Descent on the étale site

Wouter Zomervrucht, October 14, 2014

We treat two features of the étale site: descent of morphisms and descent of quasi-coherent sheaves. All will also be true on the larger fppf and fpqc sites.

1. Descent of morphisms

Let S be a scheme. Recall: the étale site $(\mathrm{Sch}/S)_{\mathrm{et}}$ is the category Sch/S together with the étale coverings; a sheaf on $(\mathrm{Sch}/S)_{\mathrm{et}}$ is a functor $(\mathrm{Sch}/S)^{\mathrm{op}} \to \mathrm{Set}$ that satisfies the sheaf property for all étale coverings.

Theorem 1.1 (Grothendieck). *Let X be a scheme over S. Then*

$$X\colon (\mathrm{Sch}/S)^{\mathrm{op}} \to \mathrm{Set}, \quad T \mapsto X(T) := \mathrm{Hom}_S(T, X)$$

is a sheaf on $(\mathrm{Sch}/S)_{\mathrm{et}}$. ♦

Corollary 1.2. *Morphisms can be defined étale-locally on the source.* ♦

Corollary 1.3. *Morphisms can be defined étale-locally on S.* ♦

The first corollary is actually a reformulation of the theorem. It generalizes the rather trivial operation of gluing morphisms along opens.

Proof. Reduction to the affine case. Let $V \to U$ be an étale covering. Since morphisms glue along opens, we may assume U is affine. Since U is quasi-compact and étale morphisms are open, there is a finite collection of affine opens $W_1, \ldots, W_n \subseteq V$ that jointly surject to U. Then also $W = \coprod_{i=1}^n W_i$ is affine and $W \to U$ is an étale covering.

Take $y \in X(V)$ with equal pullbacks to $X(V \times_U V)$. Assuming that X has the sheaf property for $W \to U$, there is a unique $x \in X(U)$ with $x|_W = y|_W$. It is the sole candidate to satisfy $x|_V = y$. We claim that this indeed is true. The identity $x|_V = y$ may be verified on an open covering. It holds certainly on each W_i. Now just observe that x does not depend on the choice of W; compare choices W, W' via $W \sqcup W'$.

Homological algebra. We have reduced to étale coverings $W \to U$ with U, W affine. Write $U = \operatorname{Spec} A$ and $W = \operatorname{Spec} B$. Then $A \to B$ is faithfully flat. Define $r\colon B \to B \otimes_A B$ by $r(b) = 1 \otimes b - b \otimes 1$. We claim that the complex

$$0 \longrightarrow A \longrightarrow B \xrightarrow{\ r\ } B \otimes_A B$$

is an exact sequence of A-modules. Since $A \to B$ is faithfully flat, it suffices to prove that

$$0 \longrightarrow B \longrightarrow B \otimes_A B \xrightarrow{\ r_B\ } B \otimes_A B \otimes_A B$$

is an exact sequence of B-modules. Here the middle map is given by $b \mapsto 1 \otimes b$ and r_B by $b \otimes c \mapsto 1 \otimes b \otimes c - b \otimes 1 \otimes c$. But now the maps

$$B \otimes_A B \to B, \qquad b \otimes c \mapsto bc,$$
$$B \otimes_A B \otimes_A B \to B \otimes_A B, \qquad b \otimes c \otimes d \mapsto c \otimes bd$$

show that the identity on this complex is null-homotopic. So the complex is exact.

Conclusion of the proof. We have to show that any commutative diagram

$$\begin{array}{ccc} W \times_U W & \longrightarrow & W \\ \downarrow & & \downarrow \\ W & \longrightarrow & U \end{array} \qquad f\colon W \to X, \quad g\colon U \dashrightarrow X$$

can be extended uniquely by a map $g\colon U \to X$. Reasoning as before, we may assume X is affine. Write $X = \operatorname{Spec} R$. Since $A \to B$ is the equalizer of the coprojections $B \to B \otimes_A B$ in A-Mod, there is a unique A-module homomorphism $g^*\colon R \to A$ compatible with f. It is automatically a ring map. ■

Example 1.4. The following functors $\mathrm{Sch}^{\mathrm{op}} \to \mathrm{Ab}$ are étale sheaves.

- $\mathbb{G}_\mathrm{a}(T) = \mathcal{O}(T)$ is representable by $\operatorname{Spec} \mathbb{Z}[t]$.
- $\mathbb{G}_\mathrm{m}(T) = \mathcal{O}(T)^\times$ is representable by $\operatorname{Spec} \mathbb{Z}[t, t^{-1}]$.
- $\mu_n(T) = \mathcal{O}(T)^\times[n]$ is representable by $\operatorname{Spec} \mathbb{Z}[t]/(t^n - 1)$. ♦

Example 1.5. Let A be an abelian group. The constant presheaf $(\mathrm{Sch}/S)^{\mathrm{op}} \to \mathrm{Ab}$, $T \mapsto A$ is not a sheaf on $(\mathrm{Sch}/S)_{\mathrm{zar}}$. Its Zariski sheafification is

$$\underline{A}\colon (\mathrm{Sch}/S)^{\mathrm{op}} \to \mathrm{Ab}, \quad T \mapsto \{\text{continuous functions } T \to A\}.$$

But $\underline{A}$ is representable by $\coprod_{a \in A} S$, so is also the étale sheafification of the constant presheaf. ♦

Many properties of morphisms are preserved under descent.

