Descent of Curves

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Introduction

Descent is the act of building a 'global object' from a collection of 'local objects'. It is a generalization of the topological concept of gluing. For example, if S is a topological space covered by open subspaces $\{U_i\}_{i \in I}$, then a space X over S may be defined by a collection of spaces X_i over U_i that are compatible on intersections: the spaces X_i are glued into a space X. The same is true if 'topological space' is replaced by 'scheme'. We say that the schemes X_i over U_i descend to a scheme X over S. The process is called *topological* or *Zariski* descent of schemes.

What happens if we replace the open immersions $U_i \to S$ by more general morphisms? Substituting fiber products for intersections, we can still ask whether compatible schemes X_i over U_i descend to a scheme X over S. In this thesis we study the generalization from topological descent where the maps $U_i \to S$ are *étale morphisms*. Étale descent behaves more wildly than its topological variant. In some cases schemes descend, in others not.

As the title suggests, we consider étale descent of schemes that are *curves* over their base schemes. Here a 'curve' means a scheme X over S that is proper, smooth of relative dimension 1, and has geometrically connected fibers. The situation then is twofold. If all schemes X_i over U_i are curves of genus g, for some fixed non-negative integer g unequal to 1, then these curves descend to a uniquely unique scheme X over S, and X is a curve of genus g as well. On the other hand, if X_i is a curve of genus 1 over U_i for all $i \in I$, that conclusion no longer applies: examples are known of a scheme S, a jointly surjective family of étale morphisms $\{U_i \to S\}_{i \in I}$, and compatible curves X_i over U_i of genus 1, where the curves do not descend to a scheme over S. Precise statements are formulated in theorems I and II on page 7.

The positive result for genus other than 1 follows from classical facts on the geometry of curves. A counterexample in genus 1 was first provided in 1968 by Michel Raynaud [R]. In this thesis we present a new counterexample. Its construction is based upon Raynaud's original work, yet different and, hopefully, accessible to a larger audience. Additionally, the construction in this thesis is more explicit.

In chapter 1 we give a self-contained introduction into descent. The theory is developed in the general context of *sites*, as we believe this modern language benefits the intuition. Simultaneously we keep track of properties of the étale site in particular. The first chapter closes with the formulation of our main theorems.

Chapter 2 establishes the proof of theorem I, descent of curves in genus $g \neq 1$. It uses descent of quasi-coherent sheaves, and very ample line bundles on curves. At several technical points we give only a sketch of the proof.

The last two chapters, which can be read independently from chapter 2, treat descent in genus 1. In chapter 3 we discuss *torsors*. Again we work mostly on general sites. Torsors are key in our construction of non-descending curves of genus 1. The construction itself is detailed in chapter 4. At the end of that chapter we shortly discuss Raynaud's counterexample and its relation to the one presented here.

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Wouter Zomervrucht

Descent 1

1.1. Representable functors

We recall the notion of a representable functor.

Definition 1.1. Let $\mathcal{C}$ be a category and $X \in \mathcal{C}$ an object. The *covariant Yoneda functor* of X is the functor $\mathsf{h}^X : \mathcal{C} \to \mathsf{Set}$ that maps Y to $\mathsf{Hom}(X, Y)$. Dually, the *contravariant Yoneda functor* of X is the functor $\mathsf{h}_X : \mathcal{C}^{\mathsf{op}} \to \mathsf{Set}$, $Y \mapsto \mathsf{Hom}(Y, X)$.

Definition 1.2. Let $\mathcal{C}$ be a category. A functor $\mathcal{F} : \mathcal{C}^{\mathsf{op}} \to \mathsf{Set}$ is *representable* if it is naturally isomorphic to the contravariant Yoneda functor of some object $X \in \mathcal{C}$. In this case, we say that X *represents* $\mathcal{F}$.

Lemma 1.3 (Yoneda). *Let $\mathcal{C}$ be a category. The functors*

$$\begin{aligned} \mathcal{C}^{\mathsf{op}} &\to \mathsf{Func}(\mathcal{C}, \mathsf{Set}), \quad X \mapsto \mathsf{h}^X, \\ \mathcal{C} &\to \mathsf{Func}(\mathcal{C}^{\mathsf{op}}, \mathsf{Set}), \quad X \mapsto \mathsf{h}_X \end{aligned}$$

are fully faithful.

Proof. By duality it suffices to consider the functor $X \mapsto \mathsf{h}^X$.

Let X and Y be objects of $\mathcal{C}$ and $\alpha : \mathsf{h}^X \to \mathsf{h}^Y$ a natural transformation. For all $T \in \mathcal{C}$ we have a map $\alpha_T : \mathsf{Hom}(X, T) \to \mathsf{Hom}(Y, T)$. Let $f : Y \to X$ be the image of id_X under α_X. It is easily verified that α is induced by f, more precisely: we have $\alpha_T(g) = g \circ f$ for all $T \in \mathcal{C}$. Moreover, f is the unique morphism $Y \to X$ with this property. So $X \mapsto \mathsf{h}^X$ is fully faithful. ∎

As a corollary to Yoneda's lemma, a representable functor is represented by a uniquely unique object.

1.2. Sites

For our study of descent, we generalize the concept of an open covering of a topological space.

Definition 1.4. Let $\mathcal{C}$ be a category. A *Grothendieck topology* on $\mathcal{C}$ is a collection of *coverings*, each covering being a family of morphisms $\{U_i \to S\}_{i \in I}$, such that
- if $U \to S$ is an isomorphism, then $\{U \to S\}$ is a covering,
- if $\{U_i \to S\}_{i \in I}$ is a covering and for all $i \in I$ also $\{V_{ij} \to U_i\}_{j \in J_i}$ is a covering, then the composition $\{V_{ij} \to S\}_{i \in I, j \in J_i}$ is a covering, and
- if $\{U_i \to S\}_{i \in I}$ is a covering and $T \to S$ is a morphism, then the fiber products $U_i \times_S T$ exist and the base change $\{U_i \times_S T \to T\}_{i \in I}$ is a covering.

A *site* is a category endowed with a Grothendieck topology.

We give some examples. The first will serve as a model for the definition. Let X be a topological space, and $\mathsf{Open}\,X$ the category whose objects are the open subsets of X and whose morphisms are the inclusions. Fiber products in $\mathsf{Open}\,X$ are just intersections. As usual, let an open covering be a family $\{V_i \to U\}_{i\in I}$ where the open subsets V_i together cover U. The open coverings make $\mathsf{Open}\,X$ into a site.

For another example, let S be a scheme. Let $\mathcal{P}$ be a class of morphisms of schemes over S that contains all isomorphisms and is stable under composition and base change. Let a covering be a family $\{h_i\colon V_i \to U\}_{i\in I}$ where each h_i is in $\mathcal{P}$ and $\bigcup_{i\in I} \operatorname{im} h_i$ equals U. These coverings constitute a topology on Sch/S.

In particular, if $\mathcal{P}$ is the class of open immersions, we get the *Zariski topology* on Sch/S and the *(large) Zariski site* $(\mathsf{Sch}/S)_{\mathsf{Zar}}$. Its coverings are the jointly surjective families of open immersions. A more interesting site arises if we let $\mathcal{P}$ be the class of étale morphisms. This topology on Sch/S is the *étale topology*, and the resulting site the *(large) étale site* on S, denoted $(\mathsf{Sch}/S)_{\text{ét}}$. The étale topology is finer than the Zariski topology: every Zariski covering is étale, but not every étale covering is Zariski. Most of the work in this thesis concerns the étale site.

There is a generalized concept of sheaves as well.

Definition 1.5. Let $\mathcal{C}$ be a site. A *sheaf of sets* on $\mathcal{C}$ is a functor $\mathcal{F}\colon \mathcal{C}^{\mathsf{op}} \to \mathsf{Set}$ such that for every covering $\{U_i \to S\}_{i\in I}$ in $\mathcal{C}$ the diagram

$$\mathcal{F}(S) \longrightarrow \prod_{i\in I} \mathcal{F}(U_i) \rightrightarrows \prod_{i,j\in I} \mathcal{F}(U_i \times_S U_j)$$

is an equalizer. Here the parallel arrows are induced by the projections $U_i \times_S U_j \to U_i$ and $U_i \times_S U_j \to U_j$, respectively.

Definition 1.6. Let $\mathcal{C}$ be a site and $\mathcal{D}$ a category. A $\mathcal{D}$-valued *presheaf* on $\mathcal{C}$ is a functor $\mathcal{C}^{\mathsf{op}} \to \mathcal{D}$. A $\mathcal{D}$-valued *sheaf* on $\mathcal{C}$ is a presheaf $\mathcal{F}\colon \mathcal{C}^{\mathsf{op}} \to \mathcal{D}$ such that $\mathsf{h}^X\mathcal{F}$ is a sheaf of sets for all objects $X \in \mathcal{D}$. A *morphism of (pre)sheaves* is a natural transformation of functors.

Consider the site $\mathsf{Open}\,X$, where X is a topological space. A sheaf on $\mathsf{Open}\,X$ is just a sheaf on X in the usual sense. As before, this example serves as a model for the definition.

In general, the category of $\mathcal{D}$-valued presheaves on a site $\mathcal{C}$ is written $\mathsf{PSh}_{\mathcal{D}}\,\mathcal{C}$, and its full subcategory of sheaves $\mathsf{Sh}_{\mathcal{D}}\,\mathcal{C}$. The index $\mathcal{D}$ is omitted in the case $\mathcal{D} = \mathsf{Set}$. Now suppose that $\mathcal{D}$ is a category with products. Since h^X commutes with limits, Yoneda's lemma 1.3 shows that a presheaf $\mathcal{F}\colon \mathcal{C}^{\mathsf{op}} \to \mathcal{D}$ is a sheaf if and only if for every covering $\{U_i \to S\}_{i\in I}$ in $\mathcal{C}$ the diagram

$$\mathcal{F}(S) \longrightarrow \prod_{i\in I} \mathcal{F}(U_i) \rightrightarrows \prod_{i,j\in I} \mathcal{F}(U_i \times_S U_j)$$

is an equalizer in $\mathcal{D}$. In particular, sheaves of sets coincide with Set-valued sheaves.

Let $S \in \mathcal{C}$ be an object. The topology on $\mathcal{C}$ induces a topology on the category $\mathcal{C}/S$ of objects over S: its coverings are the families $\{V_i \to U\}_{i\in I}$ that are coverings in $\mathcal{C}$ when we forget the structure maps to S. If $f\colon T \to S$ is a morphism, there are functors

$$f_*\colon \mathcal{C}/T \to \mathcal{C}/S, \quad g \mapsto f \circ g$$

and

$$f^{-1}\colon\ \mathsf{PSh}_{\mathcal{D}}\,\mathcal{C}/S \to \mathsf{PSh}_{\mathcal{D}}\,\mathcal{C}/T, \quad \mathcal{F} \mapsto \mathcal{F} \circ f_*^{\mathsf{op}}.$$

In other words, we can *pullback* presheaves on $\mathcal{C}/S$ to presheaves on $\mathcal{C}/T$. The pullback $f^{-1}\mathcal{F}$ of a presheaf $\mathcal{F}$ on $\mathcal{C}/S$ to $\mathcal{C}/T$ is denoted $\mathcal{F}|_T$, and similar for morphisms. The pullback of a sheaf is necessarily a sheaf.

Suppose that $\mathcal{C}$ has fiber products. Pullback of set-valued presheaves then is related to the base change functor $f^*\colon \mathcal{C}/S \to \mathcal{C}/T$. Let $\mathcal{C}/S \to \mathsf{PSh}\,\mathcal{C}/S$ and $\mathcal{C}/T \to \mathsf{PSh}\,\mathcal{C}/T$ be the covariant Yoneda embeddings from lemma 1.3. These constitute the vertical arrows of a diagram

$$\begin{tikzcd} \mathcal{C}/S \arrow[r, "f^*"] \arrow[d] & \mathcal{C}/T \arrow[d] \\ \mathsf{PSh}\,\mathcal{C}/S \arrow[r, "f^{-1}"] & \mathsf{PSh}\,\mathcal{C}/T \end{tikzcd}$$

in which the two composite diagonal functors are isomorphic. Consequently, if a (pre)sheaf $\mathcal{F}$ on $\mathcal{C}/S$ is representable by an object X over S, then the pullback $\mathcal{F}|_T$ is a (pre)sheaf on $\mathcal{C}/T$ that is representable by $X \times_S T$ over T.

We return to the case of schemes. If S is a scheme, then a sheaf on $(\mathsf{Sch}/S)_{\text{ét}}$ is also called an étale sheaf on S. As an important example, we prove that representable functors on Sch/S are étale sheaves. This fact is due to Alexander Grothendieck [SGA1, Théorème VIII.5.2].

Lemma 1.7. *Let $A \to B$ be a faithfully flat ring homomorphism and M an A-module. The diagram*

$$M \longrightarrow M \otimes_A B \rightrightarrows M \otimes_A B \otimes_A B$$

is an equalizer of A-modules.

