been constructed, then to make sure each $t \in T$ has two successors, we put two elements of $[\omega \cdot (\alpha + 1), \omega \cdot (\alpha + 2))$ above each element of $T(\alpha)$ in such a way that every element of $[\omega \cdot (\alpha + 1), \omega \cdot (\alpha + 2))$ is used.

The real problem is how to handle successors of limits. The following lemma tells us how to make sure we do this in a way such that our tree will be high enough.

3.16. Lemma. *If $\alpha > 0$ and if $\lhd_\alpha$ is as desired, then for every $s \in T_{(\alpha)}$, there is a branch B_s of length α such that $s \in B_s$.*

Proof. Let $\{\alpha_n\}_{n\in\omega}$ be an increasing sequence with $\lim_{n\to\omega} \alpha_n = \alpha$ and such that $\alpha_0 = \text{ht}(s)$. Since $\lhd_\alpha$ behaves as desired, we can recursively pick s_{n+1} at level α_{n+1} such that $s_{n+1} > s_n$. Now let $B_s = \bigcup_{n\in\omega} \hat{s_n}$. $\square$

Now, put one element from $[\omega \cdot \alpha, \omega \cdot (\alpha + 1))$ on top of each B_s in such a manner that every $t \in [\omega \cdot \alpha, \omega \cdot (\alpha + 1))$ is used. Let $\lhd = \bigcup_{\alpha\in\omega} \lhd_\alpha$. Then $(T, \lhd)$ is a splitting tree of height ω_1, which may or may not have uncountable antichains.

By Zorn's lemma, we can extend any antichain to a maximal one, so it is sufficient to make sure T has no uncountable maximal antichains. Let $A \subseteq \omega_1$ be a maximal antichain and let $\mathcal{M}(A)$ be the set of ordinals α such that $T_{(\alpha)} = \alpha$ and such that $A \cap \alpha$ is maximal in $T_{(\alpha)}$. For $\alpha \in \mathcal{M}(A)$, suppose we can construct $\lhd_{\alpha+1}$ in such a manner that each $t \in T(\alpha)$ is above some $s \in A \cap \alpha$. Suppose A is uncountable. Then we have $A \setminus T_{(\alpha)} \neq \emptyset$. However, for

$t \in A \setminus T_{(\alpha)}$ we can find a predecessor s of t at level α and so an $r \in T_{(\alpha)}$ such that $r \lhd s$ holds, contradicting the fact A is an antichain. Thus, under the assumptions mentioned above and provided that we can actually make sure each $t \in T(\alpha)$ is above some $s \in A \cap \alpha$, we have made sure A stays countable.

3.17. Lemma. *Let $A \subseteq T$ be a maximal antichain. Then $\mathcal{M}(A)$ is closed unbounded.*

Proof. Note that the α for which $T_{(\alpha)} = \alpha$ are precisely the α for which we have $\omega \cdot \alpha = \alpha$, i.e. the α in $\{\omega^\gamma : \gamma \text{ is a countable limit ordinal}\}$. Since this is the normal image of a closed unbounded set (namely, the image of the limit ordinals under the normal map $\alpha \mapsto \omega^\alpha$), this set itself is closed unbounded. Denote this set by $\mathcal{C}$.

Let $f : \omega_1 \to \omega_1$ be the map which sends ξ to $\min\{\beta \in A : \xi \text{ is comparable with } \beta\}$. Since A is a maximal antichain, this map is well-defined. Now, if α is closed under f, then all $\xi \in \alpha$ are comparable with some $\beta \in A \cap \alpha$, so $A \cap \alpha$ is a maximal antichain in α. By Lemma 3.8, the set $\mathcal{D}$ of α which are closed under f is closed unbounded.

Now, by Theorem 3.2, $\mathcal{M}(A) = \mathcal{C} \cap \mathcal{D}$ is closed unbounded, so we are done. □

Let $\{A_\alpha\}_{\alpha\in\omega_1}$ be a $\diamondsuit$-sequence. The set $S = \{A \cap \alpha = A_\alpha : \alpha \in \omega_1\}$ is stationary, so if we can construct $<_{\alpha+1}$ in such a manner that there is an $s \in A_\alpha$ below each $t \in T(\alpha)$ whenever $T_{(\alpha)} = \alpha$ and whenever A_α is a maximal antichain in α we are done: the intersection $S \cap \mathcal{M}(A)$ is non-empty, so at some point this will make sure A stays countable. That we can do this is the content of the following lemma, which is in essence a modification of Lemma 3.16.

3.18. Lemma. *Suppose $T_{(\alpha)} = \alpha$ and suppose A_α is a maximal antichain in α. Then $\lhd_{\alpha+1}$ can be constructed in such a manner that each $t \in T(\alpha)$ is above some $s \in A \cap \alpha$, while still preserving the other properties the tree must have.*

Proof. Let $s \in T_{(\alpha)}$. A_α is a maximal antichain in α, so there is an $a \in A_\alpha$ which is comparable with s. Let $\{\alpha_n^s\}_{n\in\omega}$ be an increasing sequence such that $\alpha_1^s > \mathrm{ht}(a)$, $\alpha_1^s > \mathrm{ht}(s)$ and $\lim_{n\to\omega} \alpha_n^s = \alpha$. As in Lemma 3.16, define $s_0 = s$ and s_{n+1} recursively such that $s_{n+1} \rhd s_n$ and such that s_{n+1} is at level α_{n+1}^s for all $n \in \omega$. Similarly to Lemma 3.16, for each $s \in T_{(\alpha)}$ the sequence $\{s_n\}_{n\in\omega}$ gives rise to a branch in $T_{(\alpha)}$. Now put the elements of $[\omega \cdot \alpha, \omega \cdot \alpha + \omega)$ on top of each of these branches in such a way that each element is used. □

This concludes the construction of the Souslin tree. □

4 Martin's Axiom

In this section, we will use partially ordered sets to encode information and filters to construct sets using this information.