Theorem 1.6. *Let $f\colon X \to Y$ be a morphism of schemes over S. Let $S' \to S$ be an étale surjection, and let $f'\colon X' \to Y'$ be the base change of f. If f' is*

- *an isomorphism*
- *universally open*
- *universally closed*
- *quasi-compact*
- *separated*
- *(locally) of finite type*
- *(locally) of finite presentation*
- *proper*
- *affine*
- *finite*
- *an open immersion*
- *a closed immersion*
- *flat*
- *smooth*
- *unramified*
- *étale*

then so is f.

Warning: (quasi-)projectivity is not preserved in general!

Proof. We may assume $Y = S$. Each of the properties is Zariski local on the target, so we may assume S is affine. There is an affine scheme S'', étale over S', that surjects to S. Each of the properties is stable under base change, so $f''\colon X'' \to S''$ has the property as well. Thus we may assume S' is affine. Summarizing, let a cartesian diagram

$$\begin{array}{ccc} X' & \xrightarrow{f'} & S' \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow{f} & S \end{array}$$

of schemes be given, in which p is an étale surjection of affine schemes. For each of the properties we have to show that if f' has the property, then so does f.

Isomorphism. Suppose f' is an isomorphism. Let $g' \colon S' \to X'$ be its inverse. Its two pullbacks to $S'' \to X''$ are inverse to f'', so identical, hence by theorem 1.1 g' descends to a morphism $g \colon S \to X$. Étale-locally both fg and gf are the identity. By the sheaf property, fg and gf are the identity globally, hence g is an inverse to f and f is an isomorphism.

Universally open. Suppose f' is universally open. Let $T \to S$ be any morphism, and let $U \subseteq X_T$ be open. Then $p_T^{-1}(f_T(U)) = f'_T(q_T^{-1}(U))$ is open in S'_T. Since étale surjections are topological quotients, this implies $f_T(U)$ is open in S_T.

Universally closed. Analogous.

Quasi-compact. A morphism to an affine scheme is quasi-compact if and only if the source is quasi-compact. Since q is surjective, if X' is quasi-compact then so is X.

Separated. A morphism is separated if and only if the diagonal is universally closed. The diagram

$$\begin{array}{ccc} X' & \xrightarrow{\Delta'} & X' \times_{S'} X' \\ \downarrow{\scriptstyle q} & & \downarrow \\ X & \xrightarrow{\Delta} & X \times_S X \end{array}$$

is an étale base change, so we are done.

(Locally) of finite type. By descent of quasi-compactness it suffices to consider locally of finite type morphisms. Being locally of finite type is Zariski local on the source, so we may assume that X is affine. Write $S = \operatorname{Spec} R$, $S' = \operatorname{Spec} R'$, $X = \operatorname{Spec} A$, and $A' = A \otimes_R R'$. Suppose $R' \to A'$ is of finite type, i.e. there exist $b_1, \ldots, b_n \in A'$ that generate A' as R'-algebra. Write $b_i = \sum_j a_{ij} \otimes r_{ij}$. Then the map $R[x_{ij} : i, j] \to A, x_{ij} \mapsto a_{ij}$ is surjective after the base change $- \otimes_R R'$. Since $R \to R'$ is faithfully flat, the map itself is also surjective. Hence $R \to A$ is of finite type.

(Locally) of finite presentation. Use notation as before. Suppose $R' \to A'$ is of finite presentation. Certainly $R \to A$ is of finite type. Take a surjection $R[x_1, \ldots, x_n] \to A$ with kernel I. Since $R \to R'$ is flat, the base change $R'[x_1, \ldots, x_n] \to A'$ has kernel $I \otimes_R R'$. It is finitely generated, say by $h_1, \ldots, h_m$. Write $h_i = \sum_j g_{ij} \otimes r_{ij}$. Arguing as before, I is generated by $\{g_{ij} : i, j\}$.

Proper. Proper means separated, of finite type, and universally closed.

Affine. Suppose f' is affine, i.e. X' is affine. Write $S = \operatorname{Spec} R$, $S' = \operatorname{Spec} R'$, and $X' = \operatorname{Spec} A'$. Also write $A = \mathcal{O}_X(X)$. Since f' is quasi-compact and separated, so is f. Now flat base change says $p^* f_* \mathcal{O}_X = f'_* \mathcal{O}_{X'}$, in other words $A' = A \otimes_R R'$. Consider the canonical map $X \to \operatorname{Spec} A$. The base change $X' \to \operatorname{Spec} A'$ is an isomorphism, so $X \to \operatorname{Spec} A$ is an isomorphism.

Finite. Finite is equivalent to affine and proper.

Open immersion. Suppose f' is an open immersion. Then it is universally open, so f is universally open. Replace S by $f(X)$; we need to prove f is an isomorphism. But now f and f' are surjective, so f' is an isomorphism.