Proof. Consider the base change

$$M \otimes_A B \xrightarrow{\delta^0} M \otimes_A B \otimes_A B \underset{\delta^1_1}{\overset{\delta^1_0}{\rightrightarrows}} M \otimes_A B \otimes_A B \otimes_A B.$$

Here δ^0 is defined on pure tensors by $m \otimes b \mapsto m \otimes 1 \otimes b$, and the parallel arrows are given by $\delta^1_0(m \otimes b_1 \otimes b_2) = m \otimes b_1 \otimes 1 \otimes b_2$ and $\delta^1_1(m \otimes b_1 \otimes b_2) = m \otimes 1 \otimes b_1 \otimes b_2$, respectively. Put

$$\begin{aligned} z^0 &\colon M \otimes_A B \otimes_A B \to M \otimes_A B, && m \otimes b_1 \otimes b_2 \mapsto m \otimes b_1 b_2, \\ z^1 &\colon M \otimes_A B \otimes_A B \otimes_A B \to M \otimes_A B \otimes_A B, && m \otimes b_1 \otimes b_2 \otimes b_3 \mapsto m \otimes b_2 \otimes b_1 b_3. \end{aligned}$$

Define $\delta^1 = \delta^1_1 - \delta^1_0$. Then $z^0\delta^0$ is the identity on $M \otimes_A B$, and $\delta^0 z^0 + z^1\delta^1$ is the identity on $M \otimes_A B \otimes_A B$. Hence the identity on the complex

$$0 \longrightarrow M \otimes_A B \xrightarrow{\delta^0} M \otimes_A B \otimes_A B \xrightarrow{\delta^1} M \otimes_A B \otimes_A B \otimes_A B$$

is null-homotopic at $M \otimes_A B$ and $M \otimes_A B \otimes_A B$. It follows that the complex is exact, and the base changed diagram is an equalizer. The homomorphism $A \to B$ is faithfully flat, so the original diagram is an equalizer as well. ∎

Lemma 1.8. *Let S be a scheme, $\mathcal{D}$ a category and $\mathcal{F}\colon (\mathsf{Sch}/S)^{\mathsf{op}} \to \mathcal{D}$ a functor. Then $\mathcal{F}$ is an étale sheaf if and only if $\mathcal{F}$ satisfies the sheaf property for*

- *Zariski coverings, and*
- *coverings given by a surjective étale map $\operatorname{Spec} B \to \operatorname{Spec} A$.*

Proof. The 'only if' part is obvious. Conversely, assume that $\mathcal{F}$ satisfies the sheaf property for coverings of the stated form. Let $\{V_i \to U\}_{i \in I}$ be an arbitrary étale covering. We prove that $\mathcal{F}$ satisfies the corresponding sheaf property. Since $\mathcal{F}$ satisfies the sheaf property for Zariski coverings, we can reduce to the case where $U = \operatorname{Spec} A$ is an affine scheme. The sheaf property for $\{V_i \to \operatorname{Spec} A\}_{i \in I}$ is equivalent to the sheaf property for the induced covering $\{V \to \operatorname{Spec} A\}$, where V is the disjoint union $\coprod_{i \in I} V_i$.

Take a Zariski covering $\{\operatorname{Spec} B_j \to V\}_{j \in J}$ of V by affines. Since $\operatorname{Spec} A$ is quasi-compact and étale morphisms are open, there exists a finite subset $J' \subset J$ such that $\{\operatorname{Spec} B_j \to \operatorname{Spec} A\}_{j \in J'}$ is a covering. Putting $B = \prod_{j \in J'} B_j$ we obtain $\operatorname{Spec} B = \coprod_{j \in J'} \operatorname{Spec} B_j$ as J' is finite.

Let $y \in \mathcal{F}(V)$ be such that the two pullbacks of y to $\mathcal{F}(V \times_A V)$ coincide. We have to prove that there exists a unique $x \in \mathcal{F}(\operatorname{Spec} A)$ with $y = x|_V$. Let z be the pullback of y to $\operatorname{Spec} B$. The two pullbacks of z to $\operatorname{Spec} B \otimes_A B$ coincide. By assumption $\mathcal{F}$ satisfies the sheaf property for the covering $\{\operatorname{Spec} B \to \operatorname{Spec} A\}$, so there exists a unique $x \in \mathcal{F}(\operatorname{Spec} A)$ such that $z = x|_{\operatorname{Spec} B}$. One easily proves that $x|_V$ equals y. ∎

Theorem 1.9 (Grothendieck). *Let S be a scheme and $\mathcal{F}\colon (\mathsf{Sch}/S)^{\mathsf{op}} \to \mathsf{Set}$ a functor. If $\mathcal{F}$ is representable, then $\mathcal{F}$ is an étale sheaf.*

Proof. Let X be a scheme over S. We apply lemma 1.8 to the Yoneda functor h_X.

First let $\{V_i \to U\}_{i \in I}$ be a Zariski covering. If $(f_i\colon V_i \to X)_{i \in I}$ is a tuple of morphisms that coincide on the intersections $V_i \times_U V_j$, then there exists a unique morphism $f\colon U \to X$ with $f|_{V_i} = f_i$ for all $i \in I$. Indeed, f arises from gluing the morphisms f_i together. This means precisely that h_X satisfies the sheaf property for $\{V_i \to U\}_{i \in I}$.

Now let $h\colon \operatorname{Spec} B \to \operatorname{Spec} A$ be a surjective étale morphism. Let π_0, π_1 be the projection maps $\operatorname{Spec} B \otimes_A B \to \operatorname{Spec} B$. We need to show that any morphism $f\colon \operatorname{Spec} B \to X$ satisfying $f\pi_0 = f\pi_1$ factors uniquely over $\operatorname{Spec} A$. A short argument, using that étale morphisms are open and h_X is a Zariski sheaf, reduces to the case where $X = \operatorname{Spec} R$ is affine. In that case we need to show there is a unique ring homomorphism $g^*\colon R \to A$ that makes the diagram

$$
\begin{tikzcd}
R \arrow[dr, dotted, "g^*"] \arrow[drr, bend left, "f^*"] \arrow[ddr, bend right, "f^*"'] & & \\
& A \arrow[r, "h^*"] \arrow[d, "h^*"'] & B \arrow[d, "\pi_1^*"] \\
& B \arrow[r, "\pi_0^*"'] & B \otimes_A B
\end{tikzcd}
$$

commute. Observe that $A \to B$ is faithfully flat. The special case $M = A$ of lemma 1.7 states that h^* is the equalizer of π_0^* and π_1^* in A-Mod. Then by definition we have a unique A-module homomorphism g^* as desired. Since h^* is injective, the identity $h^* g^* = f^*$ forces g^* to be, in fact, a ring homomorphism. ∎

Here is a useful application of the above theorem. Let G be a group. Let $\mathcal{F}\colon (\mathsf{Sch}/S)^{\mathsf{op}} \to \mathsf{Set}$ be the functor that maps a scheme T over S to the set of continuous functions $T \to G$, where G is endowed with the discrete topology. Then $\mathcal{F}$ is representable by the scheme $\coprod_{g \in G} S$ over S, hence $\mathcal{F}$ is an étale sheaf of sets on S. The group structure on G makes $\mathcal{F}$ naturally into a sheaf of groups. It is called the *constant sheaf* G.

1.3. Descent of sheaves

Descent problems can be posed in a very general setting, namely, for arbitrary fibered categories over a site. Here only descent of sheaves is considered. We fix some notation: if $\{U_i \to S\}_{i\in I}$ is a covering, then we abbreviate $U_{i_0} \times_S \ldots \times_S U_{i_n}$ by $U_{i_0 \ldots i_n}$.

Definition 1.10. Let $\mathcal{C}$ be a site, $\mathcal{U} = \{U_i \to S\}_{i\in I}$ a covering in $\mathcal{C}$, and $\mathcal{D}$ a category. A $\mathcal{D}$-valued *descent datum* relative to $\mathcal{U}$ consists of

- a $\mathcal{D}$-valued sheaf $\mathcal{F}_i \in \mathsf{Sh}_{\mathcal{D}}\, \mathcal{C}/U_i$ for all $i \in I$, and
- an isomorphism $\varphi_{ij}\colon \mathcal{F}_i|_{U_{ij}} \to \mathcal{F}_j|_{U_{ij}}$ in $\mathsf{Sh}_{\mathcal{D}}\, \mathcal{C}/U_{ij}$ for all $i, j \in I$,

such that the cocycle condition

$$\varphi_{ik}|_{U_{ijk}} = \varphi_{jk}|_{U_{ijk}} \circ \varphi_{ij}|_{U_{ijk}}$$

holds for all $i, j, k \in I$. A *morphism of descent data* $(\mathcal{F}_i, \varphi_{ij})_{i,j\in I} \to (\mathcal{G}_i, \psi_{ij})_{i,j\in I}$ is a tuple of morphisms $(\alpha_i\colon \mathcal{F}_i \to \mathcal{G}_i)_{i\in I}$ such that the diagram

$$\begin{tikzcd}
\mathcal{F}_i|_{U_{ij}} \arrow[r, "\alpha_i|_{U_{ij}}"] \arrow[d, "\varphi_{ij}"] & \mathcal{G}_i|_{U_{ij}} \arrow[d, "\psi_{ij}"] \\
\mathcal{F}_j|_{U_{ij}} \arrow[r, "\alpha_j|_{U_{ij}}"] & \mathcal{G}_j|_{U_{ij}}
\end{tikzcd}$$

commutes for all $i, j \in I$.

We give an important example. Let $\mathcal{F}$ be a sheaf on $\mathcal{C}/S$. Write $\mathcal{F}_i = \mathcal{F}|_{U_i}$ and $\mathcal{F}_{ij} = \mathcal{F}|_{U_{ij}}$ for the respective pullbacks. Then the sheaves $\mathcal{F}_i|_{U_{ij}}$ and $\mathcal{F}_j|_{U_{ij}}$ both equal $\mathcal{F}_{ij}$, so $(\mathcal{F}_i, \mathrm{id}_{\mathcal{F}_{ij}})_{i,j\in I}$ is a descent datum relative to $\mathcal{U}$. We denote it by $\mathcal{F}|_{\mathcal{U}}$.

Let $\mathsf{Sh}_{\mathcal{D}}\, \mathcal{U}$ be the category of $\mathcal{D}$-valued descent data relative to $\mathcal{U}$. The preceding example defines a functor

$$\mathsf{desc}\colon\ \mathsf{Sh}_{\mathcal{D}}\, \mathcal{C}/S \to \mathsf{Sh}_{\mathcal{D}}\, \mathcal{U}, \quad \mathcal{F} \mapsto \mathcal{F}|_{\mathcal{U}}.$$

A descent datum *descends* to a sheaf $\mathcal{F}$ on $\mathcal{C}/S$ if it is isomorphic in $\mathsf{Sh}_{\mathcal{D}}\, \mathcal{U}$ to $\mathcal{F}|_{\mathcal{U}}$. We then say that the descent datum is *effective*. In many cases, such $\mathcal{F}$ exists and is uniquely unique. This is intuitively clear from the definitions.

Proposition 1.11. *Let $\mathcal{C}$ be a site, $S \in \mathcal{C}$ an object, $\mathcal{U}$ a covering of S, and $\mathcal{D}$ a category. The functor* $\mathsf{desc}\colon \mathsf{Sh}_{\mathcal{D}}\, \mathcal{C}/S \to \mathsf{Sh}_{\mathcal{D}}\, \mathcal{U}$ *is fully faithful. If $\mathcal{D}$ has products and equalizers, then* desc *is an equivalence of categories.*

Proof. Write $\mathcal{U} = \{U_i \to S\}_{i\in I}$. We prove that desc is fully faithful. Let $\mathcal{F}, \mathcal{G} \in \mathsf{Sh}_{\mathcal{D}}\, \mathcal{C}/S$ be sheaves and set $\mathcal{F}_i = \mathcal{F}|_{U_i}$ and $\mathcal{G}_i = \mathcal{G}|_{U_i}$. Let $(\alpha_i)_{i\in I}$ be a morphism of descent data $\mathcal{F}|_{\mathcal{U}} \to \mathcal{G}|_{\mathcal{U}}$. We need to show that it descends to a unique morphism of sheaves $\alpha\colon \mathcal{F} \to \mathcal{G}$. In the case $\mathcal{D} = \mathsf{Set}$ this follows from the diagram

$$\begin{tikzcd}
\mathcal{F} \arrow[r] \arrow[d, dotted, "\alpha"] & \prod_{i\in I} \mathcal{F}_i(U_i \times_S -) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "(\alpha_i)_{i\in I}"] & \prod_{i,j\in I} \mathcal{F}_i(U_{ij} \times_S -) \arrow[d, "(\alpha_i)_{i,j\in I}"] \\
\mathcal{G} \arrow[r] & \prod_{i\in I} \mathcal{G}_i(U_i \times_S -) \arrow[r, shift left] \arrow[r, shift right] & \prod_{i,j\in I} \mathcal{G}_i(U_{ij} \times_S -)
\end{tikzcd}$$

whose rows are equalizers in $\mathsf{Sh}\,\mathcal{C}/S$. The diagram commutes in the sense that both subdiagrams, obtained by a consistent choice from parallel arrows, are commutative. For arbitrary $\mathcal{D}$ apply Yoneda's lemma 1.3.

Now suppose $\mathcal{D}$ has products and equalizers. We prove that desc is essentially surjective. Let $(\mathcal{F}_i, \varphi_{ij})_{i,j\in I}$ be a descent datum in $\mathsf{Sh}_{\mathcal{D}}\mathcal{U}$. For objects $T \in \mathcal{C}/S$ define $\mathcal{F}(T)$ to be the equalizer in

$$\mathcal{F}(T) \longrightarrow \prod_{i\in I} \mathcal{F}_i(U_i \times_S T) \rightrightarrows \prod_{i,j\in I} \mathcal{F}_i(U_{ij} \times_S T).$$

This makes $\mathcal{F}$ into a presheaf on $\mathcal{C}/S$. If the map $T \to S$ factors over some U_i, then the gluing maps φ_{ij} induce an isomorphism $\mathcal{F}(T) \simeq \mathcal{F}_i(T)$. We get compatible isomorphisms $\mathcal{F}|_{U_i} \simeq \mathcal{F}_i$ for all $i \in I$. It remains to show that $\mathcal{F}$ is a sheaf. For an arbitrary covering $\{V_j \to T\}_{j\in J}$ in $\mathcal{C}$ consider the diagram

$$\begin{tikzcd}
\mathcal{F}(T) \arrow[r] \arrow[d] & \prod_{j\in J} \mathcal{F}(V_j) \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \prod_{j_0,j_1\in J} \mathcal{F}(V_{j_0j_1}) \arrow[d] \\
\prod_{i\in I} \mathcal{F}(U_i \times_S T) \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & \prod_{\substack{i\in I\\ j\in J}} \mathcal{F}(U_i \times_S V_j) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & \prod_{\substack{i\in I\\ j_0,j_1\in J}} \mathcal{F}(U_i \times_S V_{j_0j_1}) \arrow[d, shift left] \arrow[d, shift right] \\
\prod_{i_0,i_1\in I} \mathcal{F}(U_{i_0i_1} \times_S T) \arrow[r] & \prod_{\substack{i_0,i_1\in I\\ j\in J}} \mathcal{F}(U_{i_0i_1} \times_S V_j) \arrow[r, shift left] \arrow[r, shift right] & \prod_{\substack{i_0,i_1\in I\\ j_0,j_1\in J}} \mathcal{F}(U_{i_0i_1} \times_S V_{j_0j_1}).
\end{tikzcd}$$

It commutes in the sense above. Its second and third row are equalizers by the isomorphisms $\mathcal{F}|_{U_i} \simeq \mathcal{F}_i$. The columns are equalizers by construction. Hence, also the first row is an equalizer. ∎

1.4. Descent of schemes

Let S be a scheme. Grothendieck's theorem 1.9, together with Yoneda's lemma 1.3, allows us to view schemes over S as representable sheaves on $(\mathsf{Sch}/S)_{\mathsf{Zar}}$ or $(\mathsf{Sch}/S)_{\text{ét}}$. We have seen in proposition 1.11 that sheaves of sets have effective descent. A natural question is: do representable sheaves descend to sheaves that are again representable? Or, rephrasing: is 'descent of schemes' effective?

