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The Hausdorff Quotient

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Contents

1 Introduction

This thesis will be concerned with the Hausdorff Quotient, also known as 'Hausdorffication', 'Hausdorfffication' or just 'maximal Hausdorff quotient'. The Hausdorff quotient of a topological space X is a Hausdorff space $H(X)$ with a quotient map $q_X : X \to H(X)$ such that for each Hausdorff topological space Y and all continuous maps $f : X \to Y$, there is a unique map $\overline{f} : H(X) \to Y$ such that $f = \overline{f} \circ q_X$.

A main motivation for studying this subject is that, for as far as I know, there is no published article about the Hausdorff Quotient (or Hausdorffication, Hausdorfffication, etc.). I could only find some comments about it on websites such as [1] and [2]. Furthermore, besides being interesting to be studied itself, the Hausdorff Quotient might be useful when one encounters certain non-Hausdorff spaces.

The structure of this thesis is as follows: in section 2, some basic properties of quotient maps and equivalence relations will be covered. The third and fourth sections will be concerned with two different constructions of the Hausdorff Quotient. Whereas the construction from section 3 is easier, the construction from section 4 is more 'constructive' and gives additional information. Section 5 extends the construction of $H(X)$ to a functor and the sixth section is concerned with the Hausdorff Quotient of products of topological spaces, thereby partially being a preparation for section 7. In this final section, it will be shown that the functor obtained in section 5 preserves homotopy, which is the main result of this thesis.

2 Equivalence Relations and Quotients

This first section, being a useful foundation for the rest of this thesis, will be concerned with some basic properties of quotient maps and equivalence relations. Most of the definitions, propositions, etc. can also be found in many books about topology such as [3].

Maps between topological spaces are assumed to be continuous. Top is the category of topological spaces, HausTop is the category of Hausdorff topological spaces and ON is the class of ordinal numbers.

Definition 2.1. Let X be a topological space, Y a set and $f : X \to Y$ a surjective map. The quotient topology on Y with respect to f is the finest topology on Y such that f is continuous. This means that $U \subset Y$ is open if and only if $f^{-1}(U)$ is open in X.

Definition 2.2. Let X, Y be topological spaces and $f : X \to Y$ a surjective map. Then f is called a quotient map if the topology on Y is the quotient topology with respect to f.

Remark 2.3. Consider any topological space X and an equivalence relation $\sim$ on X. Denote the equivalence class of any $x \in X$ by $\overline{x}$. Then $X/\sim$ is assumed to have the quotient topology with respect to the (quotient) map $q\colon X \to X/\sim,\ x \mapsto \overline{x}$.

Proposition 2.4. *Let $q\colon X \to Q$ be a quotient map of topological spaces. Let Y be a topological space and $f\colon X \to Y$ such that $q(a) = q(b) \Rightarrow f(a) = f(b)$ holds for all for all $a, b \in X$. Then there is a unique map $\overline{f}\colon Q \to Y$ such that $f = \overline{f} \circ q$.*

Proof. Define $\overline{f}\colon Q \to Y,\ q(x) \mapsto f(x)$. Then $\overline{f}$ is well-defined, since for all $x, y \in X$ with $q(y) = q(x)$, we have $f(y) = f(x)$. This $\overline{f}$ satisfies $f = \overline{f} \circ q$. It is also continuous, because for any open $U \subset Y$, $q^{-1}(\overline{f}^{-1}(U)) = f^{-1}(U)$ is open. By definition of the quotient topology on Q, this means that $\overline{f}^{-1}(U)$ is open and therefore $\overline{f}$ is continuous. It is unique, since all $\overline{x}$ should be sent to $f(x)$ in order to get $f = \overline{f} \circ q$. □

Corollary 2.5. *Let X, Y, Z be topological spaces and $f\colon X \to Y$, $g\colon X \to Z$ quotient maps such that for all $a, b \in X$, $f(a) = f(b) \Leftrightarrow g(a) = g(b)$. Then there is a unique homeomorphism $h\colon Y \to Z$ such that $g = h \circ f$. This h sends $f(x)$ to $g(x)$.*

Proof. By proposition 2.4, there is unique $\overline{g}\colon Y \to Z$ such that $g = \overline{g} \circ f$. This $\overline{g}$ sends $f(x)$ to $g(x)$. By proposition 2.4, there is also a map $\overline{f}\colon Z \to Y$, $g(x) \mapsto f(x)$. We can see that $\overline{f}$ is the inverse map of $\overline{g}$. Hence, $\overline{g}$ is a homeomorphism. □

For any quotient map $f\colon X \to Y$ of topological spaces, one can define an equivalence relation $\sim_f$ on X such that $a \sim_f b \Leftrightarrow f(a) = f(b)$ for all $a, b \in X$. Corollary 2.5 tells us that we could just as well say that Y is the unique quotient $X/\sim_f$.