4.1. Definition. Let $(P, <)$ be a partially ordered set. A subset $\mathcal{F}$ of P is called a filter if it satisfies the following conditions:

- $\mathcal{F}$ is a filter base; that is, if $F \subseteq \mathcal{F}$ is finite, then there is a $p \in \mathcal{F}$ such that $p \leq q$ for all $q \in F$.
- If $p \in \mathcal{F}$ and $q \geq p$ then $q \in \mathcal{F}$.

If Q is a filter base, then $\mathcal{F}_Q = \{p \in P : p \leq q \text{ for some } q \in Q\}$ is a filter and we say Q generates $\mathcal{F}_Q$. Elements $p, q \in P$ for which $\{p, q\}$ is a filter base are called compatible; we write $p \parallel q$ for compatible elements and $p \perp q$ for elements which are incompatible. Note that every pair of elements in a filter base is compatible.

Filters are used in topology to define convergence as follows: if (X, τ) is a topological space then a filter $\mathcal{F}$ is said to converge to x if $\mathcal{F}$ contains the filter $\mathcal{N}_x$ consisting of all the neighborhoods of x.

If $p \in \mathcal{F}$ implies $\mathcal{F}$ satisfies a certain property ϕ, we say p forces ϕ. Recall that if $p \leq q$ and if $\mathcal{F}$ is a filter, then $p \in \mathcal{F}$ implies $q \in \mathcal{F}$. Thus, p being a member of $\mathcal{F}$ is a stronger condition than membership of q; we say p extends q. Note that if p extends q and q forces ϕ, then p forces ϕ as well.

4.2. Definition. A subset $D \subseteq P$ is called dense if for every $p \in P$ there is a $d \in D$ such that $d \leq p$. If $\mathcal{D}$ is a collection of dense subsets of P, we say a filter $\mathcal{F}$ is $\mathcal{D}$-generic if $\mathcal{F} \cap D \neq \emptyset$ for all $D \in \mathcal{D}$.

Families $\mathcal{D}$ of dense subsets are crucial because, under certain circumstances, we can ensure a $\mathcal{D}$-generic filter exists. If $D \in \mathcal{D}$ and if each $p \in D$ forces a certain property ϕ, then $\mathcal{D}$-genericity implies ϕ holds as well. The following lemma is a first result on when we can expect a $\mathcal{D}$-generic filter to exist.

4.3. Lemma (Rasiowa-Sikorski). *Let $(P, <)$ be a partially ordered set and let $\mathcal{D} = \{D_n : n \in \omega\}$ be a countable family of dense subsets of P. Then there is a $\mathcal{D}$-generic filter $\mathcal{F} \subseteq P$.*

Proof. Define recursively a sequence $(d_n)_{n \in \omega}$ such that $d_{n+1} \in D_{n+1}$ and $d_n \geq d_{n+1}$ for all $n \in \omega$. Then $\{d_n : n \in \omega\}$ is a filter basis which intersects each D_n, so the filter it generates is $\mathcal{D}$-generic. □

In general, $\mathcal{D}$-generic filters need not exist.

4.4. Example. Consider the set of all finite partial functions from ω to ω_1 ordered by reverse extension; that is, the set $P = \{f : A \to \omega_1 : A \in [\omega]^{<\omega}\}$ ordered by $f \leq g$ iff $\operatorname{dom} g \subseteq \operatorname{dom} f$ and $f \restriction \operatorname{dom} g = g$. Let, for each $\alpha \in \omega_1$, C_α be the set $\{f \in P : \alpha \in \operatorname{ran} f\}$; it is easy to see each of these sets is dense since, given $f \in P$, one can easily extend it to a finite partial function g which has α in its image. Note that if $\mathcal{F}$ is a filter, each $f \in C_\alpha$ forces

α to be in the range of the function $\bigcup \mathcal{F}$. Write $\mathcal{D} = \{C_\alpha : \alpha \in \omega_1\}$. If $\mathcal{F}$ is a $\mathcal{D}$-generic filter, then $\bigcup \mathcal{F}$ is a function which has a subset of ω as its domain and ω_1 as its codomain, which is impossible. Thus, no $\mathcal{D}$-generic filter exists.

The problem in the previous example is that the partial order is, in a sense, too "wide". The following definition gives a limitation of sorts on the "width" of a partial order which proves to be sufficient to ensure the existence of $\mathcal{D}$-generic filters in certain cases.

4.5. Definition. Let $(P, <)$ be a partially ordered set. If each $A \subseteq P$ which consists of pairwise incompatible elements is countable, we say P is a ccc partial order, or that P satisfies the countable chain condition.

We are now in a position to formulate Martin's Axiom. For each cardinal number $\kappa \geq \omega$, let MA(κ) be the statement that if $\mathcal{D} = \{D_\alpha : \alpha \in \kappa\}$ is a family of at most κ dense subsets of a ccc partial order $(P, <)$ then a $\mathcal{D}$-generic filter exists.

4.6. Martin's Axiom (MA). *MA(κ) holds for all $\kappa < 2^{\aleph_0}$.*

The following theorem gives a topological application of MA(κ). A topological space X is said to be ccc, or satisfy the countable chain condition, if every disjoint family of open subsets is countable. Note that the order topology of any ccc linear order is ccc.