In the case of the Zariski topology, there is a simple, constructive answer.

Proposition 1.12. *Let S be a scheme, $\{U_i \to S\}_{i\in I}$ a Zariski covering of S, and $\mathcal{F}$ a sheaf of sets on $(\mathsf{Sch}/S)_{\mathsf{Zar}}$ such that $\mathcal{F}|_{U_i}$ is representable for all $i \in I$. Then $\mathcal{F}$ is representable.*

Proof. For all $i \in I$, fix a scheme X_i over U_i representing $\mathcal{F}|_{U_i}$, and an isomorphism $\mathcal{F}|_{U_i} \simeq \mathsf{h}_{X_i}$. There are induced isomorphisms

$$\mathsf{h}_{X_i \times_{U_i} U_{ij}} \simeq \mathsf{h}_{X_i}|_{U_{ij}} \simeq \mathcal{F}|_{U_{ij}} \simeq \mathsf{h}_{X_j}|_{U_{ij}} \simeq \mathsf{h}_{X_j \times_{U_j} U_{ij}}$$

of sheaves on U_{ij}. Yoneda's lemma 1.3 gives isomorphisms $X_i \times_{U_i} U_{ij} \simeq X_j \times_{U_j} U_{ij}$ satisfying the cocycle condition. Since the covering is Zariski, $X_i \times_{U_i} U_{ij}$ is just an open subscheme of X_i. So we can glue the schemes X_i along these open subschemes into a scheme X over S. We find an isomorphism of descent data $\mathcal{F}|_{\mathcal{U}} \simeq \mathsf{h}_X|_{\mathcal{U}}$. By proposition 1.11 we have $\mathcal{F} \simeq \mathsf{h}_X$. ∎

However, étale descent of schemes may fail. See, for instance, the example given by Donald Knutson [K, pp. 9–10]. Here we specifically study étale descent of curves.

Definition 1.13. Let S be a scheme and g a non-negative integer. By a *curve* over S we mean a scheme X over S that is proper, smooth of relative dimension 1, and has geometrically connected fibers. If all geometric fibers of X are connected curves of genus g, we also say that X is a *curve of genus g* over S.

Theorem I. *Let g be a non-negative integer unequal to 1. Let S be a scheme, $\{U_i \to S\}_{i\in I}$ an étale covering of S, and $\mathcal{F}$ an étale sheaf of sets on S such that $\mathcal{F}|_{U_i}$ is representable by a curve of genus g over U_i for all $i \in I$. Then $\mathcal{F}$ is representable by a curve of genus g over S.*

Theorem II. *There exist a scheme S, an étale covering $\{U_i \to S\}_{i\in I}$ of S, and an étale sheaf of sets $\mathcal{F}$ on S such that $\mathcal{F}|_{U_i}$ is representable by a curve of genus 1 over U_i for all $i \in I$, and such that $\mathcal{F}$ is not representable.*

Theorems I and II are the objectives of this thesis. The first is a modern formulation of a classical result; we give a proof in chapter 2. The last two chapters are dedicated to a new constructive proof of theorem II. It was first proved by Michel Raynaud [R, Exemple XIII.3.2]. In section 4.3 we compare our construction to that by Raynaud.

Descent in genus unequal to 1 2

2.1. Descent of quasi-coherent sheaves

In this chapter we show that étale descent of curves of genus $g \neq 1$ is effective. An important ingredient is the fact that quasi-coherent sheaves on schemes descend along étale coverings. This is quite remarkable: quasi-coherent sheaves are defined to be just Zariski sheaves.

We denote the category of quasi-coherent sheaves on a scheme S by $\mathsf{QCoh}\, S$. If $f\colon T \to S$ is a morphism of schemes and $\mathcal{F}$ a quasi-coherent sheaf on S, then we write $\mathcal{F}|_T$ for the quasi-coherent sheaf $f^*\mathcal{F}$ on T.

Definition 2.1. Let S be a scheme and $\mathcal{U} = \{U_i \to S\}_{i \in I}$ an étale covering of S. A *descent datum of quasi-coherent sheaves* relative to $\mathcal{U}$ consists of

- a quasi-coherent sheaf $\mathcal{F}_i \in \mathsf{QCoh}\, U_i$ for all $i \in I$, and
- an isomorphism $\varphi_{ij}\colon \mathcal{F}_i|_{U_{ij}} \to \mathcal{F}_j|_{U_{ij}}$ in $\mathsf{QCoh}\, U_{ij}$ for all $i, j \in I$,

such that the cocycle condition

$$\varphi_{ik}|_{U_{ijk}} = \varphi_{jk}|_{U_{ijk}} \circ \varphi_{ij}|_{U_{ijk}}$$

holds for all $i, j, k \in I$. A *morphism of descent data* $(\mathcal{F}_i, \varphi_{ij})_{i,j \in I} \to (\mathcal{G}_i, \psi_{ij})_{i,j \in I}$ is a tuple of morphisms $(\alpha_i\colon \mathcal{F}_i \to \mathcal{G}_i)_{i \in I}$ such that the diagram

$$\begin{tikzcd} \mathcal{F}_i|_{U_{ij}} \arrow[r, "\alpha_i|_{U_{ij}}"] \arrow[d, "\varphi_{ij}"] & \mathcal{G}_i|_{U_{ij}} \arrow[d, "\psi_{ij}"] \\ \mathcal{F}_j|_{U_{ij}} \arrow[r, "\alpha_j|_{U_{ij}}"] & \mathcal{G}_j|_{U_{ij}} \end{tikzcd}$$

commutes for all $i, j \in I$.

Let $\mathsf{QCoh}\, \mathcal{U}$ denote the category of descent data of quasi-coherent sheaves relative to $\mathcal{U}$. A quasi-coherent sheaf $\mathcal{F}$ on S induces a descent datum $\mathcal{F}|_{\mathcal{U}} \in \mathsf{QCoh}\, \mathcal{U}$. This gives a functor

$$\mathsf{desc}\colon \mathsf{QCoh}\, S \to \mathsf{QCoh}\, \mathcal{U}, \quad \mathcal{F} \mapsto \mathcal{F}|_{\mathcal{U}}.$$

Once more, the next theorem is due to Alexander Grothendieck [SGA1, Théorème VIII.1.1].

Theorem 2.2 (Grothendieck). *Let S be a scheme and $\mathcal{U}$ an étale covering of S. Then the functor* $\mathsf{desc}\colon \mathsf{QCoh}\, S \to \mathsf{QCoh}\, \mathcal{U}$ *is an equivalence of categories.*

Proof. Roughly speaking, we have to prove that QCoh is an étale 'sheaf of categories'. The correct word is 2-*sheaf* or *stack*. The upshot of this analogy is that we can apply lemma 1.8: it suffices to prove the theorem for Zariski coverings and coverings given by a surjective étale map $\mathsf{Spec}\, B \to \mathsf{Spec}\, A$. Full details can be found in [V, Definition 4.6, Lemma 4.25].

The case of Zariski coverings is well-known. Next, consider an étale covering given by $\mathsf{Spec}\, B \to \mathsf{Spec}\, A$. The theorem reduces to statements in commutative algebra, via the equivalence between $\mathsf{QCoh}(\mathsf{Spec}\, R)$ and R-Mod.

The fully faithfulness translates as follows. Let M_1 and M_2 be A-modules. Suppose that $\beta\colon M_1 \otimes_A B \to M_2 \otimes_A B$ is a homomorphism of B-modules, such that the two induced maps $M_1 \otimes_A B \otimes_A B \to M_2 \otimes_A B \otimes_A B$ are equal. We have to prove there exists a unique A-module homomorphism $\alpha\colon M_1 \to M_2$ satisfying $\beta = \alpha \otimes \mathsf{id}_B$. In the diagram of A-modules

$$\begin{tikzcd} M_1 \arrow[r] \arrow[d, dotted, "\alpha"] & M_1 \otimes_A B \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\beta"] & M_1 \otimes_A B \otimes_A B \arrow[d, "\beta \otimes \mathsf{id}_B"] \\ M_2 \arrow[r] & M_2 \otimes_A B \arrow[r, shift left] \arrow[r, shift right] & M_2 \otimes_A B \otimes_A B \end{tikzcd}$$

both subsquares on the right-hand side commute, by the assumption on β. The rows are equalizers by lemma 1.7. Hence there exists a unique map $\alpha\colon M_1 \to M_2$ that makes the left square commute. This means precisely $\beta = \alpha \otimes \mathsf{id}_B$.

The essential surjectivity translates to commutative algebra as follows. Let N be a B-module and $\varphi\colon N \otimes_A B \to B \otimes_A N$ an isomorphism of $B \otimes_A B$-modules, such that the diagram

$$\begin{tikzcd} N \otimes_A B \otimes_A B \arrow[rr, "\varphi_{02}"] \arrow[dr, "\varphi_{01}"'] & & B \otimes_A B \otimes_A N \\ & B \otimes_A N \otimes_A B \arrow[ur, "\varphi_{12}"'] & \end{tikzcd}$$

commutes. Here $\varphi_{01} = \varphi \otimes \mathsf{id}_B$ and $\varphi_{12} = \mathsf{id}_B \otimes \varphi$, and φ_{02} is obtained from φ by inserting id_B on the middle factor B. Then we have to show that there exists an A-module M and a B-module isomorphism $\beta\colon M \otimes_A B \to N$ that make a commutative square

$$\begin{tikzcd} M \otimes_A B \otimes_A B \arrow[r, "\beta \otimes \mathsf{id}_B"] \arrow[d] & N \otimes_A B \arrow[d, "\varphi"] \\ B \otimes_A M \otimes_A B \arrow[r, "\mathsf{id}_B \otimes \beta"] & B \otimes_A N \end{tikzcd}$$

where the left vertical arrow is defined on pure tensors by $m \otimes b_1 \otimes b_2 \mapsto b_1 \otimes m \otimes b_2$.

We have two A-module homomorphisms $\alpha_1, \alpha_2\colon N \to N \otimes_A B$, defined by respectively $\alpha_1(n) = n \otimes 1$ and $\alpha_2(n) = \varphi^{-1}(1 \otimes n)$. Let M to be their equalizer. Let $\beta\colon M \otimes_A B \to N$ be the B-module homomorphism defined by $m \otimes b \mapsto bm$. We obtain a diagram of B-modules

$$\begin{tikzcd} M \otimes_A B \arrow[r] \arrow[d, "\beta"] & N \otimes_A B \arrow[r, shift left, "\alpha_1 \otimes \mathsf{id}_B"] \arrow[r, shift right, "\alpha_2 \otimes \mathsf{id}_B"'] \arrow[d, "\varphi"] & N \otimes_A B \otimes_A B \arrow[d, "\varphi_{02}"] \\ N \arrow[r] & B \otimes_A N \arrow[r, shift left] \arrow[r, shift right] & B \otimes_A B \otimes_A N. \end{tikzcd}$$

Its commutativity, in the sense above, is easily verified. The first row is an equalizer since it is the base change of an equalizer by the flat map $A \to B$. The second row is an equalizer by lemma 1.7. Since both φ and φ_{02} are isomorphisms, also β must be an isomorphism. The desired compatibility between β and φ follows from the commutativity of the left-hand square in the last diagram. ∎

2.2. Projective embeddings

In this section we introduce projective bundles.

Proposition 2.3. *Let S be a scheme and $\mathcal{A}$ a quasi-coherent commutative graded $\mathcal{O}_S$-algebra. There is a uniquely unique scheme $\operatorname{Proj} \mathcal{A}$ and map $f\colon \operatorname{Proj} \mathcal{A} \to S$, such that for every affine open $U \subset S$ there is an isomorphism $\alpha_U\colon f^{-1}(U) \to \operatorname{Proj} \mathcal{A}(U)$ over U, and moreover such that if $V \subset U$ is another affine open, then the diagram*

$$\begin{tikzcd} f^{-1}(V) \arrow[r, "\alpha_V"] \arrow[d] & \operatorname{Proj} \mathcal{A}(V) \arrow[d] \\ f^{-1}(U) \arrow[r, "\alpha_U"] & \operatorname{Proj} \mathcal{A}(U) \end{tikzcd}$$

commutes.

Proof. If $V \subset U \subset S$ are affine open subschemes, we have a cartesian square

$$\begin{tikzcd} \operatorname{Proj} \mathcal{A}(V) \arrow[r] \arrow[d] & V \arrow[d] \\ \operatorname{Proj} \mathcal{A}(U) \arrow[r] & U \end{tikzcd}$$

because $\mathcal{A}$ is quasi-coherent. Now let $U, V \subset S$ be arbitrary affine open subschemes, and write $g\colon \operatorname{Proj} \mathcal{A}(U) \to U$ and $h\colon \operatorname{Proj} \mathcal{A}(V) \to V$ for the structure morphisms. If $W \subset U \cap V$ is another affine open, the cartesian square above gives canonical isomorphisms

$$g^{-1}(W) \to \operatorname{Proj} \mathcal{A}(W) \to h^{-1}(W).$$

Varying W, these isomorphisms glue into an isomorphism $\varphi_{UV}\colon g^{-1}(U \cap V) \to h^{-1}(U \cap V)$.