Closed immersion. Suppose f' is a closed immersion. Then it is affine, so f is affine. In particular X and X' are affine. Use notation as before. Since f' is a closed immersion, $R' \to A'$ is surjective. As $R \to R'$ is faithfully flat, also $R \to A$ is surjective, and f is a closed immersion.

Flat. Being flat is Zariski local on the source, so we may assume that X is affine. Let R, R', A, and A' be as before. Suppose $R' \to A'$ is flat. Let $M \to N$ be an injection of R-modules. Since $R \to R' \to A'$ is flat, $M \otimes_R A' \to N \otimes_R A'$ is injective. Since $A \to A'$ is faithfully flat, $M \otimes_R A \to N \otimes_R A$ is also injective. Hence $R \to A$ is flat.

Smooth. A morphism is smooth if and only if it is locally of finite presentation, flat, and has smooth fibers. The first two properties descend, so we are reduced to the case $S = \operatorname{Spec} k$, $S' = \operatorname{Spec} k'$ for some finite separable field extension k'/k. But for X/k locally of finite type, X/k is smooth if and only if X is geometrically regular over k.

Unramified. A morphism is unramified if and only if it is locally of finite type and the diagonal is an open immersion.

Étale. A morphism is étale if and only if it is locally of finite presentation, flat, and unramified. ■

2. Descent of quasi-coherent sheaves

Definition 2.1. Let $\mathfrak{U} = \{U_i \to S\}_{i \in I}$ be an étale covering. A *descent datum of quasi-coherent sheaves* for $\mathfrak{U}$ consists of

- a quasi-coherent sheaf $\mathcal{F}_i \in \operatorname{QCoh} U_i$ for all $i \in I$, and
- an isomorphism $\varphi_{ji} : \mathcal{F}_i|_{U_{ij}} \to \mathcal{F}_j|_{U_{ij}}$ in $\operatorname{QCoh} U_{ij}$ for all $i, j \in I$,

such that the cocycle condition $\varphi_{ki} = \varphi_{kj}\varphi_{ji}$ holds on U_{ijk} for all $i, j, k \in I$. ♦

Definition 2.2. A *morphism* $(\mathcal{F}_i, \varphi_{ji}) \to (\mathcal{G}_i, \psi_{ji})$ of descent data for $\mathfrak{U}$ consists of a morphism $\alpha_i : \mathcal{F}_i \to \mathcal{G}_i$ in $\operatorname{QCoh} U_i$ for all $i \in I$, such that $\psi_{ji}\alpha_i = \alpha_j\varphi_{ji}$ holds on U_{ij} for all $i, j \in I$. ♦

The category of descent data of quasi-coherent sheaves for $\mathfrak{U}$ is denoted $\operatorname{QCoh} \mathfrak{U}$. Let $\mathcal{F}$ be a quasi-coherent sheaf on S. Then $(\mathcal{F}|_{U_i}, \mathrm{id}_{ji})$ is a descent datum. This construction is functorial.

Theorem 2.3 (Grothendieck). *The functor* $\operatorname{QCoh} S \to \operatorname{QCoh} \mathfrak{U}$ *is an equivalence.* ♦

In other words, quasi-coherent sheaves and their morphisms descend uniquely along étale coverings. This is not true for arbitrary sheaves of modules.

The proof is very similar to that of theorem 1.1. Indeed, descent of quasi-coherent sheaves is a sheaf condition of sorts; more precisely, the theorem states that QCoh is a 2-*sheaf* or *stack* over $(\mathrm{Sch}/S)_{\mathrm{et}}$.

Remark 2.4. Corollary 1.3 states that $\mathcal{H}om_S(X, Y)$ is a sheaf on $(\mathrm{Sch}/S)_{\mathrm{et}}$. This follows directly from theorem 2.3. Indeed, we may reduce to étale coverings $V \to U$ with U, V affine. We may also assume that S, X and Y are affine. But an affine scheme over Y is a quasi-coherent algebra on Y. Descent of quasi-coherent algebras follows from that of quasi-coherent sheaves. ♦

The following proposition is an important application.

Proposition 2.5. *Let $\mathcal{F}$ be a quasi-coherent sheaf on S. Then*

$$\mathcal{F} : (\mathrm{Sch}/S)^{\mathrm{op}} \to \mathrm{Ab}, \quad T \mapsto \mathcal{F}(T) := \mathcal{F}_T(T)$$

is a sheaf on $(\mathrm{Sch}/S)_{\mathrm{et}}$.

Proof. A section $t \in \mathcal{F}(T)$ is a morphism $\mathcal{O}_T \to \mathcal{F}_T$. ■

For instance, we can speak of the *structure sheaf* on $(\mathrm{Sch}/S)_{\mathrm{et}}$. Any quasi-coherent sheaf on S has associated étale cohomology.

References

- Brochard, *Topologies de Grothendieck, descente, quotients*.
  This lecture and a bit more is covered in chapter 2.
- de Jong, *Stacks Project*.
  Descent is the topic of chapter `0238`. More in particular, theorem 1.1 is in section `023P`, theorem 1.6 in section `02YJ`, and theorem 2.3 in section `023R`.