4.7. Theorem. *If MA(κ) holds then no compact ccc Hausdorff space X can be written as a union of at most κ nowhere dense subsets.*

Proof. Let X be a ccc compact Hausdorff space and $\{X_\alpha : \alpha \in \lambda\}$ a family of $\lambda \leq \kappa$ nowhere dense sets. Consider the set P of all non-empty open subsets of X ordered by inclusion. If $U \perp V$ then $U \cap V \notin P$, so since the intersection of any two open sets is open this implies $U \cap V = \emptyset$. As such, a pairwise incompatible subset of P is just a pairwise disjoint family of open subsets of X, which is countable since X is ccc.We conclude P is a ccc partial order.

For each $\alpha \in \lambda$, define $D_\alpha = \{U \in P : \overline{U} \cap X_\alpha = \emptyset\}$. Let $U \in P$. The closure of X_α is closed nowhere dense, so its complement A_α is an open, dense set. As such, its intersection with U is non-empty. Let $x \in U \cap A_\alpha$. Since X is compact Hausdorff, it is regular, so there is some closed neighborhood K of x with $K \subseteq U \cap A_\alpha$. Now, the closure of a non-empty open subset V of K with $x \in V$ is contained in K, so it has empty intersection with X_α. Thus, $V \subseteq U$ is an element of D_α, so each D_α is dense.

Write $\mathcal{D} = \{D_\alpha : \alpha \in \lambda\}$. By MA($\kappa$) there is a $\mathcal{D}$-generic filter $\mathcal{F}$. If $\bigcap_{U \in \mathcal{F}} \overline{U} = \emptyset$ then there is some finite $F \subseteq \mathcal{F}$ with the same property by the compactness of X. For these $U \in F$ we then have $\bigcap_{U \in F} U = \emptyset$, which contradicts the filter base property of $\mathcal{F}$ since the intersection of any finite number of open sets is again open. Take $x \in \bigcap_{U \in \mathcal{F}} \overline{U}$. Since $\mathcal{F}$ has non-empty intersection with each of the D_α, there is some $U \in \mathcal{F}$ such that $\overline{U} \cap X_\alpha = \emptyset$ for all $\alpha \in \lambda$, so $x \notin \bigcup_{\alpha \in \lambda} X_\alpha$. □

As an immediate corollary of Theorem 4.7 we obtain:

4.8. Corollary. *The statement MA($2^{\aleph_0}$) is false.*

Proof. The compact interval $[0, 1]$ is a compact ccc Hausdorff space. Since $[0, 1]$ has no isolated points, each singleton is closed nowhere dense and since $|[0, 1]| = 2^{\aleph_0}$, it is the union of $2^{\aleph_0}$ closed nowhere dense sets, namely $[0,1] = \bigcup_{x\in[0,1]}\{x\}$. Thus, there is a compact ccc Hausdorff space which is the union of $2^{\aleph_0}$ closed nowhere dense subsets, so by Theorem 4.7 the statement $\mathrm{MA}(2^{\aleph_0})$ is false. □

Under (CH) we have $2^{\aleph_0} = \aleph_1$, so if $\lambda < \aleph_1$ then $\lambda \leq \omega$. Thus, Martin's Axiom holds under the Continuum Hypothesis. However, and this is where most of the upcoming proofs fail under (CH), the Continuum Hypothesis also implies $\mathrm{MA}(\lambda)$ does not hold for any $\lambda > \omega$. For this reason, Martin's Axiom is only of interest in the absence of (CH).

By the following theorem, it is possible to have both (MA) and ¬(CH). Its proof uses (iterated) forcing, an advanced technique which generalizes the concepts used in the formulation of Martin's Axiom.

4.9. Theorem. *If ZFC is consistent, then so is ZFC+(MA)+¬(CH).*

An exposition of (iterated) forcing, its numerous applications in consistency proofs and the proof of Theorem 4.9 can be found in chapters 14, 15 and 16 of [2].

This concludes our technical considerations of Martin's Axiom. In practice, Martin's Axiom is used to generalize statements which are proved with the Rasiowa-Sikorski lemma. In such a situation, one only needs to prove the new order is ccc, which may or may not be possible.

Quite often, proofs which proceed by recursion on ω pick elements in a rather arbitrary order and it is in situations like this that one may be able to transform such a proof in one using the Rasiowa-Sikorski lemma, thus opening the possibility of generalization to larger cardinals.

4.10. Example. Consider $(\omega^\omega, <^*)$, where $f <^* g$ if and only if $\{n \in \omega : f \geq g\}$ is finite. Given a sequence $\{f_n\}_{n\in\omega}$, one way to construct a function $g \in \omega^\omega$ which satisfies $f_n <^* g$ for all $n \in \omega$ is by recursion: if $g(k)$ is known for all $k < n$, let $g(n)$ be some value greater than each of the $f_k(n)$ with $k \leq n$. Now for each $k \in \omega$ we have $f_k(n) < g(n)$ for all $n \geq k$, so g is as desired.

The previous construction depends on the enumeration of the set $\mathcal{F} = \{f_n : n \in \omega\}$ and the order in which we add elements to the domain of g. We could have proceeded with any other enumeration or order and have gotten a possibly different but equally valid function. This arbitrariness is absent in the following application of the Rasiowa-Sikorski lemma.

Let $P_\mathcal{F} = \{(p, F) : p$ is a function $A \to \omega$, $A \in [\omega]^{<\omega}$ and $F \in [\mathcal{F}]^{<\omega}\}$ be ordered by $(p, F) \leq (q, G)$ if and only if $q \subseteq p$, $G \subseteq F$ and for all $f \in G, n \in \operatorname{dom} p \setminus \operatorname{dom} q$ we have $p(n) > f(n)$. Given a filter $Q \subseteq P_\mathcal{F}$, we get a function $f_Q = \bigcup\{p : (p, F) \in Q$ for some $F\}$ with co-domain ω. Note that each (p, F) forces $f_Q(n) > h(n)$ for all $h \in F$ and $n \in \operatorname{dom} f_Q \setminus \operatorname{dom} p$.