For any three affine open $U, V, W \subset S$ we have the cocycle condition $\varphi_{UW} = \varphi_{VW} \circ \varphi_{UV}$ on $U \cap V \cap W$. Hence $(\operatorname{Proj} \mathcal{A}(U), \varphi_{UV})_{U,V \in \mathcal{U}}$ is a descent datum relative to the Zariski covering $\mathcal{U}$ of S that consists of all affine open subschemes of S. By proposition 1.12 Zariski descent is effective, so the descent datum descends to a uniquely unique scheme $\operatorname{Proj} \mathcal{A}$ over S. The desired properties follow from the construction. ∎

The scheme $\operatorname{Proj} \mathcal{A}$ is the *relative Proj* of $\mathcal{A}$ over S.

Definition 2.4. Let S be a scheme and $\mathcal{E}$ an $\mathcal{O}_S$-module. The *symmetric algebra* $\operatorname{Sym} \mathcal{E}$ is the quotient of the tensor algebra $\bigoplus_{n \geqslant 0} \mathcal{E}^{\otimes n}$ by the ideal generated by local sections $x \otimes y - y \otimes x$ with $x, y \in \mathcal{E}(U)$ and $U \subset S$ open.

The symmetric algebra $\operatorname{Sym} \mathcal{E}$ is a commutative graded $\mathcal{O}_S$-algebra. If $\mathcal{E}$ is quasi-coherent, then so is $\operatorname{Sym} \mathcal{E}$ [EGA2, Corollaire 1.7.7]. Therefore, we can make the following definition.

Definition 2.5. Let S be a scheme and $\mathcal{E}$ a quasi-coherent sheaf on S. The *projective bundle* of $\mathcal{E}$ is the scheme $\mathbb{P}(\mathcal{E}) = \mathsf{Proj}(\mathsf{Sym}\,\mathcal{E})$ over S.

For example, let k be a field and V an $(n+1)$-dimensional k-vector space. Write $S = \mathsf{Spec}\,k$ and let $\mathcal{E}$ be the quasi-coherent sheaf on S associated to V. Then $\mathbb{P}(\mathcal{E})(k)$ is the set of hyperplanes in V. In fact, $\mathbb{P}(\mathcal{E})$ is non-canonically isomorphic to $\mathbb{P}^n_k$.

Projective bundles are the correct context to define projective morphisms: a morphism of schemes $X \to S$ is *projective* if X is isomorphic over S to a closed subscheme of the projective bundle $\mathbb{P}(\mathcal{E})$, for some quasi-coherent finite type sheaf $\mathcal{E}$ on S. For instance, all closed subschemes of $\mathbb{P}^n_S$ are projective over S. Conversely, every projective morphism $X \to S$ is Zariski locally on S of this form.

The construction of projective bundles commutes with base change.

Lemma 2.6. *Let $S' \to S$ be a morphism of schemes, $\mathcal{E}$ a quasi-coherent sheaf on S, and $\mathcal{E}'$ the pullback of $\mathcal{E}$ to S'. There is a canonical morphism $\mathbb{P}(\mathcal{E}') \to \mathbb{P}(\mathcal{E})$, and the square*

$$\begin{tikzcd} \mathbb{P}(\mathcal{E}') \arrow[r] \arrow[d] & S' \arrow[d] \\ \mathbb{P}(\mathcal{E}) \arrow[r] & S \end{tikzcd}$$

is cartesian.

Proof. In fact, both Proj and Sym commute with base change, up to canonical isomorphism [EGA2, 1.7.5, Proposition 3.5.3]. ■

Let S be a scheme and $\mathcal{A}$ a quasi-coherent commutative graded $\mathcal{O}_S$-algebra. For every affine open $U \subset S$ we have the Serre twisting sheaf $\mathcal{O}_{\mathsf{Proj}\,\mathcal{A}(U)}(1)$ on $\mathsf{Proj}\,\mathcal{A}(U)$. They glue into a quasi-coherent sheaf on $\mathsf{Proj}\,\mathcal{A}$, denoted again $\mathcal{O}_{\mathsf{Proj}\,\mathcal{A}}(1)$.

Definition 2.7. Let $X \to S$ be a morphism of schemes. A line bundle $\mathcal{L}$ on X is *very ample* over S if there exists a quasi-coherent sheaf $\mathcal{E}$ on S and an immersion $i\colon X \to \mathbb{P}(\mathcal{E})$ over S such that $\mathcal{L}$ is isomorphic to $i^*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$.

Proposition 2.8. *Let $f\colon X \to S$ be a quasi-compact morphism of schemes and $\mathcal{L}$ a line bundle on X that is very ample over S. Then $\mathcal{E} = f_*\mathcal{L}$ is quasi-coherent and there exists a canonical immersion $X \to \mathbb{P}(\mathcal{E})$ over S. If f is proper, then $X \to \mathbb{P}(\mathcal{E})$ is a closed immersion.*

Proof. See [EGA2, Proposition 4.4.4, Corollaire 5.4.4]. ■

If moreover $f_*\mathcal{L}$ is of finite type, then f is projective, and we obtain an embedding of X in a projective space. This is the case at least when S is locally noetherian [EGA3, Corollaire 3.2.2] of when f is of finite presentation [EGA4, Proposition 8.9.1].

2.3. Line bundles on curves

The following property of curves will be essential. Recall from definition 1.13 that curves are proper, smooth, and have geometrically connected fibers.

Theorem 2.9. *Let S be a scheme and g a non-negative integer. Let X be a curve over S of genus g. If $\mathcal{L}$ is a line bundle on X such that on every geometric fiber X_s the pullback $\mathcal{L}_s$ has degree at least $2g+1$, then $\mathcal{L}$ is very ample over S.*

Proof. We only sketch the proof. The case $S = \operatorname{Spec} k$ with k an algebraically closed field is classical: see for instance [H, Corollary IV.3.2]. So for every geometric fiber X_s the pullback $\mathcal{L}_s$ is very ample.

Since $X \to S$ is of finite presentation, we can reduce to the case where S is locally noetherian [EGA4, Proposition 8.9.1]. In that case, the proof of [EGA3, Théorème 4.7.1] can be modified to show that if $\mathcal{L}$ is very ample at a geometric fiber, then it is very ample in an open neighborhood of that fiber. Being very ample over S can be checked on a Zariski covering of S, since the map $X \to S$ is quasi-compact [EGA2, Corollaire 4.4.5]. Hence $\mathcal{L}$ is very ample over S. ∎

Let $f\colon X \to S$ be a curve of genus g. The canonical line bundle $\Omega_{X/S}$ on X restricts on every geometric fiber to a line bundle of degree $2g-2$. Hence, if the genus g is at least 2, then $\mathcal{L} = \Omega_{X/S}^{\otimes 3}$ is very ample over S by the preceding theorem. The closed immersion $X \to \mathbb{P}(f_*\mathcal{L})$ from proposition 2.8 is called the *tricanonical embedding* of X. In genus 0 the dual line bundle $\mathcal{L} = \Omega_{X/S}^{\otimes -1}$ is very ample over S.

We are now ready to prove theorem I.

Proof (Theorem I). Let g be a non-negative integer unequal to 1. Let S be a scheme, $\{U_i \to S\}_{i \in I}$ an étale covering of S, and $\mathcal{F}$ an étale sheaf of sets on S such that $\mathcal{F}|_{U_i}$ is representable by a curve of genus g over U_i for all $i \in I$. We have to prove that $\mathcal{F}$ is representable by a curve of genus g over S. Again, we can reduce to Zariski coverings and coverings given by an étale surjection $\operatorname{Spec} B \to \operatorname{Spec} A$; see [V, Example 4.39, Lemma 4.25] for more details. The Zariski case is a trivial consequence of proposition 1.12.

Consider a covering of the form $\{S' \to S\}$ with S and S' affine. Let $\mathcal{F}$ be an étale sheaf of sets on S such that $\mathcal{F}|_{S'}$ is representable by a curve $f'\colon X' \to S'$ of genus g. Fix an isomorphism $\mathcal{F}|_{S'} \simeq \mathsf{h}_{X'}$ of sheaves on S'. As in the proof of proposition 1.12, we obtain an isomorphism of schemes $\varphi\colon X' \times_S S' \to S' \times_S X'$ over $S' \times_S S'$, satisfying the cocycle condition.

Put $\mathcal{L}' = \Omega_{X'/S'}^{\otimes 3}$ if g is at least 2, and put $\mathcal{L}' = \Omega_{X'/S'}^{\otimes -1}$ if g equals 0. As discussed above, $\mathcal{L}'$ is very ample over S'. We write $\mathcal{E}' = f'_*\mathcal{L}'$. Define $\mathcal{I}'$ to be the ideal sheaf on $\mathbb{P}(\mathcal{E}')$ corresponding to the closed immersion of X' in $\mathbb{P}(\mathcal{E}')$ over S' from proposition 2.8.

The isomorphism φ induces a descent datum of quasi-coherent sheaves $(\mathcal{E}', \psi)$ relative to $\{S' \to S\}$. Applying theorem 2.2, it descends to a quasi-coherent sheaf $\mathcal{E}$ on S. The map $\mathbb{P}(\mathcal{E}') \to \mathbb{P}(\mathcal{E})$ is an étale covering by lemma 2.6. Now φ also induces a descent datum of quasi-coherent sheaves $(\mathcal{I}', \chi)$ relative to $\{\mathbb{P}(\mathcal{E}') \to \mathbb{P}(\mathcal{E})\}$. It descends to a quasi-coherent sheaf $\mathcal{I}$ on $\mathbb{P}(\mathcal{E})$. Finally, the inclusion map $\mathcal{I}' \to \mathcal{O}_{\mathbb{P}(\mathcal{E}')}$ descends to a morphism $\mathcal{I} \to \mathcal{O}_{\mathbb{P}(\mathcal{E})}$.

We prove that the latter is injective. Let $U \subset \mathbb{P}(\mathcal{E})$ be an affine open subscheme. The morphism $\mathbb{P}(\mathcal{E}') \to \mathbb{P}(\mathcal{E})$ is a base change of $S' \to S$ by the cartesian square in lemma 2.6; in particular it is faithfully flat and affine. So the inverse image $V \subset \mathbb{P}(\mathcal{E}')$ of U is also affine. Write $A = \mathcal{O}_{\mathbb{P}(\mathcal{E})}(U)$ and $B = \mathcal{O}_{\mathbb{P}(\mathcal{E}')}(V)$. The A-module homomorphism $\mathcal{I}(U) \to A$ gives rise to an injective B-module homomorphism $\mathcal{I}'(V) \to B$ by the base change $A \to B$. Since $A \to B$ is faithfully flat, also $\mathcal{I}(U) \to A$ is injective. It follows that the map $\mathcal{I} \to \mathcal{O}_{\mathbb{P}(\mathcal{E})}$ is injective.

Hence, $\mathcal{I}$ is a quasi-coherent ideal sheaf on $\mathbb{P}(\mathcal{E})$. Let X be the corresponding closed subscheme of $\mathbb{P}(\mathcal{E})$. It is easily verified that $X \to S$ is a curve of genus g that represents $\mathcal{F}$. ∎

In contrast, there is no functorial assignment of very ample line bundles on curves of genus 1. Therefore, the proof above does not apply. The situation is remedied if we consider curves of genus 1 endowed with a section, i.e. elliptic curves. The section defines a canonical very ample line bundle. It follows that étale descent of elliptic curves is effective.

Torsors 3

3.1. Actions

The first chapter was concerned mainly with the properties of single sheaves. Now we will study the actions of a sheaf of groups on another sheaf. Before we give the definition, we introduce a construction. Let $\mathcal{C}$ be a site, $\mathcal{D}$ a category, and $\mathcal{F}, \mathcal{G} \in \mathsf{Sh}_{\mathcal{D}}\,\mathcal{C}$ two $\mathcal{D}$-valued sheaves on $\mathcal{C}$. Let $\mathsf{Isom}(\mathcal{F}, \mathcal{G})$ denote the set of sheaf isomorphisms $\mathcal{F} \to \mathcal{G}$. The functor

$$\mathcal{I}som(\mathcal{F}, \mathcal{G})\colon \mathcal{C}^{\mathsf{op}} \to \mathsf{Set}, \quad T \mapsto \mathsf{Isom}(\mathcal{F}|_T, \mathcal{G}|_T)$$

is a sheaf of sets on $\mathcal{C}$ by proposition 1.11. A special case is the sheaf $\mathcal{A}ut\,\mathcal{F} = \mathcal{I}som(\mathcal{F}, \mathcal{F})$. Composition makes $\mathcal{A}ut\,\mathcal{F}$ into a sheaf of groups on $\mathcal{C}$.

Definition 3.1. Let $\mathcal{C}$ be a site and $\mathcal{A}$ a sheaf of groups on $\mathcal{C}$. Let $\mathcal{D}$ be a category and $\mathcal{F}$ a $\mathcal{D}$-valued sheaf on $\mathcal{C}$. A *left action* of $\mathcal{A}$ on $\mathcal{F}$ is a morphism of sheaves $\mathcal{A} \to \mathcal{A}ut\,\mathcal{F}$.

A left action of a sheaf of groups $\mathcal{A}$ on a sheaf of sets $\mathcal{F}$ may equivalently be defined as a morphism $\mathcal{A} \times \mathcal{F} \to \mathcal{F}$ in $\mathsf{Sh}\,\mathcal{C}$, such that for all objects $T \in \mathcal{C}$ the map $\mathcal{A}(T) \times \mathcal{F}(T) \to \mathcal{F}(T)$ is a left action of the group $\mathcal{A}(T)$ on the set $\mathcal{F}(T)$. By Yoneda's lemma 1.3, a left action of $\mathcal{A}$ on a $\mathcal{D}$-valued sheaf $\mathcal{F}$ can be given as a collection of left actions $\mathcal{A} \times \mathsf{h}^X\mathcal{F} \to \mathsf{h}^X\mathcal{F}$ for $X \in \mathcal{D}$, contravariantly functorial in X. Often this description is convenient.

Definition 3.2. Let $\mathcal{C}$ be a site, $\mathcal{A}$ a sheaf of groups on $\mathcal{C}$, and $\mathcal{D}$ a category. A $\mathcal{D}$-valued *left $\mathcal{A}$-sheaf* is a $\mathcal{D}$-valued sheaf $\mathcal{F}$ on $\mathcal{C}$ endowed with a left action of $\mathcal{A}$. A *morphism of left $\mathcal{A}$-sheaves* $\mathcal{F} \to \mathcal{G}$ is a morphism of sheaves such that for all objects $X \in \mathcal{D}$ the diagram

$$\begin{tikzcd}
\mathcal{A} \times \mathsf{h}^X\mathcal{F} \arrow[r] \arrow[d] & \mathsf{h}^X\mathcal{F} \arrow[d] \\
\mathcal{A} \times \mathsf{h}^X\mathcal{G} \arrow[r] & \mathsf{h}^X\mathcal{G}
\end{tikzcd}$$

commutes.

