The Torelli theorem

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1. Introduction

Let X be a compact connected complex manifold of dimension d. The cup product

$$\mathrm{H}^d(X,\mathbb{Z}) \times \mathrm{H}^d(X,\mathbb{Z}) \longrightarrow \mathrm{H}^{2d}(X,\mathbb{Z}) = \mathbb{Z}$$

is a bilinear form, called the *intersection pairing*. It is symmetric for d even and anti-symmetric for d odd. It is non-degenerate in the sense that $\mathrm{H}^d(X,\mathbb{Z}) \to \mathrm{H}^d(X,\mathbb{Z})^\vee$ is injective. In the case of a surface the intersection pairing is compatible with the intersection pairing already defined on $\mathrm{NS}(X) \subset \mathrm{H}^2(X,\mathbb{Z})$.

Let A and A' be free $\mathbb{Z}$-modules of finite rank, endowed with Hodge structures and bilinear forms. A *Hodge isometry* $A \to A'$ is an isomorphism that respects both the Hodge structures and the bilinear forms.

Theorem 1.1 (Torelli). *Two complex K3 surfaces X and X' are isomorphic if and only if there is a Hodge isometry $\mathrm{H}^2(X,\mathbb{Z}) \cong \mathrm{H}^2(X',\mathbb{Z})$.* ⧫

Remark 1.2. This 'Torelli theorem' is not due to Torelli, but to Shapiro–Shafarevich (algebraic case) and Burns–Rapoport (analytic case). It is named for its analogy to the original Torelli theorem for curves: two complex curves X and X' are isomorphic if and only if their Jacobians $\mathrm{Jac}\, X$ and $\mathrm{Jac}\, X'$ are isomorphic as polarized abelian varieties.

There is a classical correspondence between polarized abelian varieties and polarized Hodge structures of weight one. In this language, the theorem reads: two complex curves X and X' are isomorphic if and only if there is a Hodge isometry $\mathrm{H}^1(X,\mathbb{Z}) \cong \mathrm{H}^1(X',\mathbb{Z})$. The analogy with the Torelli theorem for K3 surfaces shows clearly. ⧫

Remark 1.3. Traditionally, the Torelli theorem for K3 surfaces is proven as follows. First one proves the theorem for Kummer surfaces, using their description via complex tori. The period points of marked Kummer surfaces are dense in the period domain Ω (see below). In the general case one needs to show that certain sequences of isomorphisms of marked K3 surfaces 'converge'. Details can be found in [1]. We follow the different approach in [3]. ⧫

2. Complex K3 surfaces

The Torelli theorem is really a statement in complex geometry. The proof requires that we consider more than just algebraic K3 surfaces.

Definition 2.1. A *complex K3 surface* is a compact connected complex manifold X of dimension 2 with $\omega_X \cong \mathcal{O}_X$ and $\mathrm{H}^1(X,\mathcal{O}_X) = 0$. ⧫

From here on, 'K3 surface' shall mean a complex K3 surface in this sense. The main example is X^{an}, for X an algebraic K3 surface over $\mathbb{C}$. Not all K3 surfaces are algebraic, i.e. of this form, but we do not wish to emphasize this. Most properties of algebraic K3 surfaces carry over to the complex case, and we shall use them freely.

3. Periods

Remark 3.1. The theory of periods originated in elliptic curves. A *marking* of a complex elliptic curve E is a choice of basis α, β for $\mathrm{H}_1(E, \mathbb{Z})$ with $(\alpha, \beta) = 1$. Let ω be a canonical differential. Then

$$\tau_1 = \int_\alpha \omega, \qquad \tau_2 = \int_\beta \omega$$

span a lattice $\langle \tau_1, \tau_2 \rangle \subset \mathbb{C}$, and $E \cong \mathbb{C}/\langle \tau_1, \tau_2 \rangle$. In fact τ_2/τ_1 lies in the upper half plane $\mathbb{H}$, the *period domain*. A family $\mathcal{E} \to S$ of marked complex elliptic curves defines a holomorphic *period map* $S \to \mathbb{H}$. ♦

Let Λ be the K3 lattice $E_8(-1)^{\oplus 2} \oplus U^{\oplus 3}$.