Proposition 2.6. *Let X, Y, Z be topological spaces and $f\colon X \to Y$, $g\colon Y \to Z$ quotient maps. Then $g \circ f$ is a quotient map.*

Proof. The map $g \circ f$ is surjective because f and g are. A subset $U \subset Z$ is open if and only if $g^{-1}(U) \subset Y$ is open, which is open if and only if $(g \circ f)^{-1}(U)$ is open in X. Hence, $(g \circ f)$ is a quotient map. □

Remark 2.7. The topology on a product of topological spaces is assumed to be the product topology, unless stated otherwise.

Definition 2.8. Let X be a topological space. The *diagonal* of X is $\operatorname{diag}(X) = \{(x, x) \in X \times X\}$.

Lemma 2.9. *Let X be a topological space and $a, b \in X$. Denote the closure of any subset $A \subset X$ by $\overline{A}$. Then $(a, b) \in \overline{\operatorname{diag}(X)}$ if and only if there are no open $U, V \subset X$ such that $a \in U$, $b \in V$ and $U \cap V = \emptyset$.*

Proof. Let $a, b \in X$. By definition, $(a, b) \in \overline{\mathrm{diag}(X)}$ if and only if for every open neighbourhood W of (a, b), $W \cap \mathrm{diag}(X) \neq \emptyset$. Because any such W is a union of basic open sets, this is true if and only if there is no basic open set $U \times V \subset X \times X$ such that $(a, b) \in U \times V \subset \mathrm{diag}(X)^{\mathsf{c}}$. This is just another formulation of 'there are no open $U, V \subset X$ such that $a \in U$, $b \in V$ and $U \cap V = \emptyset$'. □

Corollary 2.10. *Let X be a topological space. X is Hausdorff if and only if* $\mathrm{diag}(X)$ *is closed in $X \times X$.*

Since $\mathrm{diag}(X)$ is the trivial equivalence relation, this corollary motivates the question whether the equivalence relations R on X such that X/R is Hausdorff are exactly the closed equivalence relations. We will see that only one implication holds.

Lemma 2.11. *let R be an equivalence relation on X such that X/R is Hausdorff. Then R is closed as a subset of $X \times X$.*

Proof. Let $q\colon X \to X/R$ be the quotient map and let $(a, b) \in R^{\mathsf{c}}$. Then $q(a) \neq q(b)$ and since X/R is Hausdorff, there are open $U, V \subset X/R$ such that $q(a) \in U$, $q(b) \in V$ and $U \cap V = \emptyset$. By definition, $q^{-1}(U)$ and $q^{-1}(V)$ are open in X, so $q^{-1}(U) \times q^{-1}(V)$ is open in $X \times X$. Now we have $(a, b) \in q^{-1}(U) \times q^{-1}(V)$ and because $U \cap V = \emptyset$, we obtain $(q^{-1}(U) \times q^{-1}(V)) \cap R = \emptyset$. So for any $(a, b) \in R^{\mathsf{c}}$, there is an open neighbourhood of (a, b) that is a subset of R^{c}. Therefore R^{c} is open and hence R is closed. □

Definition 2.12. Let X, Y be sets and $f\colon X \to Y$ a map. A subset $C \subset X$ is *saturated* with respect to f if $f^{-1}(f(C)) = C$.

Observation 2.13. If f is a quotient map of topological spaces, the open sets of Y are exactly the images of open saturated sets of X.

The following example shows that a closed equivalence relation on a topological space does not necessarily yield a quotient that is Hausdorff.

Example 2.14. There is a space X and an equivalence relation R on X such that R is closed and X/R is not Hausdorff.

Let $X_1 = \mathbb{Q} \cap [0, 1]$, $X_2 = (0, 1]$ and $X_3 = ((0, 1] \setminus \mathbb{Q}) \cup \{0\}$. Consider $X = (X_1 \times \{1\}) \cup (X_2 \times \{2\}) \cup (X_3 \times \{3\})$ and define a relation on X as follows:

$$(x, i) \sim (y, j) \Leftrightarrow (x = y \wedge (x, y > 0 \vee i = j))$$

Let $R = \{(a, b) \in X \times X : a \sim b\}$.

R is an equivalence relation. Reflexivity and symmetry of R are clear. Let $(x, i), (y, j), (z, k) \in X$ such that $x \sim y$ and $y \sim z$. Then $x = y = z$ is true. If $x, y, z > 0$, then $x \sim z$. If $x, y, z > 0$ does not hold, then $x = y = z = 0$, so $i = j = k$ and hence $x = z$, which implies $x \sim z$. So R is transitive.