The category of $\mathcal{D}$-valued left $\mathcal{A}$-sheaves is denoted by $\mathcal{A}\text{-}\mathsf{Sh}_{\mathcal{D}}\,\mathcal{C}$. Recall from section 1.3 that sheaves can be defined 'locally' by means of descent data. The same is true of sheaves with a left action. To be precise, let $\mathcal{C}$ be a site and $\mathcal{U} = \{U_i \to S\}_{i \in I}$ a covering in $\mathcal{C}$. Let $\mathcal{A}$ be a sheaf of groups on $\mathcal{C}/S$ and write $\mathcal{A}_i = \mathcal{A}|_{U_i}$ and $\mathcal{A}_{ij} = \mathcal{A}|_{U_{ij}}$ for the respective pullbacks. Let $\mathcal{D}$ be a category. A $\mathcal{D}$-valued *$\mathcal{A}$-descent datum* relative to $\mathcal{U}$ consists of

- a $\mathcal{D}$-valued left $\mathcal{A}_i$-sheaf $\mathcal{F}_i \in \mathcal{A}_i\text{-}\mathsf{Sh}_{\mathcal{D}}\,\mathcal{C}/U_i$ for all $i \in I$, and
- an isomorphism $\varphi_{ij}\colon \mathcal{F}_i|_{U_{ij}} \to \mathcal{F}_j|_{U_{ij}}$ in $\mathcal{A}_{ij}\text{-}\mathsf{Sh}_{\mathcal{D}}\,\mathcal{C}/U_{ij}$ for all $i, j \in I$,

such that the cocycle condition

$$\varphi_{ik}|_{U_{ijk}} = \varphi_{jk}|_{U_{ijk}} \circ \varphi_{ij}|_{U_{ijk}}$$

holds for all $i, j, k \in I$. A *morphism of $\mathcal{A}$-descent data* $(\mathcal{F}_i, \varphi_{ij})_{i,j\in I} \to (\mathcal{G}_i, \psi_{ij})_{i,j\in I}$ is a tuple of morphisms $(\alpha_i \colon \mathcal{F}_i \to \mathcal{G}_i)_{i\in I}$ of left $\mathcal{A}_i$-sheaves, such that the diagram

$$\begin{tikzcd} \mathcal{F}_i|_{U_{ij}} \arrow[r, "\alpha_i|_{U_{ij}}"] \arrow[d, "\varphi_{ij}"] & \mathcal{G}_i|_{U_{ij}} \arrow[d, "\psi_{ij}"] \\ \mathcal{F}_j|_{U_{ij}} \arrow[r, "\alpha_j|_{U_{ij}}"] & \mathcal{G}_j|_{U_{ij}} \end{tikzcd}$$

commutes for all $i, j \in I$. Let $\mathcal{A}\text{-}\mathsf{Sh}_{\mathcal{D}}\,\mathcal{U}$ be the category of $\mathcal{D}$-valued $\mathcal{A}$-descent data relative to $\mathcal{U}$. As before, there is a functor $\mathsf{desc}\colon \mathcal{A}\text{-}\mathsf{Sh}_{\mathcal{D}}\,\mathcal{C}/S \to \mathcal{A}\text{-}\mathsf{Sh}_{\mathcal{D}}\,\mathcal{U}$.

Proposition 3.3. *Let $\mathcal{C}$ be a site, $S \in \mathcal{C}$ an object and $\mathcal{U}$ a covering of S. Let $\mathcal{A}$ be a sheaf of groups on $\mathcal{C}/S$ and let $\mathcal{D}$ be a category. The functor* $\mathsf{desc}\colon \mathcal{A}\text{-}\mathsf{Sh}_{\mathcal{D}}\,\mathcal{C}/S \to \mathcal{A}\text{-}\mathsf{Sh}_{\mathcal{D}}\,\mathcal{U}$ *is fully faithful. If $\mathcal{D}$ has products and equalizers, then* desc *is an equivalence of categories.*

Proof. Entirely analogous to the proof of proposition 1.11. ∎

Of course one similarly defines a *right action* as a morphism of sheaves $\mathcal{A}^{\mathsf{op}} \to \mathcal{A}ut\,\mathcal{F}$. Here $\mathcal{A}^{\mathsf{op}}$ is the sheaf of groups that sends an object $T \in \mathcal{C}$ to the opposite group $\mathcal{A}(T)^{\mathsf{op}}$. If $\mathcal{A}$ is abelian, then the identity is an isomorphism $\mathcal{A} \to \mathcal{A}^{\mathsf{op}}$, so left and right actions coincide. In this case we speak simply of an *action*.

We conclude this section with some examples. Let $\mathcal{F}, \mathcal{G} \in \mathsf{Sh}_{\mathcal{D}}\,\mathcal{C}$ be sheaves. There is a canonical left action of $\mathcal{A}ut\,\mathcal{F}$ on $\mathcal{F}$. The sheaf $\mathcal{I}som(\mathcal{F}, \mathcal{G})$ has a left action of $\mathcal{A}ut\,\mathcal{G}$ and a right action of $\mathcal{A}ut\,\mathcal{F}$. Now suppose that $\mathcal{F}$ and $\mathcal{G}$ are endowed with a left action of a sheaf of groups $\mathcal{A}$. Let $\mathsf{Isom}_{\mathcal{A}}(\mathcal{F}, \mathcal{G})$ denote the set of left $\mathcal{A}$-sheaf isomorphims $\mathcal{F} \to \mathcal{G}$. Then the functor

$$\mathcal{I}som_{\mathcal{A}}(\mathcal{F}, \mathcal{G})\colon \mathcal{C}^{\mathsf{op}} \to \mathsf{Set}, \quad T \mapsto \mathsf{Isom}_{\mathcal{A}|_T}(\mathcal{F}|_T, \mathcal{G}|_T)$$

is a sheaf of sets on $\mathcal{C}$ by proposition 3.3. Again we obtain sheaves of groups $\mathcal{A}ut_{\mathcal{A}}\,\mathcal{F}$ and $\mathcal{A}ut_{\mathcal{A}}\,\mathcal{G}$, with a right respectively left action on $\mathcal{I}som_{\mathcal{A}}(\mathcal{F}, \mathcal{G})$.

In another direction, let $\mathcal{A}$ be a sheaf of groups on $\mathcal{C}$. Write $\mathcal{F}$ for its underlying sheaf of sets. Define a morphism $\mathcal{A} \times \mathcal{F} \to \mathcal{F}$ in $\mathsf{Sh}\,\mathcal{C}$ by

$$\mathcal{A}(T) \times \mathcal{F}(T) \to \mathcal{F}(T), \quad (a, x) \mapsto ax$$

for every $T \in \mathcal{C}$. This is a left action, called the *regular left action* of $\mathcal{A}$ on itself. It is central in what follows.

3.2. Torsors and twists

To simplify the discussion, we restrict ourselves to sites with a final object. That condition is fulfilled by all sites relevant to this thesis, notably $(\mathsf{Sch}/S)_{\text{ét}}$. We remark however that, with more care, the theory in this and next section could be formulated in general.

Definition 3.4. Let $\mathcal{C}$ be a site with final object S and $\mathcal{A}$ a sheaf of groups on $\mathcal{C}$. A *left $\mathcal{A}$-torsor* is a left $\mathcal{A}$-sheaf of sets $\mathcal{T}$ for which there exists a covering $\{U_i \to S\}_{i\in I}$ of S such that for all $i \in I$ the left $\mathcal{A}|_{U_i}$-sheaf $\mathcal{T}|_{U_i}$ is isomorphic to the regular left $\mathcal{A}|_{U_i}$-sheaf.

We say that the covering $\{U_i \to S\}_{i\in I}$ *trivializes* $\mathcal{T}$, or is a *trivialization* of $\mathcal{T}$. The torsor $\mathcal{T}$ is *trivial* if it is trivialized by $\{\mathrm{id}_S\}$.

Proposition 3.5. *Let $\mathcal{C}$ be a site with final object S and $\mathcal{A}$ a sheaf of groups on $\mathcal{C}$. Any morphism of left $\mathcal{A}$-torsors is an isomorphism.*

Proof. Suppose that $\mathcal{T}_1$ and $\mathcal{T}_2$ are trivial left $\mathcal{A}$-torsors. A morphism $\alpha\colon \mathcal{T}_1 \to \mathcal{T}_2$ gives for all $T \in \mathcal{C}$ an $\mathcal{A}(T)$-equivariant map $\alpha_T\colon \mathcal{T}_1(T) \to \mathcal{T}_2(T)$, which is necessarily an isomorphism. Hence α is an isomorphism.

Now let $\mathcal{T}_1$ and $\mathcal{T}_2$ be arbitrary left $\mathcal{A}$-torsors, trivialized by coverings $\{U_i \to S\}_{i\in I}$ respectively $\{V_j \to S\}_{j\in J}$. Then $\mathcal{U} = \{U_i \times_S V_j \to S\}_{i\in I, j\in J}$ is a common refinement that trivializes both torsors. A morphism $\alpha\colon \mathcal{T}_1 \to \mathcal{T}_2$ induces an isomorphism on the corresponding $\mathcal{A}$-descent data relative to $\mathcal{U}$, because all morphisms between trivial torsors are isomorphisms. By proposition 3.3, α must be an isomorphism itself. ■

We introduce twists. Let $\mathcal{C}$ be a site with final object S and $\mathcal{A}$ a sheaf of groups on $\mathcal{C}$. Let $\mathcal{T}$ be a right $\mathcal{A}$-torsor. Let $\mathcal{D}$ be a category and $\mathcal{F}$ a $\mathcal{D}$-valued left $\mathcal{A}$-sheaf. Choose a trivialization $\mathcal{U} = \{U_i \to S\}_{i\in I}$ of $\mathcal{T}$ and choose elements $t_i \in \mathcal{T}(U_i)$ for all $i \in I$. There are unique $a_{ij} \in \mathcal{A}(U_{ij})$ such that $t_j|_{U_{ij}} \cdot a_{ij} = t_i|_{U_{ij}}$. These elements satisfy the cocycle condition $a_{ik}|_{U_{ijk}} = a_{jk}|_{U_{ijk}} \cdot a_{ij}|_{U_{ijk}}$ for all $i, j, k \in I$. Identifying them with their image under $\mathcal{A} \to \mathcal{A}ut\,\mathcal{F}$, we get a descent datum $(\mathcal{F}|_{U_i}, a_{ij})_{i,j\in I}$ relative to $\mathcal{U}$. If it descends to a sheaf on $\mathcal{C}$, that sheaf is called the *twist* of $\mathcal{F}$ by $\mathcal{T}$ and denoted $\mathcal{T} \otimes_{\mathcal{A}} \mathcal{F}$. Observe that by proposition 1.11, twists exist at least when $\mathcal{D}$ has products and equalizers, in particular in case $\mathcal{D} = \mathrm{Set}$.

Twists can also be defined by a universal property. We avoid this approach since it requires sheafification, whereas we are already familiar with descent data. One drawback is that we need the following lemma.

Lemma 3.6. *The construction of $\mathcal{T} \otimes_{\mathcal{A}} \mathcal{F}$ is functorial in $\mathcal{T}$ and $\mathcal{F}$. In particular the construction is independent of choices, up to canonical isomorphism.*

Proof. Let $\mathcal{C}$, S, $\mathcal{A}$, $\mathcal{D}$ and $\mathcal{F}$ be as above. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be right $\mathcal{A}$-torsors. Choose trivializations $\mathcal{V}_1$ and $\mathcal{V}_2$ for $\mathcal{T}_1$ and $\mathcal{T}_2$, respectively. Let $\mathcal{U} = \{U_i \to S\}_{i\in I}$ be a common refinement of $\mathcal{V}_1$ and $\mathcal{V}_2$ as in the proof of proposition 3.5. Any twist construction $\mathcal{T}_1 \otimes_{\mathcal{A}} \mathcal{F}$ corresponding to $\mathcal{V}_1$ induces a twist construction corresponding to $\mathcal{U}$ that produces a canonically isomorphic twist. The same holds for $\mathcal{T}_2 \otimes_{\mathcal{A}} \mathcal{F}$. So it suffices to consider twist constructions corresponding to the common trivialization $\mathcal{U}$.

Choose elements $s_i \in \mathcal{T}_1(U_i)$ and $t_i \in \mathcal{T}_2(U_i)$ for all $i \in I$. Let $a_{ij}, b_{ij} \in \mathcal{A}(U_{ij})$ be defined by $s_j|_{U_{ij}} \cdot a_{ij} = s_i|_{U_{ij}}$ and $t_j|_{U_{ij}} \cdot b_{ij} = t_i|_{U_{ij}}$. Given a morphism $\alpha\colon \mathcal{T}_1 \to \mathcal{T}_2$, we have to produce a canonical morphism of sheaves $\mathcal{T}_1 \otimes_{\mathcal{A}} \mathcal{F} \to \mathcal{T}_2 \otimes_{\mathcal{A}} \mathcal{F}$, or equivalently a canonical morphism of descent data $(\mathcal{F}|_{U_i}, a_{ij})_{i,j\in I} \to (\mathcal{F}|_{U_i}, b_{ij})_{i,j\in I}$ relative to $\mathcal{U}$. Proceed as follows. Define $\lambda_i \in \mathcal{A}(U_i)$ by $t_i\lambda_i = \alpha(s_i)$. Then for all $i, j \in I$ we have equalities

$$t_j|_{U_{ij}} \cdot \lambda_j|_{U_{ij}} \cdot a_{ij} = \alpha(s_j|_{U_{ij}}) \cdot a_{ij} = \alpha(s_i|_{U_{ij}}) = t_i|_{U_{ij}} \cdot \lambda_i|_{U_{ij}} = t_j|_{U_{ij}} \cdot b_{ij} \cdot \lambda_i|_{U_{ij}}$$

in $\mathcal{T}_2(U_{ij})$. This implies the identity $\lambda_j|_{U_{ij}} \cdot a_{ij} = b_{ij} \cdot \lambda_i|_{U_{ij}}$. In other words, the diagram

$$\begin{tikzcd}
\mathcal{F}|_{U_{ij}} \arrow[r, "\lambda_i|_{U_{ij}}"] \arrow[d, "a_{ij}"] & \mathcal{F}|_{U_{ij}} \arrow[d, "b_{ij}"] \\
\mathcal{F}|_{U_{ij}} \arrow[r, "\lambda_j|_{U_{ij}}"] & \mathcal{F}|_{U_{ij}}
\end{tikzcd}$$

commutes. This means that $(\lambda_i)_{i\in I}$ is a morphism of descent data $(\mathcal{F}|_{U_i}, a_{ij})_{i,j\in I} \to (\mathcal{F}|_{U_i}, b_{ij})_{i,j\in I}$. We obtain a morphism $\mathcal{T}_1 \otimes_{\mathcal{A}} \mathcal{F} \to \mathcal{T}_2 \otimes_{\mathcal{A}} \mathcal{F}$.