Lemma 3.2. *Let X be a K3 surface.*
- $\mathrm{H}^2(X, \mathbb{Z}) \cong \Lambda$.
- *There is a canonical Hodge decomposition* $\mathrm{H}^2(X, \mathbb{C}) = \mathrm{H}^{2,0}(X) \oplus \mathrm{H}^{1,1}(X) \oplus \mathrm{H}^{0,2}(X)$.
- $\dim_{\mathbb{C}} \mathrm{H}^{2,0}(X) = 1$.
- *Any non-zero $\alpha \in \mathrm{H}^{2,0}(X)$ satisfies $(\alpha, \alpha) = 0$, $(\alpha, \bar{\alpha}) > 0$, and $\alpha \perp \mathrm{H}^{1,1}(X)$.*

Proof. The first three properties we have already seen in the algebraic case. For complex K3 surfaces the existence of a Hodge decomposition relies on the fact that K3 surfaces are Kähler. This is a deep theorem [1, IV.3.1]. The last part is left as an exercise. ■

We study Hodge structures on Λ as in the lemma. Let

$$\Omega = \{x \in \mathbb{P}(\Lambda_{\mathbb{C}}) \colon (x, x) = 0, (x, \bar{x}) > 0\}$$

be the *period domain* of Λ. It is an open subset of the smooth quadric defined by $(x, x) = 0$. In particular Ω is naturally a complex manifold of dimension 20.

Lemma 3.3. *There is a bijection*

$$\left\{ \begin{array}{c} \textit{Hodge structures } \Lambda_{\mathbb{C}} = \Lambda^{2,0} \oplus \Lambda^{1,1} \oplus \Lambda^{0,2} \\ \textit{with } \dim_{\mathbb{C}} \Lambda^{2,0} = 1, \textit{ and } (\alpha, \alpha) = 0, \\ (\alpha, \bar{\alpha}) > 0 \textit{ and } \alpha \perp \Lambda^{1,1} \textit{ for non-zero } \alpha \in \Lambda^{2,0} \end{array} \right\} \quad \longleftrightarrow \quad \Omega.$$

Proof. Given a Hodge structure $\Lambda_{\mathbb{C}} = \Lambda^{2,0} \oplus \Lambda^{1,1} \oplus \Lambda^{0,2}$ as in the statement, $\Lambda^{2,0}$ defines a point in Ω. Conversely, a point $x \in \Omega$ defines a line $\Lambda^{2,0} \subset \Lambda_{\mathbb{C}}$. Let $\Lambda^{1,1}$ be the complexification of $\langle \operatorname{Re} \alpha, \operatorname{Im} \alpha \rangle^{\perp} \subset \Lambda_{\mathbb{R}}$ for any non-zero $\alpha \in \Lambda^{2,0}$. ■

A *marking* of a K3 surface X is a choice of isometry $\varphi \colon \mathrm{H}^2(X, \mathbb{Z}) \to \Lambda$. Combining lemmas 3.2 and 3.3 yields the *period map*

$$\tau \colon \{\text{marked K3 surfaces}/\cong\} \longrightarrow \Omega.$$

In the next section we make this set of marked K3 surfaces into an analytic space such that τ becomes holomorphic.

Let $p \colon \mathcal{X} \to S$ be a proper smooth family of K3 surfaces. The sheaf $\mathrm{R}^2 p_* \underline{\mathbb{Z}}$ on S has stalk precisely $\mathrm{H}^2(\mathcal{X}_s, \mathbb{Z})$ at $s \in S$. A *marking* of the family is a choice of isomorphism $\varphi \colon \mathrm{R}^2 p_* \underline{\mathbb{Z}} \to \underline{\Lambda}$ that is fiberwise an isometry. Via τ we get a period map $S \to \Omega$.

Proposition 3.4. *The period map $S \to \Omega$ is holomorphic.*

Proof. See [3, 6.2.3]. ■

As a special case, suppose S is simply connected. Choose a distinguished point $0 \in S$ and a marking $\varphi_0 \colon \mathrm{H}^2(\mathcal{X}_0, \mathbb{Z}) \to \Lambda$. Then φ_0 extends uniquely to a marking $\mathrm{R}^2 p_* \underline{\mathbb{Z}} \to \underline{\Lambda}$. We again get a holomorphic period map $S \to \Omega$.

4. Deformation theory

Let X be a K3 surface. A *deformation* of X is a proper smooth family $\mathcal{X} \to S$ with a distinguished point $0 \in S$ and a given isomorphism $\mathcal{X}_0 \cong X$. In fact we are only interested in the germ of the family around 0. A deformation $\mathcal{X} \to S$ is *universal* if any other deformation $\mathcal{X}' \to S'$ is (on germs) the pullback of $\mathcal{X} \to S$ along a unique map $S' \to S$.