To make sure f_Q has domain ω and to make sure it dominates each of the $h \in \mathcal{F}$, we consider, for each $n \in \omega$ and $h \in \mathcal{F}$, the sets $D_n = \{(p, F) \in P : n \in \operatorname{dom} p\}$ and $E_h = \{(p, F) \in P : h \in F\}$. It is easy to verify these sets are dense in P and they are countable since P is countable. Now, since each element of D_n forces $n \in \operatorname{dom} f_Q$ and each element of E_h forces $f_Q(n) > h(n)$, the filter which intersects each of the D_n and E_h yields a function which suffices.

Under the assumption of MA(λ), the construction with the Rasiowa-Sikorski lemma generalizes to families $\mathcal{G} \subseteq \omega^\omega$ of cardinality λ rather easily; we only need to replace each $\mathcal{F}$ with $\mathcal{G}$. Given that we can prove $P_\mathcal{G}$ is a ccc partial order, this proves each family $\mathcal{G} \subseteq \omega^\omega$ of cardinality λ is dominated by some $f_\mathcal{G} \in \omega^\omega$. Since the elements of $P_\mathcal{G}$ which have the same first coordinate are compatible and since there is only a countable number of possibilities for the first coordinate, this is easily seen to be the case.

We now proceed with proving the consistency of (SH). As before, we will directly derive a statement both for Souslin trees and Souslin lines. We begin with Souslin trees.

4.11. Theorem (MA($\aleph_1$)). *There are no Souslin trees.*

Proof. Suppose there is a Souslin tree. Then by Theorem 2.6 there is also a normal Souslin tree. Let $(T, <)$ be such a tree and let T^* be this tree ordered by $\geq$.

For each $\alpha \in \omega_1$, let $D_\alpha = \{t \in T^* : \mathrm{ht}(t) > \alpha\}$, where ht is taken with respect to T. If $t \in T^*$ is an element with $\mathrm{ht}(t) \leq \alpha$ then by property (1) of a normal tree there is an $s \in T^*$ such that $s > t$ and $\mathrm{ht}(s) > \alpha$, so D_α is a dense subset of T^*. Write $\mathcal{D} = \{D_\alpha : \alpha \in \omega_1\}$.

By MA($\aleph_1$), there is a $\mathcal{D}$-generic filter $\mathcal{F} \subseteq T^*$. For each $s, t \in \mathcal{F}$ there is an $r \in \mathcal{F}$ such that $r \geq s$ and $r \geq t$, so since $\hat{r}$ is well-ordered we either have $s \leq t$ or $t \leq s$. Thus, $\mathcal{F}$ is a chain and since it has non-empty intersection with each D_α, it contains elements of arbitrary height. We conclude $\mathcal{F}$ is an uncountable chain in T, contradicting the fact T is a Souslin tree. □

As for Souslin lines, we need a general statement on ccc topological spaces which is interesting in its own right.

4.12. Theorem (MA($\aleph_1$)). *Let X be a ccc topological space and let $\{U_\alpha : \alpha \in \omega_1\}$ be a family of non-empty open subsets of X. Then there is an uncountable $A \subseteq \omega_1$ such that $\{U_\alpha : \alpha \in A\}$ has the finite intersection property; that is, if $F \subseteq A$ is finite, then $\bigcap_{\alpha \in F} U_\alpha \neq \emptyset$.*

Proof. For each $\gamma \in \omega_1$, let $V_\gamma = \bigcup_{\beta \geq \gamma} U_\beta$. Note that $V_\gamma \subseteq V_\beta$ for all $\beta \leq \gamma$. We begin by proving the sequence $\{\overline{V}_\gamma\}_{\gamma \in \omega_1}$ is eventually constant. Suppose it is not. Then we can pick $\alpha_\xi > \sup_{\beta \in \xi} \alpha_\beta$ recursively in such a way that $\overline{V}_{\alpha_\xi} \neq \overline{V}_{\alpha_\beta}$ for all $\beta < \xi$. Now $\{V_{\alpha_\xi} \setminus \overline{V}_{\alpha_{\xi+1}} : \xi \in \omega_1\}$ is an uncountable family of pairwise disjoint open subsets of X, which contradicts the fact X is a ccc topological space.

Let $\alpha \in \omega_1$ be some value for which $\{\overline{V}_\gamma\}_{\gamma \geq \alpha}$ is constant. Let $P = \{U \subseteq V_\alpha : U \text{ open and } U \neq \emptyset\}$ be partially ordered by inclusion. As in the proof of Theorem 4.7, this partial order is ccc because X is. If $\mathcal{F} \subseteq P$ is a filter, let $A_\mathcal{F} = \{\gamma \in \omega_1 : (\exists U \in \mathcal{F})(U \subseteq U_\gamma)\}$. Then, since $\mathcal{F}$ is a filter basis, $\{U_\gamma : \gamma \in A_\mathcal{F}\}$ has the finite intersection property.

For each $\beta \in \omega_1$, let $D_\beta = \{U \in P : (\exists \gamma > \beta)(U \subseteq U_\gamma)\}$. Since the sequence $\{\overline{V}_\gamma\}_{\gamma \geq \alpha}$ is constant we have $\overline{V}_\beta \supseteq \overline{V}_\alpha$, so for each $U \in P$ we have $U \cap V_\beta \neq \emptyset$. As such, $U \cap U_\gamma \neq \emptyset$ for some $\gamma > \beta$, so $U \cap U_\gamma$ is an extension of U which is in D_β. We conclude D_β is dense for all $\beta \in \omega_1$.