R is closed. Let $((x, i), (y, j)) \in R^{\mathsf{c}}$. If $x \neq y$, we can find disjoint open neighbourhoods $U \ni x$ en $V \ni y$ in $[0, 1]$. For such $U, V \subset [0, 1]$, it is true that $((U \cap X_i) \times \{i\}) \times ((V \cap X_j) \times \{j\}) \cap R = \emptyset$ and $((x,i),(y,j)) \in ((U \cap X_i) \times \{i\}) \times ((V \cap X_j) \times \{j\})$. Now suppose $x = y$. Then $x = y = 0$ and $i \neq j$ because $((x,i),(y,j)) \in R^{\mathsf{c}}$. Without loss of generality we can assume $i = 1$ en $j = 3$. Now observe that $((0,1),(0,3)) \in (X_1 \times \{1\}) \times (X_3 \times \{3\})$ and $((X_1 \times \{1\}) \times (X_3 \times \{3\})) \cap R = \emptyset$. Also notice that $(X_1 \times \{1\}) \times (X_3 \times \{3\})$ is an open set.

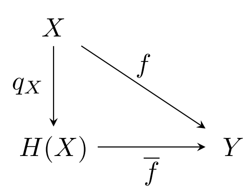
So every point in R^{c} has an open neighbourhood that is a subset of R^{c}, so R^{c} is open. Therefore, R is closed.

X/R is not Hausdorff. Let $q\colon X \to X/R$ be the quotient map. It will be shown that $q((0,1))$ and $q((0,3))$ cannot be separated by open sets, although $q((0,1)) \neq q((0,3))$. Let U, V be open neighbourhoods of $q((0,1))$ resp. $q((0,3))$. $F := q^{-1}(U)$ and $G := q^{-1}(V)$ are saturated open neighbourhoods of $(0,1)$ resp. $(0,3)$. So F contains $([0,\epsilon) \cap X_1) \times \{1\}$ for some $\epsilon > 0$ and G contains $([0,\delta) \cap X_3) \times \{3\}$ for some $\delta > 0$. Because they are saturated, F contains $((0, \min(\epsilon,\delta)) \cap \mathbb{Q}) \times \{2\}$ and G contains $((0, \min(\epsilon,\delta)) \setminus \mathbb{Q}) \times \{2\}$. We can see that any open set $S \subset (0,1)$ containing $((0, \min(\epsilon,\delta)) \cap \mathbb{Q})$ must contain some element of $((0, \min(\epsilon,\delta)) \setminus \mathbb{Q})$, so $F \cap G \neq \emptyset$. Therefore, $U \cap V \neq \emptyset$.

3 Construction of the Hausdorff Quotient

The construction in this section is inspired by [1] .

Theorem 3.1. *For every $X \in \mathrm{Top}$, there is a Hausdorff space $H(X)$ and a map $q_X\colon X \to H(X)$, such that for every $Y \in \mathrm{HausTop}$ and $f\colon X \to Y$, there is a unique $\overline{f}\colon H(X) \to Y$ such that the following diagram commutes:*



.

This $\overline{f}$ sends $q_X(x)$ to $f(x)$.

Proof. First recall the axiom schema of specification, which roughly states that one can obtain a set by selecting all elements from a certain set that satisfy some well-formed formula. Because of this axiom, $R_X = \{(a,b) \in X \times X : (\forall Y \in \mathrm{HausTop})(\forall f\colon X \to Y)(f(a) = f(b))\}$ is a set, even though one quantifies over HausTop, which is a class and not a set.[1] R_X is an equivalence relation: it

[1]It is possible to avoid such a quantification, but it would make the definition unnecessarily complex.

is clearly reflexive, symmetric and transitive. Let $H(X) = X/R_X$, with quotient map q_X. For any Hausdorff space Y and all maps $f\colon X \to Y$, we have $q_X(a) = q_X(b) \Rightarrow f(a) = f(b)$. By proposition 2.4, $\overline{f}\colon H(X) \to Y,\ q_X(x) \mapsto f(x)$ is the required unique map.

Now let $q_X(a), q_X(b) \in H(X)$ with $q_X(a) \neq q_X(b)$. Then there is a Hausdorff space Y and a map $f\colon X \to Y$ such that $f(a) \neq f(b)$. Choose open neighbourhoods $U \ni f(a)$ and $V \ni f(b)$ such that $U \cap V = \emptyset$. Then $\overline{f}^{-1}(U) \ni q_X(a)$ and $\overline{f}^{-1}(V) \ni q_X(b)$ are disjoint open sets, which proves that $H(X)$ is Hausdorff. □

Definition 3.2. Let X be a topological space. $H(X)$, as constructed in theorem 3.1, is called the *Hausdorff quotient* of X.

