Infinite Galois theory and localic groups

Wouter Zomervrucht

Diamant symposium, November 30, 2018

MAF
Moduli and Automorphic Forms

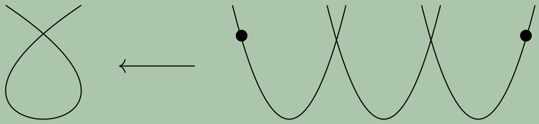
Berlin Mathematical School

Universiteit Leiden & Freie Universität Berlin

Let X be a connected scheme. The *étale fundamental group* $\pi^{\text{et}}(X)$ is an important invariant. It classifies finite étale covers.

Example

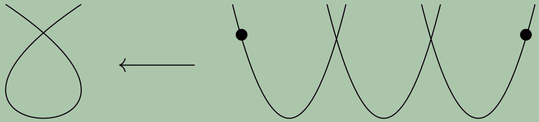
Take for X the nodal curve $\mathbb{P}^1/\{0 \sim \infty\}$ over $\mathbb{C}$. Its (connected) covers are of the form



We have $\pi^{\text{et}}(X) = \widehat{\mathbb{Z}}$.

Example

Take for X the nodal curve $\mathbb{P}^1/\{0 \sim \infty\}$ over $\mathbb{C}$. Its (connected) covers are of the form



We have $\pi^{\text{et}}(X) = \widehat{\mathbb{Z}}$.

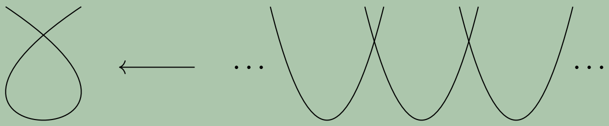
The étale fundamental group is profinite and satisfies

$$\text{FCov}(X) \approx \pi^{\text{et}}(X)\text{-FSet}.$$

The *pro-étale fundamental group* $\pi^{\text{proet}}(X)$ classifies *geometric covers*: étale maps that have the valuative criterion of properness.

Example

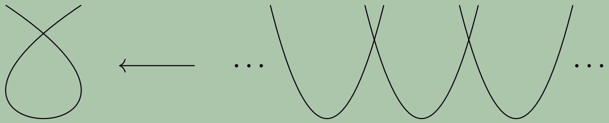
Now there is also an infinite connected cover



We have $\pi^{\text{proet}}(X) = \mathbb{Z}$.

Example

Now there is also an infinite connected cover



We have $\pi^{\text{proet}}(X) = \mathbb{Z}$.

The pro-étale fundamental group is … and satisfies

$$\mathrm{Cov}(X) \approx \pi^{\text{proet}}(X)\text{-Set}.$$

Theorem (Bhatt–Scholze)

Assume X is connected and locally noetherian. Continuous representations $\pi^{\mathrm{proet}}(X) \to \mathrm{GL}_n(\mathbb{Q}_l)$ are the same as $\mathbb{Q}_l$-local systems on X.

Example

For the nodal curve a local system is given by an element of $\mathrm{GL}_n(\mathbb{Q}_l)$ that determines gluing at the node. The theorem is false for the compact group $\pi^{\mathrm{et}}(X)$.

Example

If G is a topological group, then $(G\text{-FSet}, \text{forget})$ is a profinite Galois category.

A *profinite Galois category* is a category $\mathcal{C}$ equipped with a functor $F\colon \mathcal{C} \to \text{FSet}$ such that

- $\mathcal{C}$ has finite limits and finite colimits,
- F preserves finite limits and finite colimits,
- F is conservative,
- $\mathcal{C}$ is generated under finite colimits by connected objects.

Its *fundamental group* is $\pi(\mathcal{C}, F) := \text{Aut}(F)$.

A *profinite Galois category* is a category $\mathcal{C}$ equipped with a functor $F\colon \mathcal{C} \to \mathrm{FSet}$ such that

- $\mathcal{C}$ has finite limits and finite colimits,
- F preserves finite limits and finite colimits,
- F is conservative,
- $\mathcal{C}$ is generated under finite colimits by connected objects.

Its *fundamental group* is $\pi(\mathcal{C}, F) := \mathrm{Aut}(F)$.