Suppose that $\mathcal{T}_3$ is a third right $\mathcal{A}$-torsor and $\beta\colon \mathcal{T}_2 \to \mathcal{T}_3$ a morphism. Without loss of generality $\mathcal{T}_3$ is trivialized by $\mathcal{U}$. Again choose elements $r_i \in \mathcal{T}_3(U_i)$ for all $i \in I$, and define $\mu_i, \nu_i \in \mathcal{A}(U_i)$ by $r_i\mu_i = \beta(t_i)$ and $r_i\nu_i = \beta\alpha(s_i)$. Then as above, $(\mu_i)_{i\in I}$ defines a morphism $\mathcal{T}_2 \otimes_{\mathcal{A}} \mathcal{F} \to \mathcal{T}_3 \otimes_{\mathcal{A}} \mathcal{F}$, and $(\nu_i)_{i\in I}$ defines a map $\mathcal{T}_1 \otimes_{\mathcal{A}} \mathcal{F} \to \mathcal{T}_3 \otimes_{\mathcal{A}} \mathcal{F}$. The identity $\nu_i = \mu_i\lambda_i$ now proves that $\mathcal{T} \otimes_{\mathcal{A}} \mathcal{F}$ is functorial in $\mathcal{T}$.

Functoriality in $\mathcal{F}$ is immediate since the choices made in the construction are all independent from $\mathcal{F}$. ■

By construction the pullbacks $\mathcal{F}|_{U_i}$ and $(\mathcal{T} \otimes_{\mathcal{A}} \mathcal{F})|_{U_i}$ are isomorphic for all $i \in I$. It is in fact this 'local isomorphism' that characterizes twists.

Proposition 3.7. *Let $\mathcal{C}$ be a site with final object S and $\mathcal{U} = \{U_i \to S\}_{i\in I}$ a covering of S. Let $\mathcal{D}$ be a category. Suppose that $\mathcal{F}$ and $\mathcal{G}$ are $\mathcal{D}$-valued sheaves on $\mathcal{C}$ such that $\mathcal{F}|_{U_i}$ is isomorphic to $\mathcal{G}|_{U_i}$ for all $i \in I$. Then $\mathcal{I}som(\mathcal{F}, \mathcal{G})$ is a right $\mathcal{A}ut\,\mathcal{F}$-torsor trivialized by $\mathcal{U}$, and $\mathcal{G}$ is isomorphic to the twist $\mathcal{I}som(\mathcal{F}, \mathcal{G}) \otimes_{\mathcal{A}ut\,\mathcal{F}} \mathcal{F}$.*

Proof. Fix isomorphisms $\alpha_i\colon \mathcal{F}|_{U_i} \to \mathcal{G}|_{U_i}$ for all $i \in I$. The maps

$$\mathsf{Aut}\,\mathcal{F}|_T \to \mathsf{Isom}(\mathcal{F}|_T, \mathcal{G}|_T), \quad \sigma \mapsto \alpha_i|_T \circ \sigma$$

for objects $T \in \mathcal{C}/U_i$ constitute an isomorphism $\mathcal{A}ut\,\mathcal{F}|_{U_i} \to \mathcal{I}som(\mathcal{F}|_{U_i}, \mathcal{G}|_{U_i})$ of right $\mathcal{A}ut\,\mathcal{F}|_{U_i}$-sheaves. So $\mathcal{I}som(\mathcal{F}, \mathcal{G})$ is a right $\mathcal{A}ut\,\mathcal{F}$-torsor trivialized by $\mathcal{U}$.

Let $\sigma_{ij} \in \mathsf{Aut}\,\mathcal{F}|_{U_{ij}}$ be given by $\alpha_j|_{U_{ij}} \circ \sigma_{ij} = \alpha_i|_{U_{ij}}$. We get a commutative diagram

$$\begin{tikzcd}
\mathcal{F}|_{U_{ij}} \arrow[r, "\alpha_i|_{U_{ij}}"] \arrow[d, "\sigma_{ij}"] & \mathcal{G}|_{U_{ij}} \arrow[d, "\mathsf{id}"] \\
\mathcal{F}|_{U_{ij}} \arrow[r, "\alpha_j|_{U_{ij}}"] & \mathcal{G}|_{U_{ij}}
\end{tikzcd}$$

hence $(\alpha_i)_{i\in I}$ is an isomorphism of descent data $(\mathcal{F}|_{U_i}, \sigma_{ij})_{i,j\in I} \to \mathcal{G}|_{\mathcal{U}}$. Therefore $\mathcal{G}$ is isomorphic to the twist $\mathcal{I}som(\mathcal{F}, \mathcal{G}) \otimes_{\mathcal{A}ut\,\mathcal{F}} \mathcal{F}$. ■

Suppose $\mathcal{A}$ is a sheaf of abelian groups acting on $\mathcal{F}$. Then $\mathcal{A} \to \mathcal{A}ut\,\mathcal{F}$ actually maps into the subsheaf $\mathcal{A}ut_{\mathcal{A}}\,\mathcal{F}$ of automorphisms of $\mathcal{F}$ as $\mathcal{A}$-sheaf. Let $\mathcal{T}$ be an $\mathcal{A}$-torsor. By the functoriality in lemma 3.6 there is a map $\mathcal{A}ut_{\mathcal{A}}\,\mathcal{F} \to \mathcal{A}ut(\mathcal{T} \otimes_{\mathcal{A}} \mathcal{F})$. Hence the twist $\mathcal{T} \otimes_{\mathcal{A}} \mathcal{F}$ again has a canonical $\mathcal{A}$-action.

We compute this $\mathcal{A}$-sheaf $\mathcal{T} \otimes_{\mathcal{A}} \mathcal{F}$ in case $\mathcal{F}$ is an $\mathcal{A}$-torsor as well. Let $\mathcal{U} = \{U_i \to S\}_{i\in I}$ be a common trivialization of $\mathcal{T}$ and $\mathcal{F}$. As usual, choose $t_i \in \mathcal{T}(U_i)$ and take $a_{ij} \in \mathcal{A}(U_{ij})$ satisfying $t_j|_{U_{ij}} \cdot a_{ij} = t_i|_{U_{ij}}$. Observe that $\mathcal{T}|_{\mathcal{U}} = (\mathcal{A}|_{U_i}, a_{ij})_{i,j\in I}$ as $\mathcal{A}$-descent data relative to $\mathcal{U}$. Similarly write $\mathcal{F}|_{\mathcal{U}} = (\mathcal{A}|_{U_i}, b_{ij})_{i,j\in I}$. Then the twist $\mathcal{T} \otimes_{\mathcal{A}} \mathcal{F}$ is given by the $\mathcal{A}$-descent datum $(\mathcal{A}|_{U_i}, a_{ij}b_{ij})_{i,j\in I}$. We see that $\mathcal{T} \otimes_{\mathcal{A}} \mathcal{F}$ is an $\mathcal{A}$-torsor itself, trivialized by $\mathcal{U}$. In fact, this leads to a group structure.

Prior to the precise statement, we establish some notation. For a sheaf of abelian groups $\mathcal{A}$, the set of isomorphism classes of $\mathcal{A}$-torsors is denoted by $\mathsf{Tors}\,\mathcal{A}$. If $\mathcal{U}$ is a covering of the final object S, then $\mathsf{Tors}(\mathcal{U}, \mathcal{A})$ denotes the subset of isomorphism classes of $\mathcal{A}$-torsors that are trivialized by $\mathcal{U}$. From this point onwards, the group structure on an abelian sheaf of groups is written additively.

Proposition 3.8. *Let $\mathcal{C}$ be a site with final object S and $\mathcal{A}$ a sheaf of abelian groups on $\mathcal{C}$. The twist operation $\otimes_{\mathcal{A}}$ makes* $\mathsf{Tors}\,\mathcal{A}$ *into an abelian group, with identity element the class of trivial $\mathcal{A}$-torsors. If $\mathcal{U}$ is a covering of S, then* $\mathsf{Tors}(\mathcal{U},\mathcal{A}) \subset \mathsf{Tors}\,\mathcal{A}$ *is a subgroup.*

Proof. We first prove that $\mathsf{Tors}(\mathcal{U},\mathcal{A})$ is an abelian group. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be $\mathcal{A}$-torsors with trivialization $\mathcal{U} = \{U_i \to S\}_{i\in I}$. We have seen that $\mathcal{T}_1$ and $\mathcal{T}_2$ are given by $\mathcal{A}$-descent data $(\mathcal{A}|_{U_i}, a_{ij})_{i,j\in I}$ and $(\mathcal{A}|_{U_i}, b_{ij})_{i,j\in I}$ for suitable $a_{ij}, b_{ij} \in \mathcal{A}(U_{ij})$. Moreover, the twist $\mathcal{T}_1 \otimes_{\mathcal{A}} \mathcal{T}_2$ is then given by the $\mathcal{A}$-descent datum $(\mathcal{A}|_{U_i}, a_{ij} + b_{ij})_{i,j\in I}$. Because $\mathcal{A}$ is abelian, it immediately follows that $\mathsf{Tors}(\mathcal{U},\mathcal{A})$ is an abelian group under the operation $\otimes_{\mathcal{A}}$.

Any finite number of $\mathcal{A}$-torsors has a common trivialization. Therefore the preceding arguments also prove that $\mathsf{Tors}\,A$ is an abelian group. Since $\otimes_{\mathcal{A}}$ is independent from the choice of trivialization, $\mathsf{Tors}(\mathcal{U},\mathcal{A}) \subset \mathsf{Tors}\,\mathcal{A}$ is a subgroup. ∎

The inverse of a torsor $\mathcal{T}$ in $\mathsf{Tors}\,\mathcal{A}$ can also be described directly as follows. There is a canonical isomorphism $\mathcal{A} \simeq \mathcal{A}ut_{\mathcal{A}}\,\mathcal{A}$, which induces an action of $\mathcal{A}$ on $\mathcal{I}som_{\mathcal{A}}(\mathcal{T},\mathcal{A})$. Under this action, $\mathcal{I}som_{\mathcal{A}}(\mathcal{T},\mathcal{A})$ is an $\mathcal{A}$-torsor, and the twist $\mathcal{I}som_{\mathcal{A}}(\mathcal{T},\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{T}$ is a trivial torsor.

3.3. Cohomology

The group $\mathsf{Tors}(\mathcal{U},\mathcal{A})$ can be computed as a Čech cohomology group.

Definition 3.9. Let $\mathcal{C}$ be a site and $\mathcal{U} = \{U_i \to S\}_{i\in I}$ a covering in $\mathcal{C}$. Let $\mathcal{A}$ be a sheaf of abelian groups on $\mathcal{C}$. Define for $n \geqslant 0$ the abelian group

$$C^n(\mathcal{U},\mathcal{A}) = \prod_{i_0,\dots,i_n\in I} \mathcal{A}(U_{i_0\dots i_n}).$$

The maps

$$C^n(\mathcal{U},\mathcal{A}) \to C^{n+1}(\mathcal{U},\mathcal{A}), \quad (a_{i_0\dots i_n})_{i_0,\dots,i_n\in I} \mapsto \left(\sum_{k=0}^{n+1} (-1)^k a_{i_0\dots\hat{i}_k\dots i_{n+1}}|_{U_{i_0\dots i_{n+1}}}\right)_{i_0,\dots,i_{n+1}\in I}$$

constitute a cochain complex $C^\bullet(\mathcal{U},\mathcal{A})$ called the *Čech complex* of $\mathcal{A}$ relative to $\mathcal{U}$. Its cohomology, written $\mathsf{H}^\bullet(\mathcal{U},\mathcal{A})$, is the *Čech cohomology* of $\mathcal{A}$ relative to $\mathcal{U}$.

We have $\mathsf{H}^0(\mathcal{U},\mathcal{A}) = \mathcal{A}(S)$ by the sheaf property.

Proposition 3.10. *Let $\mathcal{C}$ be a site with final object S and $\mathcal{U}$ a covering of S. Let $\mathcal{A}$ be a sheaf of abelian groups on $\mathcal{C}$. There is a canonical isomorphism* $\mathsf{H}^1(\mathcal{U},\mathcal{A}) \simeq \mathsf{Tors}(\mathcal{U},\mathcal{A})$.

Proof. Let Z be the kernel of $C^1(\mathcal{U},\mathcal{A}) \to C^2(\mathcal{U},\mathcal{A})$ and B the image of $C^0(\mathcal{U},\mathcal{A}) \to C^1(\mathcal{U},\mathcal{A})$. We construct a map $\rho\colon Z \to \mathsf{Tors}(\mathcal{U},\mathcal{A})$ as follows. Write $\mathcal{U} = \{U_i \to S\}_{i\in I}$. Elements of Z are tuples $(a_{ij})_{i,j\in I}$ with $a_{ij} \in \mathcal{A}(U_{ij})$ for $i,j \in I$, satisfying the cocycle condition

$$a_{jk}|_{U_{ijk}} - a_{ik}|_{U_{ijk}} + a_{ij}|_{U_{ijk}} = 0$$

for all $i,j,k \in I$. Let $\rho(a_{ij})_{i,j\in I}$ be the isomorphism class of $\mathcal{A}$-torsors corresponding to the $\mathcal{A}$-descent datum $(\mathcal{A}|_{U_i}, a_{ij})_{i,j\in I}$. This makes ρ into a surjective group homomorphism.