Proposition 3.3. *The equivalence relation R_X from theorem 3.1 is the smallest equivalence relation S on X such that X/S is Hausdorff. In other words,*

$$R_X = \bigcap \{S \subset X \times X : S \textit{ is an equivalence relation and } X/S \textit{ is Hausdorff}\}\,.$$

Proof. Let $P := \bigcap\{S \subset X \times X : \text{S is an equivalence relation and } X/S \text{ is Hausdorff}\}$. Because X/R_X is Hausdorff, $R_X \supset P$ holds. On the other hand, for all equivalence relations S on X such that X/S is Hausdorff, $X \to X/S$ factors through X/R_X and hence $R_X \subset S$. So $R_X \subset P$. □

4 Alternative Construction

In this section, an alternative construction of $H(X)$ will be presented. Again, $H(X)$ will be the quotient X/R_X, but R_X will be constructed in a way that can be thought of as being more 'constructive' or 'from below'. The idea of this construction is based on [2].

Definition 4.1. Let X be a topological space. Define the following relation on X:

$$\begin{aligned} a \sim b \Leftrightarrow\ & (\exists n \geq 1)(\exists x_1, ..., x_n \in X) \\ & [a = x_1 \wedge b = x_n \wedge (\forall i < n)((x_i, x_{i+1}) \in \overline{\text{diag}(X)})]\,. \end{aligned}$$

This relation will be denoted by

$$r_X = \{(a, b) \in X \times X : a \sim b\}\,.$$

Proposition 4.2. *The relation r_X is an equivalence relation.*

Proof. Let $a, b, c \in X$. Take $n = 1$, $x_1 = a$ in order to see that $a \sim a$. Now suppose $a \sim b$. Then there are $x_1, x_2, ..., x_n$ such that $a = x_1$, $b = x_n$ and $(\forall i < n)((x_i, x_{i+1}) \in \overline{\text{diag}(X)})$. For each $1 \leq i \leq n$, let $y_i = x_{n+1-i}$. Because of lemma 2.9, $(y_i, y_{i+1}) \in \overline{\text{diag}(X)}$ holds for each $1 \leq i < n$. Since $y_1 = x_n = b$ and $y_n = x_1 = a$, we obtain $b \sim a$.

Suppose $a \sim b$ and $b \sim c$. Then there are $x_1, ..., x_n$ such that $a = x_1$, $b = x_n$ and $(\forall i < n)((x_i, x_{i+1}) \in \overline{\text{diag}(X)})$ and there are $y_1, ..., y_m$ such that $b = y_1$, $c = y_m$ and $(\forall i < m)((y_i, y_{i+1}) \in \overline{\text{diag}(X)})$. Let $z_i = x_i$ if $i \leq n$ and $z_i = y_{i+1-n}$ if $n < i \leq n + m - 1$. It follows directly that $a = z_1$, $c = z_{n+m-1}$ and $(\forall i < n + m - 1)((z_i, z_{i+1}) \in \overline{\text{diag}(X)})$. So $a \sim c$. □

Remark 4.3. The relation r_X is the smallest equivalence relation containing $\overline{\text{diag}(X)}$, since any transitive relation containing $\overline{\text{diag}(X)}$ must also contain r_X.

Example 4.4. In general, $\overline{\text{diag}(X)}$ itself is not transitive. Let $X = \{1, 2, 3, 4, 5\}$ and define a topology on X by the basis $\{\{1, 2\}, \{2\}, \{2, 3, 4\}, \{4\}, \{4, 5\}\}$. Then 1 and 5 can be separated by open sets, so it follows that $(1, 5) \notin \overline{\text{diag}(X)}$ by lemma 2.9. However, $(1, 3) \in \overline{\text{diag}(X)}$ and $(3, 5) \in \overline{\text{diag}(X)}$ do hold.

Lemma 4.5. *A topological space X is Hausdorff if and only if r_X is the trivial equivalence relation.*

Proof. By definition, r_X is trivial if and only if $\overline{\text{diag}(X)} = \text{diag}(X)$, so by corollary 2.10 r_X is trivial if and only if X is Hausdorff. □

Proposition 4.6. *For any topological space X, the inclusion $r_X \subset R_X$ holds.*

Proof. R_X is reflexive, so $\text{diag}(X) \subset R_X$. R_X is closed by lemma 2.11, so $\overline{\text{diag}(X)} \subset R_X$. Because of remark 4.3, the inclusion follows. □

Example 4.7. Let X be an infinite topological space with the cofinite topology, i.e. $U \subset X$ is open if and only if $U = \emptyset$ or U^{c} is finite. Then there are no points that can be separated by open sets, so $r_X = X \times X$. Since $r_X \subset R_X$, this means that $R_X = X \times X$ and $H(X)$ consists of only one point.