Theorem 4.1 (Kuranishi–Kodaira). *Any K3 surface X has a universal deformation $\mathcal{X} \to \mathrm{Def}(X)$. It is a universal deformation for each of its fibers. The deformation space $\mathrm{Def}(X)$ is (the germ of) a smooth complex manifold of dimension 20.*

Proof. See [2]. We just remark that the dimension 20 originates from isomorphisms $\mathrm{T}_0\mathrm{Def}(X) = \mathrm{H}^1(X, \mathcal{T}_X) \cong \mathrm{H}^1(X, \Omega^1_X)$. ∎

The deformation space $\mathrm{Def}(X)$ is simply connected, so given a marking of X we have a period map $\mathrm{Def}(X) \to \Omega$ as in proposition 3.4.

Theorem 4.2 (local Torelli). *Let (X, φ) be a marked K3 surface. The period map $\mathrm{Def}(X) \to \Omega$ is a local isomorphism on $\mathrm{Def}(X)$.*

Proof. Since $\mathcal{X} \to \mathrm{Def}(X)$ is a universal deformation for each of its fibers, we only have to verify that $\mathrm{Def}(X) \to \Omega$ is an isomorphism at the distinguished point $0 \in \mathrm{Def}(X)$. Both $\mathrm{Def}(X)$ and Ω have dimension 20, so it suffices to show that the tangent map at 0 is bijective.

We have already seen $\mathrm{T}_0\mathrm{Def}(X) = \mathrm{H}^1(X, \mathcal{T}_X)$. The point $\tau(X, \varphi) \in \Omega$ corresponds to a line $\Lambda^{2,0} \subset \Lambda_{\mathbb{C}}$, and we have

$$\mathrm{T}_{\tau(X,\varphi)}\Omega = \mathrm{Hom}(\Lambda^{2,0}, (\Lambda^{2,0})^{\perp}/\Lambda^{2,0}) = \mathrm{Hom}(\Lambda^{2,0}, \mathrm{H}^1(X, \Omega^1_X)) \cong \mathrm{H}^1(X, \Omega^1_X).$$

It follows from Griffiths transversality [3, 6.2.4] that the tangent map $\mathrm{H}^1(X, \mathcal{T}_X) \to \mathrm{H}^1(X, \Omega^1_X)$ is an isomorphism induced by a choice of non-zero $\alpha \in \Lambda^{2,0}$. ∎

Now we glue all deformation spaces together. Let $\mathcal{M}$ be the moduli functor of marked K3 surfaces, i.e. $\mathcal{M}(S)$ is the set of marked proper smooth families of K3 surfaces over S, up to isomorphism. It has a fine moduli space: there is an analytic space M and a natural bijection between $\mathcal{M}(S)$ and $M(S) = \mathrm{Hom}(S, M)$.

As a set, let $M = \{\text{marked K3 surfaces}/\cong\}$. We give it a complex structure as follows. Let (X, φ) be a marked K3 surface. The induced map $\mathrm{Def}(X) \to M$ is injective by the local Torelli theorem 4.2. Since the deformation $\mathcal{X} \to \mathrm{Def}(X)$ is universal for each of its fibers, the complex structures on $\mathrm{Def}(X)$ glue into a complex structure on M. By the same argument, the universal deformations glue together into a universal marked family $p\colon \mathcal{X} \to M$. Now M is a fine moduli space for $\mathcal{M}$. Applying once more proposition 3.4 we see that the period map $\tau\colon M \to \Omega$ is holomorphic.

Remark 4.3. The moduli space M is a smooth analytic space of dimension 20. It is an espace étalé over Ω.[1] However, it is not Hausdorff [1, VIII.12.2]. This turns out to be an issue in what follows, so we pass to a 'Hausdorffification'. ♦

Proposition 4.4. *There is a quotient $M \to \tilde{M}$ to a Hausdorff complex manifold $\tilde{M}$ that precisely identifies inseparable points, such that the period map $\tau\colon M \to \Omega$ factors as*

$$M \longrightarrow \tilde{M} \xrightarrow{\tilde{\tau}} \Omega.$$

Proof. See [4]. Note that the existence of such a Hausdorff quotient is not a general fact, and relies on the moduli interpretation of M. ∎

[1] Thanks to Bas Edixhoven for pointing this out.

5. Twistor lines

Let $W \subset \Lambda_{\mathbb{R}}$ be a *positive* 3-*space*, i.e. a 3-dimensional subspace on which the bilinear form is positive definite. Let

$$L_W = \mathbb{P}(W_{\mathbb{C}}) \cap \Omega$$

be the associated *twistor line*. It is a smooth quadric in $\mathbb{P}(W_{\mathbb{C}})$, hence isomorphic to the complex projective line. A twistor line is *generic* if $W^{\perp} \cap \Lambda = 0$.