Suppose $(a_{ij})_{i,j\in I}$ is in the kernel of ρ, that is, $(\mathcal{A}|_{U_i}, a_{ij})_{i,j\in I}$ defines a trivial $\mathcal{A}$-torsor. Then there exist elements $b_i \in \mathcal{A}(U_i)$ for $i \in I$ such that

$$\begin{tikzcd} \mathcal{A}|_{U_{ij}} \arrow[r, "b_i|_{U_{ij}}"] \arrow[d, "a_{ij}"] & \mathcal{A}|_{U_{ij}} \arrow[d, "\mathsf{id}"] \\ \mathcal{A}|_{U_{ij}} \arrow[r, "b_j|_{U_{ij}}"] & \mathcal{A}|_{U_{ij}} \end{tikzcd}$$

is a commutative square for all $i, j \in I$. In other words, there exist $b_i \in \mathcal{A}(U_i)$ satisfying the identities $a_{ij} = b_i|_{U_{ij}} - b_j|_{U_{ij}}$. This means precisely that $(a_{ij})_{i,j\in I}$ is in B. Hence we have $\ker \rho = B$, and ρ induces an isomorphism $Z/B \to \mathsf{Tors}(\mathcal{U}, \mathcal{A})$. ∎

One can also construct the 'absolute' cohomology groups $\mathsf{H}^\bullet(\mathcal{A})$ that do not depend on a particular covering, with maps $\mathsf{H}^\bullet(\mathcal{U}, \mathcal{A}) \to \mathsf{H}^\bullet(\mathcal{A})$ for every covering $\mathcal{U}$. There is a canonical isomorphism $\mathsf{H}^1(\mathcal{A}) \simeq \mathsf{Tors}\,\mathcal{A}$, compatible with the isomorphisms $\mathsf{H}^1(\mathcal{U}, \mathcal{A}) \simeq \mathsf{Tors}(\mathcal{U}, \mathcal{A})$. Consult [G, III.3.6] for more details. Here we do not require this generality.

In the situation of definition 3.9, suppose that I is endowed with a total order $<$. Define the abelian groups

$$C^n_<(\mathcal{U}, \mathcal{A}) = \prod_{\substack{i_0,\ldots,i_n\in I \\ i_0<\ldots<i_n}} \mathcal{A}(U_{i_0\ldots i_n})$$

with obvious maps $C^n_<(\mathcal{U}, \mathcal{A}) \to C^{n+1}_<(\mathcal{U}, \mathcal{A})$. Then $C^\bullet_<(\mathcal{U}, \mathcal{A})$ is again a cochain complex, called the *small Čech complex* of $\mathcal{A}$ relative to $\mathcal{U}$. In general its cohomology $\mathsf{H}^\bullet_<(\mathcal{U}, \mathcal{A})$ is different from the 'full' cohomology $\mathsf{H}^\bullet(\mathcal{U}, \mathcal{A})$. However, on the Zariski site they coincide.

Lemma 3.11. *Let S be a scheme and $\mathcal{U}$ a Zariski covering of S endowed with a total order $<$. Let $\mathcal{A}$ be a sheaf of abelian groups on $(\mathsf{Sch}/S)_{\mathsf{Zar}}$. Then $\mathsf{H}^\bullet(\mathcal{U}, \mathcal{A})$ is canonically isomorphic to $\mathsf{H}^\bullet_<(\mathcal{U}, \mathcal{A})$.*

Proof. The single ingredient is the fact that if $U \to S$ is an open immersion, then the two projections $U \times_S U \to U$ coincide and are isomorphisms. A complete proof, however, would be long and technical. Details can be found in [S, Proposition I.3.2]. ∎

3.4. Representability of torsors

In this section we return to the étale site $(\mathsf{Sch}/S)_{\text{ét}}$. In particular, we discuss conditions under which torsors on $(\mathsf{Sch}/S)_{\text{ét}}$ are representable, or are not representable. We introduce some terminology.

Definition 3.12. Let $\mathcal{C}$ be a category. A *group object* in $\mathcal{C}$ is a functor $\mathcal{C}^{\mathsf{op}} \to \mathsf{Grp}$ such that the induced functor $\mathcal{C}^{\mathsf{op}} \to \mathsf{Set}$ is representable. A group object is *commutative* if the functor $\mathcal{C}^{\mathsf{op}} \to \mathsf{Grp}$ maps into the subcategory $\mathsf{Ab} \subset \mathsf{Grp}$.

Equivalently, a group object is a functor $\mathcal{C}^{\mathsf{op}} \to \mathsf{Grp}$, together with an object $A \in \mathcal{C}$ and an isomorphism from the induced functor $\mathcal{C}^{\mathsf{op}} \to \mathsf{Set}$ to the Yoneda functor h_A. A group object in the category Sch/S is called a *group scheme* over S.

Definition 3.13. Let S be a scheme. An *abelian scheme* over S is a proper smooth group scheme over S with geometrically connected fibers.

Lemma 3.14. *The group scheme structure on abelian schemes is commutative.*

Proof. See [MFK, Corollary 6.5]. ■

We are interested in group schemes that are curves. Recall from definition 1.13 that we suppose curves to be proper and smooth with geometrically connected fibers. An *elliptic curve* over a scheme S is a curve over S endowed with the structure of a group scheme. From the definitions, elliptic curves are abelian schemes. Moreover, elliptic curves are necessarily curves of genus 1.

We are now able to state the main result of this section. Its proof is beyond the scope of this thesis.

Theorem 3.15 (Raynaud). *Let S be a local scheme and A an abelian scheme over S. Let $\mathcal{T}$ be an étale A-torsor and $\mathcal{U}$ an étale covering of S that trivializes $\mathcal{T}$.*

- *Suppose S is normal. Then $\mathcal{T}$ is representable if and only if the class of $\mathcal{T}$ in $\mathsf{H}^1(\mathcal{U}, A)$ is torsion. In the representable case, $\mathcal{T}$ is represented by a projective scheme over S.*
- *Suppose S is regular. Then $\mathcal{T}$ is representable by a projective scheme over S.*

Proof. See [R, Proposition XIII.2.6]. ■

The preceding theorem is key in proving theorem II. Indeed, it suggests the following approach. Suppose we construct an étale torsor $\mathcal{T}$ under an elliptic curve E over S, for some normal local scheme S. Note that S must not be too nice, according to the second part of the theorem. Suppose moreover that $\mathcal{T}$ has infinite order in $\mathsf{H}^1(\mathcal{U}, E)$, where $\mathcal{U} = \{U_i \to S\}_{i \in I}$ is a trivialization of $\mathcal{T}$. Then $\mathcal{T}$ is a non-representable sheaf of sets on S. However, for all $i \in I$ the pullback $\mathcal{T}|_{U_i}$ is representable by $E \times_S U_i$, which is a curve of genus 1 over U_i. Hence, $\mathcal{T}$ is an example of theorem II.

Raynaud's original proof of theorem II followed this pattern [R, Exemple XIII.3.2]. In the next chapter we present a new construction, applying the same method. We compare the two constructions in section 4.3.

Descent in genus 1

4

4.1. Setup

In this chapter we construct an example of non-effective étale descent of curves of genus 1, that is, we prove theorem II. The first section is preparatory, establishing the required objects and their basic properties. Section 4.2 then constructs a non-representable torsor and finishes the proof.

Definition 4.1. We introduce the following objects:
- k a field of characteristic unequal to 2,
- $E_0 = \mathrm{Proj}\, k[x, y, z]/(f)$ an elliptic curve in $\mathbb{P}^2_k$, where f is an irreducible cubic over k,
- $C = \mathrm{Spec}\, k[x, y, z]/(f)$ the affine cone over E_0,
- $s \in C$ the closed point corresponding to $(0, 0, 0) \in C(k)$,
- $S = \mathrm{Spec}\, \mathcal{O}_{C,s}$ the localization of C at s,
- $E = E_0 \times_k S$ the elliptic curve in $\mathbb{P}^2_S$ defined by f.

Pictorially, s is the top of the cone C. The scheme S, which is the intersection of all open neighborhoods of s, will serve as base scheme. It consists of all points of C that contain s in their closure.

Lemma 4.2. *The scheme S is local, normal and non-regular.*

Proof. By construction S is local. The cone C is regular at all points except s. In particular, S is not regular. We apply Serre's criterion for normality [M, Theorem 23.8] to prove that S is normal: a locally noetherian connected scheme X is normal if and only if
- X is regular in codimension 1, and
- for every point $p \in X$ we have $\mathrm{depth}\, \mathcal{O}_{X,p} \geqslant \min(2, \dim \mathcal{O}_{X,p})$.

Here the *depth* of a noetherian local ring R is the largest integer n for which there exist elements $a_1, \ldots, a_n$ in the maximal ideal of R such that multiplication by a_i is injective on $R/(a_1, \ldots, a_{i-1})$ for $i = 1, \ldots, n$. Our scheme S is noetherian and connected. Its single non-regular point s has codimension 2, so the first condition is satisfied. Since S is a complete intersection over k we have $\mathrm{depth}\, \mathcal{O}_{S,p} = \dim \mathcal{O}_{S,p}$ for any point $p \in S$, see [M, Theorem 21.3]. We conclude that S is normal. ■

Definition 4.3. We introduce the following objects:
- $u \in \mathcal{O}^\times_{C,s}$ not a square,
- $S' = \mathrm{Spec}\, \mathcal{O}_{C,s}[\sqrt{u}]$,

chosen such that $S' \to S$ is a finite étale covering that splits above s,
- σ the automorphism of S' over S induced by $\sqrt{u} \mapsto -\sqrt{u}$,
- $s_0, s_1 \in S'$ the closed points above s,
- $U_0 = S' \backslash \{s_1\}$ and $U_1 = S' \backslash \{s_0\}$,
- $U_{01} = U_0 \cap U_1$,

under the further condition that all morphisms $U_0 \to E_0$ over $\mathrm{Spec}\, k$ are constant.

We verify that these conditions can be fulfilled; the only choice is in u. Write $A = \mathcal{O}_S(S)$ and $B = \mathcal{O}_{S'}(S')$. First we show that for any choice of $u \in A^\times$ the map $S' \to S$ is a finite étale covering. As A-modules we have $B \simeq A \oplus A\sqrt{u}$, which proves that S' is finite over S. The presentation $B = A[t]/(t^2 - u)$ makes $A \to B$ standard étale, because $\operatorname{char} k$ is not 2 and u is a unit. Therefore S' is étale over S. Let $\mathfrak{p} \subset A$ be a prime ideal. The corresponding fiber in S' is $\operatorname{Spec}(A/\mathfrak{p})[t]/(t^2 - v)$, where v is the image of u in $A/\mathfrak{p}$. In particular the fiber is non-empty, so $S' \to S$ is surjective.

Now put $u = 1 + x \in A^\times$. We claim that it satisfies the conditions in definition 4.3. For one thing, u is not a square in A, as A is a graded ring and the highest non-zero homogeneous component of u has the odd degree 1. We have $u \equiv 1$ modulo the maximal ideal (x, y, z) of A. Hence the fiber of $S' \to S$ above s equals $\operatorname{Spec}(k \times k)$, consisting of two closed points.

The following lemma helps us prove that all k-morphisms $U_0 \to E_0$ are constant.

Lemma 4.4. *Suppose k is algebraically closed. There is a dense subset of U_0 consisting of points $p \in U_0$ that contain s_0 in their closure and have residue field $\kappa(p) \simeq k(t)$.*

Proof. Take $a, b, c \in k$ with $f(a, b, c) = 0$ and $a \neq 0$. Let

$$L = \operatorname{Spec} k[x, y, z]/(bx - ay, cy - bz, az - cx)$$

be the corresponding line in C. Then L is isomorphic to $\mathbb{A}^1_k$, with inclusion $i\colon L \to C$ given by

$$i^*\colon k[x, y, z]/(f) \to k[\lambda], \qquad \begin{cases} x \mapsto a\lambda, \\ y \mapsto b\lambda, \\ z \mapsto c\lambda. \end{cases}$$

We deduce $L \times_C S' = \operatorname{Spec} k[\lambda]_{(\lambda)}[t]/(t^2 - 1 - a\lambda)$. Since a is non-zero, $1 + a\lambda$ is not a square in $k[\lambda]_{(\lambda)}$. Therefore $k[\lambda]_{(\lambda)}[t]/(t^2 - 1 - a\lambda)$ is an integral domain. We conclude that $L \times_C S'$ is an integral closed subscheme of S' with function field $k(t)$. Let p be the generic point of $L \times_C S'$. Then p lies in U_0, its closure contains s_0, and its residue field is $k(t)$. So p has the desired properties. Since k is algebraically closed, the set of points $p \in U_0$ obtained as above for varying a, b, c is dense in U_0. ∎

Let $\alpha\colon U_0 \to E_0$ be a morphism over $\operatorname{Spec} k$. We may assume that k is algebraically closed, after base change. Let $p \in U_0$ be as in lemma 4.4. Its image $\alpha(p)$ can not be the generic point of E_0, since there is no map from the function field of E_0 to $k(t)$. So the image of p is a closed point $q \in E_0$. Since s_0 lies in the closure of p, we also have $\alpha(s_0) = q$. Varying p, we see that α maps a dense subset of U_0 to q. The preimage of q is closed, so in fact α is constant. Hence, objects as in definition 4.3 exist.

Lemma 4.5. *The square*

$$\begin{tikzcd} S' \sqcup S' \arrow[r, "\mathrm{id}, \sigma"] \arrow[d, "\mathrm{id}, \mathrm{id}"] & S' \arrow[d] \\ S' \arrow[r] & S \end{tikzcd}$$

is cartesian.

Proof. Write $A = \mathcal{O}_S(S)$ and $B = \mathcal{O}_{S'}(S')$. Then we have an isomorphism $B \otimes_A B \simeq B \times B$, and the claim immediately follows. ∎

Next, we construct another elliptic curve $\tilde{E}$ over S. By a common abuse of notation, write S' for the Yoneda functor $\mathsf{h}_{S'/S} \colon (\mathsf{Sch}/S)^{\mathsf{op}} \to \mathsf{Set}$. The automorphism sheaf $\mathcal{A}ut\, S'$ is often called the *Galois group* of S' over S. Observe that $\mathcal{A}ut\, S'$ is isomorphic to the constant sheaf $\mathbb{Z}/2\mathbb{Z}$, as defined at the end of section 1.2. In particular, $\mathcal{A}ut\, S'$ is abelian.