Definition 4.8. Let X be a topological space. For each ordinal number α, define a topological space $h^\alpha(X)$, an equivalence relation r_X^α on X and a map $q_X^\alpha \colon X \to h^\alpha(X)$ in the following way: when r_X^α is defined, we let $h^\alpha(X) = X/r_X^\alpha$ with quotient map $q_X^\alpha \colon X \to h^\alpha(X)$. With this convention, define r_X^α recursively:

$$\begin{aligned} r_X^0 &= \text{diag}(X) & \\ r_X^{\alpha+1} &= \{(a, b) \in X \times X : (q_X^\alpha(a), q_X^\alpha(b)) \in r_{h^\alpha(X)}\} & \text{for all } \alpha \in \text{ON} \\ r_X^\alpha &= \textstyle\bigcup_{\beta<\alpha} r_X^\beta & \text{if } \alpha \text{ is a limit}. \end{aligned}$$

Here $r_{h^\alpha(X)}$ is the equivalence relation as defined in definition 4.1. Notice that $r_X^\alpha \subset r_X^\beta$ holds for any two ordinals α, β with $\alpha < \beta$, provided that all r_X^γ with $\gamma < \beta$ are well-defined equivalence relations. So inductively it follows that r_X^β is an equivalence relation for any limit β, since it is a union of an ascending chain of equivalence relations. Therefore, all r_X^α are well-defined equivalence relations.

Observation 4.9. Since r_X^0 is trivial, $h^0(X) = X$ and hence $h^1(X) = X/r_X$. For any two ordinals α, β with $\alpha < \beta$, $r_X^\alpha \subset r_X^\beta$ holds, so there is a unique map $f_{\alpha,\beta} \colon h^\alpha(X) \to h^\beta(X)$ such that $q_X^\beta = f_{\alpha,\beta} \circ q_X^\alpha$ (proposition 2.4). Furthermore, because of proposition 2.6, $q^1_{h^\alpha(X)} \circ q_X^\alpha$ is a quotient map, so corollary 2.5 tells us that there is a (unique) homeomorphism $g \colon h^{\alpha+1}(X) \to h^1(h^\alpha(X))$ such that $g \circ q_X^{\alpha+1} = q^1_{h^\alpha(X)} \circ q_X^\alpha$. This g sends $q_X^{\alpha+1}(x)$ to $(q^1_{h^\alpha(X)} \circ q_X^\alpha)(x)$.

Proposition 4.10. *Let X be a topological space. For any $\alpha \in \mathrm{ON}$, $r_X^\alpha \subset R_X$.*

Proof. By induction: for $\alpha = 0$ it is clear. Now let $\alpha+1$ be a successor such that $r_X^\alpha \subset R_X$. The map $X \to H(X)$ factors through $h^\alpha(X)$ by proposition 2.4. The resulting map $h^\alpha(X) \to H(X)$ factors through $H(h^\alpha(X))$ and since $r_{h^\alpha(X)} \subset R_{h^\alpha(X)}$, $h^\alpha(X) \to H(X)$ also factors through $h^\alpha(X)/r_{h^\alpha(X)} \cong h^{\alpha+1}(X) = X/r_X^{\alpha+1}$. Thus $r_X^{\alpha+1} \subset R_X$ holds.

Let α be a limit ordinal and assume that $r_X^\beta \subset R_X$ for all $\beta < \alpha$. Then $r_X^\alpha = \bigcup_{\beta<\alpha} r^\beta \subset R_X$. $\square$

Lemma 4.11. *There is some ordinal α such that $r_X^\alpha = r_X^{\alpha+1}$.*

Proof. All r_X^α are elements of $\mathcal{P}(X \times X)$. So $|\mathcal{P}(X \times X)| \geq |\{r_X^\alpha : \alpha < \beta\}|$ holds for any $\beta \in \mathrm{ON}$. Let $\beta \in \mathrm{ON}$ such that $|\beta| > |\mathcal{P}(X \times X)|$. Because $|\mathcal{P}(X \times X)| \geq |\{r_X^\alpha : \alpha < \beta\}|$, there must be some $\alpha, \gamma < \beta$ with $\alpha < \gamma$ such that $r_X^\alpha = r_X^\gamma$. Since $r_X^\alpha \subset r_X^{\alpha+1} \subset r_X^\gamma$, this yields $r_X^\alpha = r_X^{\alpha+1}$. $\square$

Construction 4.12. Let α be the smallest ordinal such that $r_X^\alpha = r_X^{\alpha+1}$. By definition of $r_X^{\alpha+1}$, $r_{h^\alpha(X)}$ is trivial. Hence $h^\alpha(X)$ is Hausdorff by lemma 4.5. By proposition 4.10, $r_X^\alpha \subset R_X$, so because of proposition 3.3, $R_X = r_X^\alpha$. So $H(X) = h^\alpha(X)$.