Theorem

The functor F lifts to an equivalence $\overline{F}\colon \mathcal{C} \to \pi(\mathcal{C}, F)$-FSet.

Example

If G is a topological group, then $(G\text{-Set}, \text{forget})$ is an infinite Galois category.

An *infinite Galois category* is a category $\mathcal{C}$ equipped with a functor $F\colon \mathcal{C} \to \text{Set}$ such that

- $\mathcal{C}$ has finite limits and all colimits,
- F preserves finite limits and all colimits,
- F is conservative,
- $\mathcal{C}$ is generated under colimits by connected objects.

Its *fundamental group* is $\pi(\mathcal{C}, F) := \text{Aut}(F)$.

An *infinite Galois category* is a category $\mathcal{C}$ equipped with a functor $F\colon \mathcal{C} \to \mathrm{Set}$ such that

- $\mathcal{C}$ has finite limits and all colimits,
- F preserves finite limits and all colimits,
- F is conservative,
- $\mathcal{C}$ is generated under colimits by connected objects.

Its *fundamental group* is $\pi(\mathcal{C}, F) := \mathrm{Aut}(F)$.

Theorem

If *$(\mathcal{C}, F)$* ***is tame****, then F lifts to an equivalence $\overline{F}\colon \mathcal{C} \to \pi(\mathcal{C}, F)$-Set.*

Example

Let $(G_i)_{i\in\mathcal{I}}$ be a cofiltered system of groups with surjective transition maps. Then

$$(\mathcal{C},F) := \left(\operatorname*{colim}_{i\in\mathcal{I}^{\mathrm{op}}} (G_i\text{-Set}), \text{forget}\right)$$

is an infinite Galois category with fundamental group

$$\pi(\mathcal{C},F) = \lim_{i\in\mathcal{I}} G_i.$$

But that limit can be trivial even if the G_i are not!

$\rightsquigarrow$ $(\mathcal{C},F)$ is **not tame**
$\rightsquigarrow$ $\overline{F}\colon \mathcal{C} \to \pi(\mathcal{C},F)$-Set is **not an equivalence**

Example

Let $(\mathcal{C}, F)$ and $(\mathcal{C}, G)$ be infinite Galois categories.

$$\mathcal{C} \xrightarrow{F} \mathrm{Set} \qquad \mathcal{C} \xrightarrow{G} \mathrm{Set}$$

Is $F \cong G$?

Solution: work with a better category of spaces.

Replace **topological spaces** by **locales**.

locales

=

topology without points

A *frame* is a partially ordered set L such that

- L has arbitrary suprema, or *joins*, $\bigvee_{a \in A} a$,
- L has finite infima, or *meets*, $\bigwedge_{b \in B} b$,
- L satisfies the distributive law

$$\left(\bigvee_{a \in A} a\right) \wedge b = \bigvee_{a \in A} (a \wedge b)$$

  for all $A \subseteq L$ and $b \in L$.

In particular L has a minimum $\mathbf{0}$ and a maximum $\mathbf{1}$.

A *frame homomorphism* is a function $L \to M$ that preserves all joins and finite meets.

A *locale* is the dual of a frame:

$$\mathrm{Loc} := \mathrm{Frm}^{\mathrm{op}}.$$

The underlying frame of a locale X is denoted $\mathcal{O}(X)$.

Example

If X is a topological space, $\mathcal{O}(X)$ is a frame. Let X_{loc} be the locale with $\mathcal{O}(X_{\mathrm{loc}}) = \mathcal{O}(X)$. This makes a functor

$$\mathrm{Top} \xrightarrow{\mathrm{loc}} \mathrm{Loc}.$$

Example

The *localic point* is the locale $*$ with frame $\{0, 1\}$.

Let X be a locale.

- A *point* of X is a morphism $* \to X$.
- The *spectrum* of X is the set $|X| := \mathrm{Hom}_{\mathrm{Loc}}(*, X)$. It comes equipped with a natural topology.

This makes a functor

$$\mathrm{Loc} \xrightarrow{|-|} \mathrm{Top}.$$

Theorem (Stone duality)

There is an adjunction

$$\mathrm{Top} \underset{|-|}{\overset{\mathrm{loc}}{\rightleftarrows}} \mathrm{Loc} \quad (\perp)$$

that restricts to an equivalence

$$\{\text{sober spaces}\} \longleftrightarrow \{\text{spatial locales}\}.$$

Example

If G is a localic group, then $(G\text{-Set}, \text{forget})$ is an infinite Galois category.