Proposition 5.1. *For any two points $x, y \in \Omega$ there exists $x = x_0, \ldots, x_n = y$ and generic twistor lines $L_{W_1}, \ldots, L_{W_n}$ with $x_{i-1}, x_i \in L_{W_i}$ for $i = 1, \ldots, n$.*

Proof. For $x \in \Omega$ choose $\alpha \neq 0$ in the associated $\Lambda^{2,0} \subset \Lambda_{\mathbb{C}}$, and set $P(x) = \langle \operatorname{Re} \alpha, \operatorname{Im} \alpha \rangle$. It is an oriented positive plane in $\Lambda_{\mathbb{R}}$. In fact P is a bijection between Ω and the set of oriented positive planes in $\Lambda_{\mathbb{R}}$.

Since Ω is connected, it suffices to show that the equivalence classes of 'connected' points are open. Choose $x \in \Omega$ and write $P(x) = \langle a, b \rangle$. There exists $c \in \Lambda_{\mathbb{R}}$ such that the 3-space $W_1 = \langle a, b, c \rangle$ is positive; we may take $c^{\perp} \cap \Lambda = 0$. For $x' \in \Omega$ close enough to x, also the corresponding a', b' are close to a, b, and $W_2 = \langle a, b', c \rangle$ and $W_3 = \langle a', b', c \rangle$ are still positive 3-spaces. By choice of c, the twistor lines L_{W_1}, L_{W_2} and L_{W_3} are generic. Write $x_2 = P^{-1}\langle a, c \rangle$ and $x_3 = P^{-1}\langle b', c \rangle$. We have

$$x, x_2 \in L_{W_1}, \quad x_2, x_3 \in L_{W_2}, \quad x_3, x' \in L_{W_3}$$

and we are done. ■

We would like to lift twistor lines along the period map $\tau \colon M \to \Omega$. However, this is problematic by remark 4.3. Instead we use the Hausdorffification $\tilde{M}$ from proposition 4.4.

Theorem 5.2. *Let (X, φ) be a marked K3 surface and $L_W \subset \Omega$ a generic twistor line containing $\tau(X, \varphi)$. There is a unique commutative diagram*

$$\begin{tikzcd} & L_W \arrow[dl, "i"'] \arrow[d, hook] \\ \tilde{M} \arrow[r, "\tilde{\tau}"'] & \Omega \end{tikzcd}$$

with (X, φ) in the image of i.

Proof. The period map $\tau \colon M \to \Omega$ is a local isomorphism on M by the local Torelli theorem 4.2. The same is true for $\tilde{\tau} \colon \tilde{M} \to \Omega$. Therefore there is a small open neighborhood of $\tau(X, \varphi) \in L_W$ that lifts to $\tilde{M}$. Since $\tilde{M}$ is Hausdorff, this lift is unique.

The fact that this lift extends to all of L_W is a deep theorem relying on the existence of hyperkähler structures on K3 surfaces. We omit it. See [3, 7.3.9]. ■

Recall that the period map $\tau \colon M \to \Omega$ is a local isomorphism on M by the local Torelli theorem 4.2. Now we can say more.

Theorem 5.3. *The period map $\tilde{\tau} \colon \tilde{M} \to \Omega$ is a covering space. The period map $\tau \colon M \to \Omega$ is surjective and generically injective on each connected component of M.*

Proof. Let $M_0 \subseteq M$ be a connected component. We show that $\tau\colon M_0 \to \Omega$ is surjective. After proposition 5.1 it suffices to show that if $x, y \in \Omega$ lie on a generic twistor line L_W, then $x \in \tau(M_0)$ if and only if $y \in \tau(M_0)$. This follows from theorem 5.2, since $M \to \tilde{M}$ preserves connected components.

To see that $\tilde{\tau}$ is a covering space, we use the following criterion. Let $\pi\colon X \to Y$ be a local homeomorphism on X between Hausdorff topological manifolds. Then π is a covering space if and only if for each open ball $B \subseteq Y$ and each connected component $C \subseteq \pi^{-1}(\bar{B})$ we have $\pi(C) = \bar{B}$ In the case of $\tilde{\tau}$ this is a 'local' version of the surjectivity of the period map, which can be proven analogously. See [3, 7.4.3] or [4] for details.