The following lemma might be useful for explicitly computing Hausdorff quotients.

Lemma 4.13. *Let X be a topological space such that all equivalence classes of r_X are finite and there are are only finitely many equivalence classes with more than 1 element. Then $H(X) = h^1(X)$.*

Proof. Let $a, b \in X$ such that $q_X^1(a) \neq q_X^1(b)$. For any $x \in X$, let $\overline{x}$ be the equivalence class of x with respect to r_X. Let Y be the union of all equivalence classes $\overline{x}$ with more than 1 element such that $\{a, b\} \cap \overline{x} = \emptyset$. For each $y \in Y$ and each $x \in \overline{a}$, there is an open neighbourhood $U_{x,y}$ of x such that $y \notin U_{x,y}$, because $(x, y) \notin \overline{\mathrm{diag}(X)}$. Now $U := \bigcup_{x\in\overline{a}} \bigcap_{y\in Y} U_{x,y}$ is an open neighbourhood of $\overline{a}$ such that $Y \cap U = \emptyset$. Similarly, there is an open neighbourhood V of B such that $Y \cap V = \emptyset$.

Each pair of elements a', b' with $a' \in \overline{a}$ and $b' \in \overline{b}$ can be separated by open neighbourhoods $U_{a',b'}$ and $V_{a',b'}$, because $(a', b') \notin \overline{\mathrm{diag}(X)}$. Therefore, $\overline{a}$ and $\overline{b}$ can be separated by open neighbourhoods $S := \bigcap_{b'\in\overline{b}} \bigcup_{a'\in\overline{a}} U_{a',b'}$ and

$T := \bigcup_{b' \in \bar{b}} \bigcap_{a' \in \overline{a}} V_{a',b'}$. Now a and b are separated by the open saturated neighbourhoods $U \cap S$ and $V \cap T$. Therefore, $q_1(a)$ and $q_1(b)$ are separated by the open neighbourhoods $q_1(U \cap S)$ and $q_1(V \cap T)$. Hence $h^1(X)$ is Hausdorff and $H(X) = h^1(X)$. □

Corollary 4.14. *If X is a finite topological space, then $H(X) = h^1(X)$.*

Example 4.15. Let $X = (\mathbb{R} \times \{0,1\})/{\sim}$, where $\sim$ is an equivalence relation on $\mathbb{R} \times \{0,1\}$ such that $(x,0) \sim (x,1)$ for all $x \in \mathbb{R} \setminus \{0\}$. Then X can be considered as the real line with a double origin and $\{(x,0),(x,1)\}$ is the only equivalence class of r_X that has more than one element. So $H(X) = h^1(X) \cong \mathbb{R}$.

Example 4.16. In general, $H(X) = h^1(X)$ does not hold. Let $X = \mathbb{N}_{\geq 1} \cup \{\infty\}$ and let the topology on X be given by a basis that consists of all sets of the form $\{2n-1\}$, $[2n-1, 2n+1]$, or $[2n-1, \infty]$, with $n \in \mathbb{N}_{\geq 1}$. Notice that the intersection of two basic sets is either empty or again a basic set, so this is indeed a basis for a topology on X. Since $(n, n+1) \in \overline{\operatorname{diag}(X)}$ and $(n, \infty) \notin \overline{\operatorname{diag}(X)}$ for each $n \geq 1$, the equivalence classes of r_X are $\mathbb{N}_{\geq 1}$ and $\{\infty\}$. However, the latter is not open in X, so 1 and ∞ cannot be separated by saturated open neighbourhoods (with respect to q^1_X). Therefore, $q^1_X(1)$ and $q^1_X(\infty)$ cannot be separated by open sets, whereas $q^1_X(1) \neq q^1_X(\infty)$. Hence $h^1(X)$ is not Hausdorff and $h^1(X) \neq H(X)$. (In this case, $h^2(X) = H(X)$ does hold since $h^1(X)$ is finite.)

5 The Hausdorff Functor

In this section, the general construction of $H(X)$ from a topological space X will be extended to a functor $H\colon \text{Top} \to \text{HausTop}$.

Construction 5.1. The functor $H\colon \text{Top} \to \text{HausTop}$ is defined as follows: for any $X \in \text{HausTop}$, $H(X)$ is the Hausdorff quotient as defined in definition 3.2. For any $f\colon X \to Y$ with $X, Y \in \text{Top}$, we let $H(f)$ be the unique map $H(X) \to H(Y)$ such that $H(f) \circ q_X = q_Y \circ f$. This map exists and it is unique by theorem 3.1, since $q_Y \circ f\colon X \to H(Y)$ is a map from X to a Hausdorff space.