Note that if $\mathcal{F}$ is a filter, each $U \in D_\beta$ forces $A_\mathcal{F}$ to contain some $\gamma > \beta$. Write $\mathcal{D} = \{D_\beta :$

$\beta \in \omega_1\}$. By MA($\aleph_1$) there is a $\mathcal{D}$-generic filter $\mathcal{F}$. For this $\mathcal{F}$ we have that $A_\mathcal{F}$ is unbounded in ω_1 and therefore uncountable, so $\{U_\gamma : \gamma \in A_\mathcal{F}\}$ is as desired. □

4.13. Corollary (MA($\aleph_1$)). *If X and Y are ccc topological spaces, then $X \times Y$ with the product topology is ccc as well.*

Proof. Let $\{U_\alpha \times V_\alpha : \alpha \in \omega_1\}$ be an uncountable family of disjoint sets, where U_α and V_α are non-empty open subsets of X respectively Y for all $\alpha \in \omega_1$. By Theorem 4.12, there is an uncountable $A \subseteq \omega_1$ such that $\{U_\alpha : \alpha \in A\}$ has the finite intersection property. We have $U_\alpha \cap U_\beta \neq \emptyset$ for all $\alpha \neq \beta$ in A. Since $U_\alpha \times V_\alpha$ and $U_\beta \times V_\beta$ are disjoint, this implies $V_\alpha \cap V_\beta = \emptyset$ for all $\alpha \neq \beta$ in A, contradicting the fact Y is a ccc topological space. □

4.14. Proposition. *If L is a Souslin line, then L^2 with the product topology is not a ccc topological space.*

Proof. Let L' and $2^{\omega_1} \to (L')^2$ be defined as in the proof of Theorem 2.7. Since the tree consisting of the $s \in 2^{<\omega_1}$ for which $[a_s, b_s]$ is non-degenerate is of height ω_1, there is for each $\alpha \in \omega_1$ an $s_\alpha \in 2^{\omega_1}$ such that $I_\alpha = (a_{s_\alpha}, b_{s_\alpha})$ is non-empty. By construction, the intervals $I^0_\alpha = (a_{s_\alpha \cdot 0}, b_{s_\alpha \cdot 0})$ and $I^1_\alpha = (a_{s_\alpha \cdot 1}, b_{s_\alpha \cdot 1})$ are non-empty and have empty intersection. Define $U_\alpha = I^0_\alpha \times I^1_\alpha$ and consider $A = \{U_\alpha\}_{\alpha \in \omega_1}$. If $\alpha < \beta$, we distinguish two cases. If $I_\alpha \cap I_\beta = \emptyset$, then $I^0_\alpha \cap I^0_\beta = \emptyset$, so $U_\alpha \cap U_\beta = \emptyset$. If $I_\alpha \cap I_\beta \neq \emptyset$ then, since each pair of intervals in $\{[a_s, b_s] : s \in 2^{\omega_1}\}$ is either nested or disjoint, we have $I_\beta \subseteq I_\alpha$. Without loss of generality we have $I_\beta \subseteq I^0_\alpha$, which implies $I^1_\alpha \cap I^1_\beta = \emptyset$, so $U_\alpha \cap U_\beta = \emptyset$. We conclude A is a family of ω_1 disjoint open subsets, so L^2 is not a ccc topological space. □

In the end, the non-existence of Souslin trees was a rather easy consequence of MA($\aleph_1$). As we know, Souslin trees are a special kind of Aronszajn trees. The following theorem gives a very specific characterization of Aronszajn trees and denies the existence of Souslin trees in a much stronger way. As a bonus, its proof, which will be given at the end, is an excellent showcase of both the power and the typical usage of Martin's Axiom.

4.15. Theorem (MA($\aleph_1$)). *Every Aronszajn tree is the union of a countable number of antichains.*

To prove this theorem, we need some combinatorial results. The way the following lemmas are used in the proof of Theorem 4.15 is typical and demonstrates the power of these techniques. We call a collection W a Δ-system if there is some $d \in W$ (called the root of W) such that $v \cap w = d$ for all $v \neq w \in W$.

4.16. Lemma (Pigeonhole principle). *If κ is a regular cardinal, $\lambda < \kappa$ and $f : \kappa \to \lambda$ is a function, then there is some $\alpha \in \lambda$ such that $|f^{-1}\{\alpha\}| = \kappa$.*

Proof. Suppose not. We have $\bigcup_{\alpha \in \lambda} f^{-1}\{\alpha\} = \kappa$, so κ is then a union of $< \kappa$ sets of cardinality $< \kappa$, contradicting the regularity of κ. □

4.17. Lemma (Δ-system lemma). *If W is an uncountable family of finite sets, then there is an uncountable Δ-system $A \subseteq W$.*

Proof. Let $W' = \{w_\alpha : \alpha \in \omega_1\} \subseteq W$ be some subset of W of cardinality ω_1. Applying Lemma 4.16 to $f : \omega_1 \to \omega, \alpha \mapsto |w_\alpha|$ we can find some $n \in \omega$ such that the subset W''

of W' consisting of sets of cardinality n is uncountable. Thus, we are done if we can prove the lemma for uncountable families U consisting of sets of cardinality n. We proceed by induction on n.

For $n = 1$, we can simply take U itself since the intersection of two distinct singleton sets is empty, which implies U is a Δ-system with root $\emptyset$. If the theorem has been proven for some $m \geq 1$, let $n = m + 1$.

If there is a $u \in \bigcup U$ such that u is an element of uncountably many $v \in U$, then $U' = \{v \setminus \{u\} : v \in U, u \in v\}$ is uncountable, so by the induction hypothesis there is an uncountable Δ-system $U'' \subseteq U'$, say with root d. Now the set $\{v \cup \{u\} : v \in U''\} \subseteq U$ is an uncountable Δ-system with root $d \cup \{u\}$.