The problem in topology was inadequacy of limits.

Theorem

If $Y = \lim_{i \in \mathcal{I}} X_i$ is a cofiltered limit of locales with epimorphic transition maps, then the projections $Y \to X_i$ are epimorphic as well.

Let X and Y be sets.
We make a *locale of functions* denoted $\ell\mathrm{Hom}(X, Y)$.

The set $\mathrm{Hom}(X, Y)$ usually has the *compact-open* topology with open subbase $\langle x|y\rangle := \{f \in \mathrm{Hom}(X, Y) : f(x) = y\}$.

The locale $\ell\mathrm{Hom}(X, Y)$ has opens $\langle x|y\rangle$ for $x \in X$ and $y \in Y$, such that

- $\langle x|y_1\rangle \wedge \langle x|y_2\rangle = \mathbf{0}$ for all $x \in X$ and $y_1 \neq y_2$,
- $\bigvee_{y\in Y}\langle x|y\rangle = \mathbf{1}$ for all $x \in X$.

We define $\ell\mathrm{Hom}(X, Y)$ to be **universal** for these properties.

The locale $\ell\mathrm{Hom}(X, Y)$ has opens $\langle x|y\rangle$ for $x \in X$ and $y \in Y$, such that

- $\langle x|y_1\rangle \wedge \langle x|y_2\rangle = \mathbf{0}$ for all $x \in X$ and $y_1 \neq y_2$,
- $\bigvee_{y\in Y}\langle x|y\rangle = \mathbf{1}$ for all $x \in X$.

We define $\ell\mathrm{Hom}(X, Y)$ to be **universal** for these properties.

Example

A point of $\ell\mathrm{Hom}(X, Y)$ is a morphism $* \to \ell\mathrm{Hom}(X, Y)$, so a function $h\colon X \times Y \to \{\mathbf{0}, \mathbf{1}\}$. The subset $S = h^{-1}(\mathbf{1})$ satisfies

- (x, y_1) and (x, y_2) are not both in S if $y_1 \neq y_2$,
- for all $x \in X$ there is $y \in Y$ with $(x, y) \in S$.

So S defines a function $X \to Y$.

Define $\ell\mathrm{Isom}(X, Y)$ to be the universal locale with opens $\langle x|y\rangle$ for $x \in X$ and $y \in Y$, such that

- $\langle x|y_1\rangle \wedge \langle x|y_2\rangle = \mathbf{0}$ for all $x \in X$ and $y_1 \neq y_2$,
- $\bigvee_{y\in Y}\langle x|y\rangle = \mathbf{1}$ for all $x \in X$,
- $\langle x_1|y\rangle \wedge \langle x_2|y\rangle = \mathbf{0}$ for all $x_1 \neq x_2$ and $y \in Y$,
- $\bigvee_{x\in X}\langle x|y\rangle = \mathbf{1}$ for all $y \in Y$.

Theorem

If X and Y are infinite, then $\ell\mathrm{Isom}(X, Y)$ is non-trivial.

Let $(\mathcal{C}, F)$ be an infinite Galois category.

One introduces the *localic fundamental group*

$$\pi^\ell(\mathcal{C}, F) := \ell\mathrm{Aut}(F)$$

similar to the constructions before.

Theorem (Joyal–Tierney, Moerdijk, Dubuc)

The functor F lifts to an equivalence $\overline{F}\colon \mathcal{C} \to \pi^\ell(\mathcal{C}, F)$-Set.

Theorem

Let $(\mathcal{C}, F)$ and $(\mathcal{C}, G)$ be infinite Galois categories.

$$\mathcal{C} \xrightarrow{F} \mathrm{Set} \qquad \mathcal{C} \xrightarrow{G} \mathrm{Set}$$

The locale $\ell\mathrm{Isom}(F, G)$ is a $\pi^\ell(\mathcal{C}, G)$-$\pi^\ell(\mathcal{C}, F)$-bitorsor.
It is trivial if and only if $F \cong G$.