It needs to be checked that this H is indeed a functor. It is clear that it sends identity morphisms to identity morphisms: $\text{id}_{H(X)}$ is the unique map $h\colon H(X) \to H(X)$ such that $h \circ q_X = q_X \circ \text{id}_X$ for all $X \in \text{Top}$. So $H(\text{id}_x) = \text{id}_{H(X)}$. Composition is also preserved by H: let $X, Y, Z \in \text{Top}$ and let $f\colon X \to Y$, $g\colon Y \to Z$ be maps. Then $(H(g) \circ H(f)) \circ q_X = H(g) \circ (q_Y \circ f) = q_z \circ (g \circ f)$. Hence, $H(g) \circ H(f)$ is the unique map $h\colon H(X) \to H(Y)$ such that $h \circ q_X = q_z \circ (g \circ f)$. So $H(g) \circ H(f) = H(g \circ f)$.

Definition 5.2. The functor H from construction 5.1 is called the *Hausdorff Functor*.

6 The Hausdorff Quotient and Products

The following main question that will be answered in this thesis is whether the Hausdorff Functor preserves homotopy. This section provides the theorems required for proving that this is true.

Theorem 6.1. *Let X, Y be topological spaces. There is a continuous bijection*

$$H(X \times Y) \to H(X) \times H(Y),\ q_{X\times Y}((x,y)) \mapsto (q_X(x), q_Y(y))\,.$$

Proof. Consider the following maps:

$$q_X \circ \pi_1 \colon X \times Y \to H(X),\ (x,y) \mapsto q_X(x)$$
$$q_Y \circ \pi_2 \colon X \times Y \to H(Y),\ (x,y) \mapsto q_Y(y)\,.$$

By the universal property of the product $H(X) \times H(Y)$, this induces a map $N\colon X \times Y \to H(X) \times H(Y),\ (x,y) \mapsto (q_X(x), q_Y(y))$. Since $H(X) \times H(Y)$ is Hausdorff as a product of two Hausdorff spaces, this map factors through $H(X \times Y)$, so there is a map $\overline{N}\colon H(X \times Y) \to H(X) \times H(Y)$ that sends $q_{X\times Y}((x,y))$ to $(q_X(x), q_Y(y))$. Because N is surjective, $\overline{N}$ is also surjective.

Now let $x, x' \in X$, $y, y' \in Y$ such that $q_X(x) = q_X(x')$ and $q_Y(y) = q_Y(y')$ and let $f\colon X \times Y \to Z$ with Z Hausdorff. Consider these maps:

$$g\colon X \to X \times Y,\ a \mapsto (a, y)$$
$$h\colon Y \to X \times Y,\ b \mapsto (x', b)\,.$$

It follows that:

$$f(x,y) = (f \circ g)(x) = (f \circ g)(x') = f(x', y) = (f \circ h)(y) = (f \circ h)(y') = f(x', y')\,.$$

So $q_{X\times Y}((x,y)) = q_{X\times Y}((x',y'))$. Therefore $\overline{N}$ is injective and hence a bijection. □

Corollary 6.2. *Let X, Y be topological spaces such that $H(X \times Y)$ is compact. Then $\overline{N}\colon H(X \times Y) \to H(X) \times H(Y),\ q_{X\times Y}((x,y)) \mapsto (q_X(x), q_Y(y))$ is a homeomorphism.*

Proof. $\overline{N}$ is a continuous bijection from a compact space to a Hausdorff space. Hence it is a homeomorphism. □

Corollary 6.3. *Let X, Y be compact topological spaces. Then $\overline{N}\colon H(X \times Y) \to H(X) \times H(Y),\ q_{X\times Y}((x,y)) \mapsto (q_X(x), q_Y(y))$ is a homeomorphism.*

Proof. $H(X \times Y) = q_{X\times Y}(X \times Y)$ is compact, because it is a continuous image of the compact set $X \times Y$. Therefore, by corollary 6.2, $\overline{N}$ is a homeomorphism. □

The following lemma can be found in a more general form in [6].

Lemma 6.4. *Let $f\colon X \to Y$ be a quotient map of topological spaces and let $I = [0,1]$ be the unit interval. Then $F\colon X \times I \to Y \times I, (x,t) \mapsto (f(x),t)$ is a quotient map.*

Proof. Let $U \subset X \times I$ be a saturated open set and let $(x,t) \in U$. U is open, so there is a basic open neighbourhood $V \times W \subset U$ of (x,t). But there is also a closed interval $W' \subset W$ that is a neighbourhood of t. Define the set $V' := \{v \in X : \{v\} \times W' \subset U\}$.

Claim 1: V' is saturated (with respect to f). Let $a \in V'$ and $b \in X$ with $f(a) = f(b)$. For any $w \in W'$, we have $F(b,w) = F(a,w)$, so $(b,w) \in U$ because U is saturated. Therefore we get $\{b\} \times W' \subset U$ and hence $b \in V'$.