Let $\tilde{M}_0$ be the connected component of $\tilde{M}$ corresponding to M_0. Since $\tilde{M}_0 \to \Omega$ is a connected covering space of a simply connected space, it is in fact an isomorphism. Now use that $M_0 \to \tilde{M}_0$ is generically injective. ■

6. Monodromy

Let (X, φ) be a marked K3 surface. A proper smooth family $\mathcal{X} \to S$ of K3 surfaces over a connected base S, with a distinguished point $0 \in S$ and a given isomorphism $\mathcal{X}_0 \cong X$, induces a monodromy representation

$$\pi_1(S, 0) \longrightarrow \mathrm{O}(\Lambda).$$

The *monodromy group* of X is the subgroup $\mathrm{Mon}(X) \subseteq \mathrm{O}(\Lambda)$ generated by the monodromy of all such families.

Remark 6.1. Up to conjugation, the subgroup $\mathrm{Mon}(X) \subseteq \mathrm{O}(\Lambda)$ is independent of (X, φ). This follows from the fact that any two K3 surfaces X, X' are deformation equivalent: there exists a proper smooth family $\mathcal{X} \to S$ of K3 surfaces over a connected base S, such that X and X' occur as fibers. See [1, VIII.8.6]. ♦

Proposition 6.2. $\mathrm{Mon}(X) = \mathrm{O}^+(\Lambda)$.

Proof. See [3, 7.5.5]. ■

Proposition 6.3. *The moduli space M has two connected components, interchanged by $-\mathrm{id} \in \mathrm{O}(\Lambda)$.*

Proof. As noted in remark 6.1 all K3 surfaces are deformation equivalent. So it suffices to prove that for any two markings φ, φ' on a K3 surface X, precisely one of (X, φ') and $(X, -\varphi')$ lies in the connected component of (X, φ).

Write $\psi = \varphi' \circ \varphi^{-1}$. The quotient $\mathrm{O}(\Lambda)/\mathrm{O}^+(\Lambda)$ is generated by $-\mathrm{id}$, so by proposition 6.2 $\pm\psi$ lies in $\mathrm{Mon}(X)$. Write $\pm\psi$ as $\psi_n \circ \ldots \circ \psi_1$, with ψ_i coming from the monodromy representation of a family $\mathcal{X}_i \to S_i$. These families yield a path

$$(X, \varphi) \rightsquigarrow (X, \psi_1 \circ \varphi) \rightsquigarrow \ldots \rightsquigarrow (X, \psi_n \circ \ldots \circ \psi_1 \circ \varphi) = (X, \pm\varphi')$$

in M, hence (X, φ) and $(X, \pm\varphi')$ are in the same connected component.

To see that there are really two components, we use theorem 5.3. The element $-\mathrm{id}$ acts nowhere trivially on M but trivially on Ω. To ensure that τ is generically injective on each connected component of M, there must be two. ■

The *Kähler cone* $\mathcal{K}_X$ of a K3 surface X is the connected component of

$$\left\{\alpha \in \mathrm{NS}(X)_{\mathbb{R}} \colon (\alpha, \alpha) > 0, (\alpha, D) > 0 \text{ for every effective divisor } D \text{ on } X\right\}$$

containing some Kähler class (hence all). In the algebraic case $\mathcal{K}_X$ coincides with the ample cone.

Theorem 6.4 (Torelli, v2). *Let X and X' be K3 surfaces and $\psi\colon \mathrm{H}^2(X',\mathbb{Z}) \to \mathrm{H}^2(X,\mathbb{Z})$ a Hodge isometry. Then $\psi(\mathcal{K}_{X'}) = \pm\mathcal{K}_X$, and there is a unique isomorphism $f\colon X \to X'$ inducing $\pm\psi$.*

Proof. Choose a marking φ on X and set $\varphi' = \psi \circ \varphi$. Then $\tau(X,\varphi) = \tau(X',\varphi') = \tau(X',-\varphi')$. There is a unique sign $\pm$ such that (X,φ) and $(X',\pm\varphi')$ are in the same connected component $M_0 \subset M$. By theorem 5.3, the map $M_0 \to \Omega$ is generically injective, so 'generically' we are done. If (X,φ) and $(X',\pm\varphi')$ are not isomorphic, then at least they are inseparable in M. A closer look at the map $M \to \tilde{M}$ shows that still $X \cong X'$.

For the existence of f inducing $\pm\psi$ we refer to [3, 7.5.3]. The unicity follows from the fact that $\mathrm{Aut}(X) \to \mathrm{O}(\Lambda)$ is injective. The statement about the Kähler cones is immediate. ∎

References

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