If there is no such $u \in \bigcup U$ then each $u \in \bigcup U$ belongs to only countably many $v \in U$. Therefore, we can recursively construct a pairwise disjoint subfamily of U as follows: if X_ξ is given for $\xi < \alpha$, then $U_\alpha = \bigcup_{\xi<\alpha} \bigcup_{x \in X_\xi} \{v \in U : x \in v\}$ is a countable union of countable sets, so it is a countable subset of U. Pick $X_\alpha \in U \setminus U_\alpha$. □

4.18. Theorem (Ramsey). *Let $n, k \in \omega$. If $F : [\omega]^n \to k$ is a function then there is an infinite $H \subseteq \omega$ such that F is constant on $[H]^n$.*

Proof. We proceed by induction on n.

For $n = 1$, this is just the Pigeonhole principle for $\kappa = \omega$ and $\lambda = k$.

Suppose the theorem holds for some $n \in \omega$. Let $F : [\omega]^{n+1} \to k$. For each $a \in \omega$ and infinite $S \subseteq \omega$, define $F_a : [S \setminus \{a\}]^n \to k$, $A \mapsto F(A \cup \{a\})$. By the induction hypothesis, there an $H_a^S \subseteq S \setminus \{a\}$ such that F_a is constant on $[H_a^S]^n$.

Define S_m and a_m as follows: $S_0 = \omega$,$a_0 = 0$ and if S_m and a_m are known, let $S_{m+1} = H_{a_m}^{S_m}$ and $a_{m+1} = \min\{x \in S_{m+1} : x > a_m\}$. Then for each $m \in \omega$, F_{a_m} is constant on $[\{a_x : x > m\}]^n$ since for each $x > m$ we have $S_x \subseteq S_m$. For each $m \in \omega$, take $G(a_m)$ to be this constant value and take $H \subseteq \{a_m : m \in \omega\}$ such that G is constant on $[\{a_m : m \in \omega\}]^n$. Now F is constant on $[H]^{n+1}$ by definition of G, so H is as desired. □

The statement in Theorem 4.18 is often written as $\omega \to (\omega)^n_k$. For our proof of Theorem 4.8, we only need $\omega \to (\omega)^2_2$, which can be restated in terms of graphs: since a graph G with ω nodes is just a pair (E, ω), where E is a subset of $[\omega]^2$, a graph can be identified with a function $[\omega]^2 \to 2$ which maps a pair $\{a, b\}$ to 1 if and only if $\{a, b\} \in E$. By Theorem 3.11 there is an infinite $H \subseteq \omega$ such that f is constant on $[H]^2$, i.e. there is a subset $H \subseteq \omega$ such that either all pairs of nodes in H are connected by an edge or none are.

Proof of Theorem 4.15. Let $(T, <)$ be an Aronszajn tree. We will construct a strictly isotone function $f : T \to \mathbb{Q}$, i.e. a function such that $t_1 < t_2$ implies $f(t_1) < f(t_2)$. Then by contraposition each of the $f^{-1}\{q\}$ are antichains and we have $T = \bigcup_{q \in \mathbb{Q}} f^{-1}\{q\}$.

Let P be the set of finite approximations of f, that is, functions g with domain a finite subset of T which satisfy the implication $t_1 < t_2 \Rightarrow g(t_1) < g(t_2)$. Define $g \leq h$ if $\operatorname{dom} h \subseteq \operatorname{dom} g$ and $g \restriction \operatorname{dom} h = h$, i.e. if g is an extension of h. Note that we have $g_1 \parallel g_2$ if there is an approximation $h \in P$ which extends both g_1 and g_2.

Define $D_t = \{g \in P : t \in \text{dom } g\}$. This set is dense since we can extend any function $g \in P$ to one with $t \in \operatorname{dom} g$: simply send t to some element of $\mathbb{Q}$ which is greater than all the $g(s)$ with $s < t$ in $\operatorname{dom} g$ and lesser than all the $g(r)$ with $r > t$ in $\operatorname{dom} g$. Write $\mathcal{D} = \{D_t : t \in T\}$.

Suppose P is a ccc partial order. Since T is of cardinality ω_1, MA($\aleph_1$) is applicable, so there is a $\mathcal{D}$-generic filter $\mathcal{F}$. Now $\bigcup \mathcal{F}$ is a function which is strictly isotone and which has domain T, i.e. it is the desired function f. Thus, we are done if we can prove P satisfies the countable chain condition.

Let $\{f_\alpha\}_{\alpha \in \omega_1} \subseteq P$. We will prove there are $\alpha, \beta \in \omega_1$ such that f_α and f_β are compatible. For this, we make repeated use of the pigeonhole principle and the Δ-system lemma to find an uncountable subset of ω_1 in which our functions satisfy certain properties.

1. Applying the pigeonhole principle to $\omega_1 \to \omega, \alpha \mapsto |\operatorname{dom} f_\alpha|$, we find an uncountable $A \subseteq \omega_1$ and an $n \in \omega$ such that $|\operatorname{dom} f_\alpha| = n$ for all f_α with $\alpha \in A$.