Claim 2: V' is open. Let $a \in V'$. Because $\{a\} \times W' \subset U$, there is an open neighbourhood $A_w \times B_w$ of (a,w) for each $w \in W'$. The set of all B_w now covers W'. W' is compact, so it has a finite subcover $\{B_j : j \in J\}$ for some finite $J \subset W'$. We see that $(\bigcap_{j\in J} A_j) \times W' \subset (\bigcap_{j\in J} A_j) \times (\bigcup_{j\in J} B_j) \subset U$ holds. So $v \in V'$ holds for all $v \in \bigcap_{j\in J} A_j$. Hence a has an open neighbourhood $\bigcap_{j\in J} A_j \subset V'$, so V' is open.

Now observe that $V' \times W'^{\circ} \subset U$ is a saturated open neighbourhood of (x,t). Thus U is a union of basic open sets: $U = \bigcup_{u\in U}(A_u \times B_u)$, where $A_u \times B_u$ is an open neighbourhood of u and all A_u are saturated. Therefore $F(U) = F(\bigcup_{u\in U}(A_u \times B_u)) = \bigcup_{u\in U}(f(A_u) \times B_u)$, which is open. So F sends saturated open sets to open sets and is hence a quotient map. □

Theorem 6.5. *Let X be a topological space, let $I = [0,1]$ be the unit interval. The canonical map $C\colon H(X \times I) \to H(X) \times I,\ q_{X\times I}((x,y)) \mapsto (q_X(x), y)$ is a homeomorphism.*

Proof. Since I is Hausdorff, $H(I) = I$ holds. So by theorem 6.1, C is a well-defined bijective (continuous) map. Because of lemma 6.4, $X \times I \to H(X) \times I$, $(x,t) \mapsto (q_X(x), t)$ is a quotient map. By corollary 2.5, C is a homeomorphism. □

7 Preservation of Homotopy

In this final section, it will be proved that the Hausdorff Functor preserves homotopy. This can be stated otherwise by introducing the homotopy category and saying that H induces a functor in this category.

The unit interval $[0,1]$ will be denoted by I.

Definition 7.1. Two maps $f, g\colon X \to Y$ of topological spaces are called *homotopic* if there is a map $F\colon X \times I \to Y$ such that $F((x,0)) = f(x)$ and $F((x,1)) = g(x)$ for all $x \in X$. In this case, F is called a *homotopy* between f and g and the fact that f and g are homotopic will be denoted by $f \simeq g$.

A proof for the following well-known fact can be found in [5].

Fact 7.2. For any $X, Y \in \mathrm{Top}$, being homotopic is an equivalence relation on the set of continuous functions $X \to Y$.

Lemma 7.3. *Let $f, f' \colon X \to Y$ and $g, g' \colon Y \to Z$ be maps between topological spaces and assume $f \simeq f'$ and $g \simeq g'$. Then $g \circ f \simeq g' \circ f'$.*

Proof. Because $f \simeq f'$ and $g \simeq g'$, there are homotopies $F \colon X \times I \to Y$ and $G \colon Y \times I \to Z$ such that $F((x,0)) = f(x)$, $F((x,1)) = f'(x)$, $G((y,0)) = g(y)$ and $G((y,1)) = g'(y)$ for all $x \in X$ and $y \in Y$. Define $E \colon X \times I \to Z$, $(x,t) \mapsto G(F((x,t),t)$. This E is clearly continuous and we have $E((x,0)) = (g \circ f)(x)$ and $E((x,1)) = (g' \circ f')(x)$ for all $x \in X$. So $g \circ f \simeq g' \circ f'$. $\square$

Definition 7.4. The *homotopy category* hTop is a category whose objects are those from Top and whose morphisms are the homotopy classes of the morphisms from Top. Composition of morphisms is done by composing representatives and then taking the homotopy class of this composition. This is well-defined by lemma 7.3. The identity morphisms are the homotopy classes of the identity morphisms in Top.

Proposition 7.5. *The functor H induces a functor in the homotopy category. That is, whenever $f, g \colon X \to Y$ are homotopic maps of topological spaces, then $H(f)$ and $H(g)$ are also homotopic.*

Proof. Let $F \colon X \times I \to Y$ be a homotopy between f and g. It easily follows that $H(F) \colon H(X \times I) \to H(Y)$ is a homotopy between $H(f)$ and $H(g)$, after identifying $H(X \times I)$ with $H(X) \times I$.

Now let $[f]$ denote the homotopy class of any morphism f in Top. Since $[H(f)] = [H(g)]$ whenever $f \simeq g$, we can define a functor H_h in the homotopy category by $H_h(X) := H(X)$ for all $X \in \mathrm{Top}$ and $H_h([f]) := [H(f)]$ for all morphisms f in Top. $\square$

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