The canonical action of $\mathcal{A}ut\, S'$ on S' makes S' into a torsor with trivialization $\{S' \to S\}$. On the other hand, $\mathcal{A}ut\, S'$ acts, via its identification with the constant sheaf $\mathbb{Z}/2\mathbb{Z}$, on E by multiplication with -1.

Definition 4.6. We define $\tilde{E} = S' \otimes_{\mathcal{A}ut\, S'} E$.

A priori $\tilde{E}$ is just a sheaf of groups. However, we proved at the end of chapter 2 that $\tilde{E}$ is representable by an elliptic curve over S, which we denote again by $\tilde{E}$. Here we can actually describe $\tilde{E}$ directly. Remark first that, since $\mathsf{char}\, k$ is not 2, after a change of variables we may assume that E is given by a cubic of the form $y^2z - x^3 - ax^2z - bxz^2 - cz^3$.

Proposition 4.7. *Suppose that $E \subset \mathbb{P}^2_S$ is given by $f = y^2z - x^3 - ax^2z - bxz^2 - cz^3$. Then $\tilde{E}$ is the elliptic curve in $\mathbb{P}^2_S$ given by $\tilde{f} = uy^2z - x^3 - ax^2z - bxz^2 - cz^3$.*

Proof. Lemma 4.5 gives $S'(S' \times_S S') = S'(S')^2$ and $(\mathcal{A}ut\, S')(S' \times_S S') = (\mathsf{Aut}\, S')^2$. The two pullbacks of $\mathsf{id} \in S'(S')$ to $S'(S' \times_S S')$ differ by $(\mathsf{id}, \sigma) \in (\mathsf{Aut}\, S')^2$. Hence the twist $\tilde{E}$ is given by the descent datum $(E_{S'}, (\mathsf{id}, \sigma))$ relative to $\{S' \to S\}$.

Write E_1 for the elliptic curve in $\mathbb{P}^2_S$ defined by $\tilde{f}$. We prove that E_1 is isomorphic to $\tilde{E}$. Let T be a scheme over S. Following the proof of proposition 1.11, $\tilde{E}(T)$ consists of precisely those $e \in E(S' \times_S T)$ for which the respective pullbacks (e, e) and $(e, \sigma^* e)$ to $E(S' \times_S T)^2$ differ by the action of (id, σ). We obtain

$$\tilde{E}(T) = \Big\{ e \in E(S' \times_S T) : \sigma^* e = -e \Big\}.$$

An element e of $E(S' \times_S T)$ can be given as a line bundle $\mathcal{L}$ on $S' \times_S T$ together with global sections $r_0, r_1, r_2 \in \mathcal{L}(S' \times_S T)$ that generate $\mathcal{L}$ and satisfy $f(r_0, r_1, r_2) = 0$. In this description, the condition $\sigma^* e = -e$ means that the tuple $(\mathcal{L}, \sigma^* r_0, \sigma^* r_1, \sigma^* r_2)$ is isomorphic to $(\mathcal{L}, r_0, -r_1, r_2)$. Equivalently, the tuple $(\mathcal{L}, r_0, r_1/\sqrt{u}, r_2)$ defines an element d of $E_1(S' \times_S T)$ invariant under σ^*. The invariance means that d comes from an element of $E_1(T)$.

We have a bijection from $\tilde{E}(T)$ to $E_1(T)$. In fact it is a group isomorphism. The isomorphism is clearly functorial in T, hence $\tilde{E} \simeq E_1$. ∎

4.2. A non-torsion torsor

We are about to construct an $\tilde{E}$-torsor on S with trivialization $\mathcal{U} = \{U_0 \to S\}$. The following lemma is useful in the computation of $\mathsf{H}^1(\mathcal{U}, \tilde{E})$.

Lemma 4.8. *There is an isomorphism $U_0 \times_S U_0 \simeq U_0 \sqcup U_{01}$ under which the projection maps π_0, π_1 are given by*

$$\pi_0 : \begin{tikzcd} U_0 \arrow[dr] & \sqcup & U_{01} \arrow[dl] \\ & U_0 & \end{tikzcd} \qquad \pi_1 : \begin{tikzcd} U_0 \arrow[dr] & \sqcup & U_{01} \arrow[dl, squiggly] \\ & U_0 & \end{tikzcd}$$

where a squiggly arrow denotes application of σ. There is also an isomorphism

$$U_0 \times_S U_0 \times_S U_0 \simeq U_0 \sqcup U_{01} \sqcup U_{01} \sqcup U_{01}$$

under which the projection maps $\pi_{01}, \pi_{02}, \pi_{12}$ are given by

$$\pi_{01}\colon \begin{tikzcd} U_0 \arrow[drr] & \sqcup & U_{01} \arrow[d] & \sqcup & U_{01} \arrow[d] & \sqcup & U_{01} \arrow[dll] \\ & & U_0 & \sqcup & U_{01} & & \end{tikzcd} \quad \pi_{02}\colon \begin{tikzcd} U_0 \arrow[drr] & \sqcup & U_{01} \arrow[drr] & \sqcup & U_{01} \arrow[dll] & \sqcup & U_{01} \arrow[dll] \\ & & U_0 & \sqcup & U_{01} & & \end{tikzcd} \quad \pi_{12}\colon \begin{tikzcd} U_0 \arrow[drr] & \sqcup & U_{01} \arrow[drr] & \sqcup & U_{01} \arrow[d, squiggly] & \sqcup & U_{01} \arrow[dllll, squiggly] \\ & & U_0 & \sqcup & U_{01}. & & \end{tikzcd}$$

Proof. The statements are direct consequences of lemma 4.5. ∎

We can now compute the cohomology group $\mathsf{H}^1(\mathcal{U}, \tilde{E})$. Let $\tilde{E}(U_{01})^-$ denote the subgroup of $\tilde{E}(U_{01})$ of elements that are anti-invariant under σ^*, that is, elements satisfying $\sigma^* e = -e$.

Proposition 4.9. *The cohomology group $\mathsf{H}^1(\mathcal{U}, \tilde{E})$ is isomorphic to $\tilde{E}(U_{01})^-$.*

Proof. Lemma 4.8 yields

$$\begin{aligned} \tilde{E}(U_0 \times_S U_0) &\simeq \tilde{E}(U_0) \times \tilde{E}(U_{01}), \\ \tilde{E}(U_0 \times_S U_0 \times_S U_0) &\simeq \tilde{E}(U_0) \times \tilde{E}(U_{01})^3 \end{aligned}$$

with projection maps

$$\begin{aligned} &\pi_0^*\colon \tilde{E}(U_0) \to \tilde{E}(U_0) \times \tilde{E}(U_{01}), && e \mapsto (e, e|_{U_{01}}), \\ &\pi_1^*\colon \tilde{E}(U_0) \to \tilde{E}(U_0) \times \tilde{E}(U_{01}), && e \mapsto (e, \sigma^* e|_{U_{01}}), \\ &\pi_{01}^*\colon \tilde{E}(U_0) \times \tilde{E}(U_{01}) \to \tilde{E}(U_0) \times \tilde{E}(U_{01})^3, && (d,e) \mapsto (d, d|_{U_{01}}, e, e), \\ &\pi_{02}^*\colon \tilde{E}(U_0) \times \tilde{E}(U_{01}) \to \tilde{E}(U_0) \times \tilde{E}(U_{01})^3, && (d,e) \mapsto (d, e, d|_{U_{01}}, e), \\ &\pi_{12}^*\colon \tilde{E}(U_0) \times \tilde{E}(U_{01}) \to \tilde{E}(U_0) \times \tilde{E}(U_{01})^3, && (d,e) \mapsto (d, e, \sigma^* e, \sigma^* d|_{U_{01}}). \end{aligned}$$

We consider the Čech complex $C^\bullet(\mathcal{U}, \tilde{E})$ in degrees 0, 1 and 2:

$$\tilde{E}(U_0) \xrightarrow{\quad \delta^0 \quad} \tilde{E}(U_0) \times \tilde{E}(U_{01}) \xrightarrow{\quad \delta^1 \quad} \tilde{E}(U_0) \times \tilde{E}(U_{01})^3.$$

Let Z be the kernel of $\delta^1 = \pi_{12}^* - \pi_{02}^* + \pi_{01}^*$. We find

$$Z = \Big\{ (d,e) \in \tilde{E}(U_0) \times \tilde{E}(U_{01}) : d = 0,\ \sigma^* e = -e \Big\} \simeq \tilde{E}(U_{01})^-.$$

We claim that δ^0 is the zero map. An element $e \in \tilde{E}(U_0)$ maps to $(0, \sigma^* e - e)$. There are isomorphisms $\tilde{E}(U_0) \simeq E(U_0) \simeq E_0(U_0)$, so by construction e is constant. In particular, $\sigma^* e$ equals e and $\delta^0(e) = 0$. Now $\mathsf{H}^1(\mathcal{U}, \tilde{E})$ is isomorphic to Z and that concludes the proof. ∎

At last, we construct an $\tilde{E}$-torsor on S. Consider first the tautological element $\tau_0 \in E(C \backslash \{s\})$ that maps a point (a,b,c) on $C(k) \backslash \{(0,0,0)\}$ to the point $(a : b : c) \in E(k)$. More precisely, τ_0 is given by the trivial line bundle on $C \backslash \{s\}$ and the generating global sections x, y, z.

We have maps $E(C\backslash\{s\}) \to E(S\backslash\{s\}) \to E(U_{01}) \to \tilde{E}(U_{01})$. The last map comes from the non-canonical isomorphism $E|_{S'} \simeq \tilde{E}|_{S'}$ and is described in the proof of proposition 4.7. Let $\tau \in \tilde{E}(U_{01})$ be the image of τ_0 under this composition. It is defined by the trivial line bundle on U_{01} and the generating global sections $x, y/\sqrt{u}, z$. We see that τ is anti-invariant under σ^*.

By propositions 3.10 and 4.9, we have an $\tilde{E}$-torsor $\mathcal{T}$ associated with τ. We can actually describe $\mathcal{T}$ explicitly:

$$\mathcal{T}(T) = \left\{ e \in \tilde{E}(U_0 \times_S T) : \sigma^* e|_{U_{01} \times_S T} - e|_{U_{01} \times_S T} = \tau|_{U_{01} \times_S T} \right\}$$

for every scheme T over S, with the obvious $\tilde{E}$-action.

Proposition 4.10. *The $\tilde{E}$-torsor $\mathcal{T}$ is not representable.*

Proof. Multiples of τ are not zero in $\tilde{E}(U_{01})^-$, so $\mathcal{T}$ has infinite order in $\mathsf{H}^1(\mathcal{U}, \tilde{E})$. Combine this with theorem 3.15 and lemma 4.2. ∎

As explained at the end of chapter 3, this concludes the proof of theorem II.

4.3. Raynaud's construction

The first example of a non-representable étale torsor under an elliptic curve was provided by Raynaud [R, Exemple XIII.3.2]. Although the details are very different from our exposition above, the constructions are comparable on a higher level. Raynaud himself provides an outline for his construction, as follows:

- S a normal noetherian local scheme,
- $S' \to S$ a connected étale covering of degree 2 with two closed points $s_0, s_1 \in S'$,
- $U_0 = S'\backslash\{s_1\}$ and $U_1 = S'\backslash\{s_0\}$ and $U_{01} = U_0 \cap U_1$,
- E an elliptic curve over S
- $E' = E \times_S S'$ its base change to S',
- $\tau' \in E'(U_{01})$ with no multiple of τ' of the form $d|_{U_{01}} - e|_{U_{01}}$ with $d \in E'(U_0)$, $e \in E'(U_1)$,
- $\mathcal{T}'$ the E'-torsor with Zariski trivialization $\mathcal{V}' = \{U_0 \to S', U_1 \to S'\}$ defined by τ'.

Lemma 3.11 states that $\mathcal{T}'$ is fully defined by τ'. By construction, the class of $\mathcal{T}'$ in $\mathsf{H}^1(\mathcal{V}', E')$ is not torsion.

Let A be the *Weil restriction* of E' to S. As sheaf, A is defined by $A(T) = E'(T \times_S S')$ for every scheme T over S; in fact, this sheaf A is representable by a 2-dimensional abelian scheme over S. Similarly let $\mathcal{T}$ be the Weil restriction of $\mathcal{T}'$ to S. It is an A-torsor with étale trivialization $\mathcal{V} = \{U_0 \to S, U_1 \to S\}$, whose class in $\mathsf{H}^1(\mathcal{V}, A)$ is not torsion.

Let $\tilde{E}$ be the twist of E as before. There is an exact sequence of abelian schemes

$$0 \longrightarrow E \longrightarrow A \longrightarrow \tilde{E} \longrightarrow 0$$

which induces an exact sequence of cohomology groups

$$0 \longrightarrow \mathsf{H}^1(\mathcal{V}, E) \longrightarrow \mathsf{H}^1(\mathcal{V}, A) \longrightarrow \mathsf{H}^1(\mathcal{V}, \tilde{E}).$$

Since $\mathsf{H}^1(\mathcal{V}, A)$ contains an element of infinite order, either $\mathsf{H}^1(\mathcal{V}, E)$ or $\mathsf{H}^1(\mathcal{V}, \tilde{E})$ contains an element of infinite order. By theorem 3.15, there exists either a non-representable E-torsor or a non-representable $\tilde{E}$-torsor.

After this outline, Raynaud continues to provide data that satisfy the requirements above. His techniques are different from our construction, and more involved. However, our construction can be modified to fit the given outline.

The construction in this thesis has two improvements upon the outline above. First, the detour via torsors on S' is omitted; the redundant étale covering $\mathcal{V} = \{U_0 \to S, U_1 \to S\}$ is reduced to $\mathcal{U} = \{U_0 \to S\}$. Second, Raynaud proves that there exists a non-representable torsor under either E or $\tilde{E}$. In our construction, we actually obtain a non-representable torsor under $\tilde{E}$, and we write it down explicitly.

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