Bhargava's cube law and cohomology

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235. Si forma $AXX + 2BXY + CYY \ldots F$ transit in productum e duabus formis $axx + 2bxy + cyy \ldots f$, et $a'x'x' + 2b'x'y' + c'y'y' \ldots f'$ per substitutionem talem $X = pxx' + p'xy' + p''yx' + p'''yy'$, $Y = qxx' + q'xy' + q''yx' + q'''yy'$ (quod breuitatis causa in sequentibus semper ita exprimemus: Si F transit in ff' per substitutionem p, p', p'', p'''; q, q', q'', q''' *)), dicemus simpliciter, formam F *transformabilem* esse in ff'; si insuper haec transformatio ita est comparata, vt sex numeri $pq' - qp'$, $pq'' - qp''$, $pq''' - qp'''$, $p'q'' - q'p''$, $p'q''' - q'p'''$, $p''q''' - q''p'''$ diuisorem communem non habeant: formam F e formis f, f' *compositam* vocabimus.

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  binary quadratic forms
- **Peter Gustav Lejeune Dirichlet (1839)**
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A *binary quadratic form* is an expression

$$q = ax^2 + bxy + cy^2, \qquad a, b, c \in \mathbb{Z}.$$

It is *primitive* if $\gcd(a, b, c) = 1$.

The group $\mathrm{SL}_2(\mathbb{Z})$ acts on binary quadratic forms by variable substitution. The *discriminant*

$$\Delta q = b^2 - 4ac$$

is invariant under this action.

Let $D \equiv 0, 1 \bmod 4$. We define

$$Q_D(\mathbb{Z}) = \{\text{primitive binary quadratic forms of discriminant } D\}.$$

Theorem (Gauss)

For any two $q_1, q_2 \in Q_D(\mathbb{Z})$ there exists a third $q \in Q_D(\mathbb{Z})$ and forms $u, v \in \mathbb{Z}[x_1, y_1, x_2, y_2]_{1,1}$ such that

$$q_1(x_1, y_1) \cdot q_2(x_2, y_2) = q(u, v).$$

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Example

Suppose $D \equiv 0 \bmod 4$. Then

- $[x^2 - \frac{D}{4}y^2] = 0$,
- $[ax^2 + bxy + cy^2]^{-1} = [ax^2 - bxy + cy^2]$.

Let $\mathcal{O}_D$ be the unique *quadratic order* of discriminant D.

Example

Suppose $D \neq 1$ is squarefree. Then $\mathcal{O}_D = \mathbb{Z}[\frac{1+\sqrt{D}}{2}]$ is the maximal order in $\mathbb{Q}(\sqrt{D})$.

The *class group* of $\mathcal{O}_D$ is

$$\mathrm{Cl}(\mathcal{O}_D) = \frac{\{\text{invertible fractional ideals}\}}{\{\text{invertible principal ideals}\}}.$$

There is also a *narrow* or *oriented class group* $\mathrm{Cl}^+(\mathcal{O}_D)$ which fits into a short exact sequence

$$1 \longrightarrow \{\pm 1\}/\mathrm{N}(\mathcal{O}_D^\times) \longrightarrow \mathrm{Cl}^+(\mathcal{O}_D) \longrightarrow \mathrm{Cl}(\mathcal{O}_D) \longrightarrow 1.$$

Example

If D is negative, $\mathrm{Cl}^+(\mathcal{O}_D) = \{\pm 1\} \times \mathrm{Cl}(\mathcal{O}_D)$.

Theorem (Dirichlet)

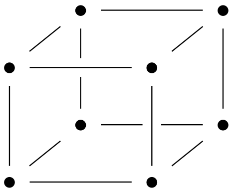
$Q_D(\mathbb{Z})/\mathrm{SL}_2(\mathbb{Z}) \cong \mathrm{Cl}^+(\mathcal{O}_D)$.

Roughly, $[ax^2 + bxy + cy^2]$ corresponds to $[\mathbb{Z} \oplus \frac{-b+\sqrt{D}}{2a}\mathbb{Z}]$.

Example

If D is negative, $\mathrm{Cl}^+(\mathcal{O}_D) = \{\pm 1\} \times \mathrm{Cl}(\mathcal{O}_D)$. The subgroup $\mathrm{Cl}(\mathcal{O}_D) \subset \mathrm{Cl}^+(\mathcal{O}_D)$ corresponds to *positive definite* forms.

A *cube* is a $2 \times 2 \times 2$-matrix of integers



The group $G(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z})$ acts on cubes.
For instance, the first factor acts by

$$(\square, \square)\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = (\alpha\square + \gamma\square, \beta\square + \delta\square).$$

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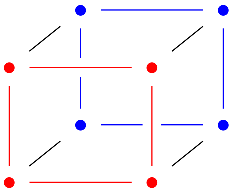

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$$(\color{red}{\square}, \color{blue}{\square})\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) = (\alpha\color{red}{\square} + \gamma\color{blue}{\square}, \beta\color{red}{\square} + \delta\color{blue}{\square}).$$

Associated to a cube w are three binary quadratic forms $q_1(w), q_2(w), q_3(w)$. For instance,

$$q_1(\square, \square) = \det(\square x_1 + \square y_1).$$

The discriminants satisfy

$$\Delta q_1(w) = \Delta q_2(w) = \Delta q_3(w)$$

and this number is the *discriminant* Δw of the cube.

A cube w is *projective* if $q_1(w), q_2(w), q_3(w)$ are primitive. We define $W_D(\mathbb{Z}) = \{\text{projective cubes of discriminant } D\}$.