2. If the range A' of $h : A \to [T]^n, \alpha \mapsto \operatorname{dom} f_\alpha$ is countable, use the pigeonhole principle to find an uncountable subset $B \subseteq A$ such that $\operatorname{dom} f_\alpha = \operatorname{dom} f_\beta$ for all $\alpha, \beta \in B$; then B is a Δ-system with root $R = \operatorname{dom} f_\alpha$. Otherwise, apply the Δ-system lemma to $A' = \{\operatorname{dom} f_\alpha : \alpha \in A\}$ to find a finite $R \subseteq T$ and an uncountable subset $B' \subseteq A'$ such that $\operatorname{dom} f_\alpha \cap \operatorname{dom} f_\beta = R$ for all $\operatorname{dom} f_\alpha$ and $\operatorname{dom} f_\beta$ with $\alpha, \beta \in B'$. Now take $B = h^{-1}[B']$.

3. Apply the pigeonhole principle to $B \to \mathbb{Q}^R, \alpha \mapsto f_\alpha \restriction R$ to find an uncountable $C \subseteq B$ with $f_\alpha \restriction R = h$, where h is some function $R \to \mathbb{Q}$. This guarantees all the f_α with $\alpha \in C$ are defined and agree on R.

4. Write $\operatorname{dom} f_\alpha = \{s_{\alpha,i} : i < n\}$ and $R = \{s_{\alpha,i} : i < k\}$, where $k = |R|$. Apply the pigeonhole principle to $C \to \mathbb{Q}^n, \alpha \mapsto g_\alpha$, where $g_\alpha(i) = f_\alpha(s_{\alpha,i})$ to find a function $g : n \to \mathbb{Q}$ and an uncountable $D \subseteq C$ such that $g_\alpha = g$ for all $\alpha \in D$. We then have $f_\alpha(s_{\alpha,i}) = f_\beta(s_{\beta,i})$ for all $\alpha, \beta \in D$ and all $i < n$.

5. For each $\alpha \in D$, let m_α be the minimum of $\{\text{ht}(s_{\alpha,i}) : k \leq i < n\}$ and M_α the maximum. Since the $s_{\alpha,i}$ are distinct from the $s_{\beta,j}$ and since each level of T is countable, there can only be a countable number of $\alpha \in \omega_1$ with $m_\alpha = \gamma$ for any given γ. As such, the set $\{\beta : m_\beta \leq \delta\} = \bigcup_{\gamma \leq \delta} \{\beta : m_\beta = \gamma\}$ is countable for every $\delta \in \omega_1$. Define recursively a map $q : D \to D$ as follows: if $q(\xi)$ has been determined for all $\xi \in \beta \cap D$ for some $\beta \in D$, take $q(\beta)$ to be some element of D for which $m_{q(\beta)}$ is greater than $\sup_{\xi \in \beta \cap D} M_q(\xi)$. The image $E \subseteq D$ is then an uncountable subset such that for all $\alpha < \beta$ in E we have $\text{ht}(s_{\alpha,i}) < \text{ht}(s_{\beta,j})$ for all $k \leq i, j < n$.

6. For each $i \geq k$ the set $\{s_{\alpha,i} : \alpha \in E\}$ is uncountable. By MA($\aleph_1$), no Souslin trees exist, so there is some uncountable $E_k \subseteq E$ such that $\{s_{\alpha,k} : \alpha \in E_k\}$ is an antichain. Now define for each $k \leq i < n - 2$ recursively E_{i+1} to be an uncountable subset of E_i such that $\{s_{\alpha,i+1} : \alpha \in E_{i+1}\}$ is an antichain. Then for $M = E_{n-1}$, each $\{s_{\alpha,i} : \alpha \in E_i\}$ is an antichain.

Now, since all of the properties mentioned at the end of each step is preserved on taking subsets, M satisfies all of them. Let (i, j) be a pair of natural numbers with $k \leq i, j < n$.

Let M' be a countably infinite subset of M and let G be the graph in which nodes $\alpha < \beta$ are connected by an edge if $s_{\alpha,i} < s_{\beta,j}$. Then $\alpha < \beta < \gamma$ can not be all connected by an edge: if they were, we would have $s_{\alpha,i} < s_{\gamma,j}$ and $s_{\beta,i} < s_{\gamma,j}$. But since T is a tree, the predecessors of $s_{\gamma,j}$ are well-ordered, so $s_{\alpha,i} < s_{\beta,i}$ by (5), contradicting (6). Therefore, by Ramsey's theorem we can find an infinite subset $H \subseteq M'$ such that no nodes in H are connected by an edge.

Let $m = (n-k)^2$, let $u : m \to (n \setminus k)^2$ be a bijection and let i_x respectively j_x be the first respectively second coordinate of $u(x)$. Using the above, let M_0 be an infinite subset of M such that $s_{\alpha,i_0} \not< s_{\beta,j_0}$ for all $\alpha < \beta$ in M and define M_{x+1} recursively for $x < m-1$ to be an infinite subset of M_x such that $s_{\alpha,i_{x+1}} \not< s_{\beta,i_{x+1}}$ for all $\alpha < \beta$ in M_x. Let $M^* = M_m$, $\alpha, \beta \in M^*$ and define $S_\alpha = \operatorname{dom} f_\alpha$ and $S_\beta = \operatorname{dom} f_\beta$. Then $k : S_\alpha \cup S_\beta \to \mathbb{Q}$, where $k(s) = f_\alpha(s)$ for $s \in S_\alpha$ and $k(s) = f_\beta(s)$ for $s \in S_\beta$ is as desired. As such P is a ccc partial order, which concludes the proof. □

5 Appendix: The Dedekind Completion

In this appendix we define the Dedekind completion of a non-empty dense linearly ordered set without endpoints (which we will refer to as a dne linearly ordered set or a dne linear order) and show the countable chain condition and separability of any such set is preserved under this completion.

5.1. Definition. Let $(L,<)$ be a dne linear order. A Dedekind cut D is a non-empty initial segment of L which has no maximum and which is not equal to L. The Dedekind completion $\hat{L}$ of L is the set of Dedekind cuts ordered by inclusion.