Theorem (Bhargava)

For any cube $w \in W_D(\mathbb{Z})$ the identity

$$[q_1(w)] + [q_2(w)] + [q_3(w)] = 0$$

holds in $Q_D(\mathbb{Z})/\mathrm{SL}_2(\mathbb{Z})$. Conversely, if

$$[q_1] + [q_2] + [q_3] = 0$$

holds in $Q_D(\mathbb{Z})/\mathrm{SL}_2(\mathbb{Z})$, there is a cube $w \in W_D(\mathbb{Z})$ satisfying $q_1(w) = q_1$, $q_2(w) = q_2$, and $q_3(w) = q_3$.

Theorem (Bhargava)

There is a unique group law on $W_D(\mathbb{Z})/G(\mathbb{Z})$ such that the maps

$$q_1, q_2, q_3 \colon W_D(\mathbb{Z})/G(\mathbb{Z}) \longrightarrow Q_D(\mathbb{Z})/\mathrm{SL}_2(\mathbb{Z})$$

are group homomorphisms.

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Goal: explain class groups geometrically.

- group scheme SL_2 acting on $Q_D \subset \mathbb{A}^3$
- group scheme $G = \mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{SL}_2$ acting on $W_D \subset \mathbb{A}^8$

We use *arithmetic invariant theory* and *flat cohomology*.

Example

Let SL_2 act on $(\mathbb{P}^1, \mathcal{O}(1))$. On global sections we retrieve the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathcal{O}(2)(\mathbb{P}^1) = \mathbb{Z}[x, y]_2$.

Let G act on $(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(1, 1, 1))$. On global sections we get an action of $G(\mathbb{Z})$ on

$$\mathcal{O}(1, 1, 1)(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1) = \mathbb{Z}[x_1, y_1, x_2, y_2, x_3, y_3]_{1,1,1}.$$

Identifying cubes with $1, 1, 1$-forms, this is the action above.

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Example

Let $w \in W_D(\mathbb{Z})$ be a cube. The fibers of

$$\mathrm{Z}(w) \xrightarrow{\pi_1} \mathbb{P}^1$$

are degenerate precisely above $\mathrm{Z}(q_1(w))$.

Principle of transitive actions

Let $\mathcal{C}/S$ be a site with final object S. Let G be a sheaf of groups acting *transitively* on a sheaf of sets X. Let $x \in X(S)$ be a global section and $H \subseteq G$ the stabilizer of x.

The short exact sequence of sheaves of pointed sets

$$1 \longrightarrow H \longrightarrow G \xrightarrow{\cdot x} X \longrightarrow 1$$

induces a longer exact sequence

$$1 \longrightarrow H(S) \longrightarrow G(S) \longrightarrow X(S) \xrightarrow{\delta} \mathrm{H}^1(S,H) \longrightarrow \mathrm{H}^1(S,G)$$

where $\delta(y)$ is the transporter $G_{y,x}$.

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If moreover

- H is abelian,
- $\mathrm{H}^1(S, G) = 1$,

then $G(S) \backslash X(S)$ has a $\mathrm{H}^1(S, H)$-torsor structure independent of the choice of x.

Let $\mathbb{T}_D$ be the *norm one unit group* with respect to $\mathbb{Z} \to \mathcal{O}_D$.
That is, if $\mathcal{O}_D = \mathbb{Z}[\tau]/(\tau^2 - b\tau + c)$, then

$$\begin{aligned} \mathbb{T}_D &= \{(u,v) : \mathrm{N}(u + v\tau) = 1\} \\ &= \{(u,v) : u^2 + buv + cv^2 = 1\}. \end{aligned}$$

One has $\mathrm{H}^1_{\mathrm{fppf}}(\mathbb{Z}, \mathbb{T}_D) = \mathrm{Cl}^+(\mathcal{O}_D)$.

If $H \subset \mathrm{SL}_2$ is the stabilizer of $x^2 + bxy + cy^2$ in $Q_D(\mathbb{Z})$, then

$$H^\flat \cong \mathbb{T}_D.$$

Here $H^\flat$ is the scheme-theoretic closure of the generic fiber.

Theorem

The set $Q_D(\mathbb{Z})/\mathrm{SL}_2(\mathbb{Z})$ is canonically a torsor under $\mathrm{H}^1_{\mathrm{fppf}}(\mathbb{Z}, \mathbb{T}_D)$.

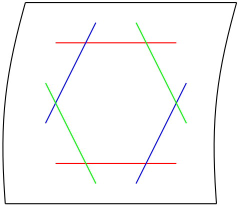
The same is true if we replace $\mathbb{Z}$ by any Dedekind domain of characteristic not 2.

What is the stabilizer $H \subset G$ of a cube $w \in W_D(\mathbb{Z})$?

Generically, the projection

$$\mathrm{Z}(w) \xrightarrow{\pi_{23}} \mathbb{P}^1 \times \mathbb{P}^1$$

is a blowup in two points. So $\mathrm{Z}(w)$ is a *degree* 6 *del Pezzo surface*. It contains a hexagon of six -1-curves.



We find

$$H^\flat \cong \ker\left(\mathbb{T}_D \times \mathbb{T}_D \times \mathbb{T}_D \xrightarrow{\cdot} \mathbb{T}_D\right).$$

Theorem

The set $W_D(\mathbb{Z})/G(\mathbb{Z})$ is canonically a torsor under $\mathrm{H}^1_{\mathrm{fppf}}(\mathbb{Z}, \mathbb{T}_D)^2$.

The same is true if we replace $\mathbb{Z}$ by any Dedekind domain of characteristic not 2.