Note that since L is densely ordered and has no minimum, the set $\hat{a} = \{l \in L : l < a\}$ is a Dedekind cut for all $a \in L$ and note that $\hat{a} \subsetneq \hat{b}$ iff $a < b$. Therefore, one can embed L in $\hat{L}$ by mapping a to $\hat{a}$. Denote this embedding by ι.

If there is an order-embedding $L \to X$ and if $(X,<)$ is a complete dne linearly ordered set, then X is said to be a completion of L. By the following theorem, the Dedekind completion of L is, in a sense, the smallest example of a completion of L.

5.2. Theorem. *Let $(L,<)$ be a dne linear order and let $\hat{L}$ be its Dedekind completion. Then $\hat{L}$ is a completion of L and $\iota[L]$ is dense in $\hat{L}$. If X is another completion of L and $f : L \to X$ is an embedding, then there is an order-embedding $\overline{f} : \hat{L} \to X$ which satisfies $f = \overline{f} \circ \iota$.*

Proof. Let $A, B \in \hat{L}$. If $A \neq B$, then without loss of generality there is an $a \in A$ such that $a \notin B$. Since B is an initial segment, it having elements $b \in B$ with $b > a$ would imply $a \in B$, so $B \subseteq \hat{a} \subsetneq A$ since A has no maximum. We conclude $\hat{L}$ is in fact linearly ordered by inclusion.

If $A \subsetneq B$, then $B \setminus A$ is non-empty, so let $l \in B \setminus A$. Since B has no maximum, there is some $b \in B$ with $b > l$. Now, since this b is still not a maximum, we have $A \subsetneq \hat{l} \subsetneq B$, so $\hat{L}$ is densely ordered and $\iota[L]$ is dense in $\hat{L}$. This also implies $\hat{L}$ has no endpoints.

Suppose $\mathcal{A} \subseteq \hat{L}$ is bounded by D. Then $\bigcup \mathcal{A} \subseteq D \subsetneq L$ is clearly an upper bound for $\mathcal{A}$ and if $C \subsetneq \bigcup \mathcal{A}$ then $C \subsetneq A$ for some $A \in \mathcal{A}$, so C is not an upper bound of $\mathcal{A}$. We conclude $\bigcup \mathcal{A}$ is the supremum of $\mathcal{A}$, so $\hat{L}$ is a complete dense linear order without endpoints and therefore a completion of L.

Now let $(X,<)$ be another completion of L and let $f : L \to X$ be an embedding. Let $\overline{f} : \hat{L} \to X$ be the map which sends $D \in \hat{L}$ to $f(\sup D)$ if $\sup D$ exists and to $\sup f[D]$ if it does not. This is an order-embedding, as one can verify with a case by case analysis. Let $l \in L$. Since $\sup \hat{l} = l$ holds we have $(\overline{f} \circ \iota)(l) = \overline{f}(\hat{l}) = f(\sup \hat{l}) = f(l)$, so $\overline{f}$ is as desired. □

The stability of the countable chain condition and separability under taking the Dedekind completion follows directly from the following theorem since $\iota[L]$ is dense in $\hat{L}$.

5.3. Theorem. *Let $(L,<)$ be a dne linear order and suppose A is dense in L. Then:*

- *If A satisfies the countable chain condition, then so does L.*

- *If the induced order topology on A is separable, then so is the order topology on L.*

Proof. We begin by proving the subspace topology on A coincides with the topology induced by the induced order on A. First of all, each interval $(a,b) \subseteq A$ with $a, b \in A$ in the induced order is equal to $A \cap (a,b)$ in L, which proves one inclusion. For the other inclusion, it is sufficient to check the non-empty intervals $(l,m) \subseteq \hat{L}$ since they form a basis for the order topology on L, so suppose we have $a \in (l,m) \cap A$. Then, since A is dense in L, the sets $(l,a) \cap A$ and $(a,m) \cap A$ are non-empty. Now for each $a_1 \in (l,a) \cap A$ and $a_2 \in (a,m) \cap A$ we have $a \in (a_1, a_2)$, so $(l,m) \cap A$ is open, as was to be shown.

Suppose A satisfies the countable chain condition and suppose $\{U_i\}_{i \in I}$ is a family of disjoint non-empty open intervals of L. Then each $A \cap U_i$ is non-empty since A is dense in L, so $\{U_i \cap A\}_{i \in I}$ is a family of non-empty pairwise disjoint open subsets of A, which is countable since A is a ccc linear order. We conclude L is a ccc linear order as well.

Now suppose the induced order topology on A is separable and let $Q \subseteq A$ be a countable dense subset of A. Then Q is dense in L as well since the subspace topology on A coincides with the topology induced from the induced order on A and since denseness of subspaces is transitive. □

Note that the Dedekind completion of $\mathbb{Q}$ is isomorphic to $\mathbb{R}$ by Theorem 5.3 and Theorem 1.1. This comes as no surprise since one can construct the real numbers by equipping $\hat{\mathbb{Q}}$ with a field structure in an appropriate manner.

Theorem 5.3 implies the Dedekind completion of any separable dne linearly ordered set $(L, <)$ is isomorphic to $\mathbb{R}$. If A is a separable linearly ordered set then $\mathbb{Q} \times A$ with the lexicographical order is a dne linear order, so any separable linearly ordered set can be embedded in $\mathbb{R}$. By a similar argument for ccc linearly ordered sets, we get another equivalent formulation of Souslin's Hypothesis:

5.4. Souslin's Hypothesis (SH). *Suppose $(L, <)$ is a ccc linear order. Then there is an order-embedding $\iota : L \to \mathbb{R}$.*

This concludes our examination of the Dedekind completion.

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