Moduli spaces and cone complexes

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Overview

The main reference is [1], from which we cover sections 2–4. For a different point of view on cones also consult sections 2–3 from [5].

First we introduce cones, extended cones, cone complexes, and generalized cone complexes. We recall some facts about the moduli stack $\overline{\mathcal{M}}_{g,n}$ of stable algebraic curves and its stratification. Then we analogously describe the coarse moduli space $\overline{M}_{g,n}^{\mathrm{trop}}$ of stable tropical curves with its stratification. It has the structure of a generalized extended cone complex. The stratifications of $\overline{\mathcal{M}}_{g,n}$ and $\overline{M}_{g,n}^{\mathrm{trop}}$ are, in a certain sense, dual to each other. We shall see later that $\overline{M}_{g,n}^{\mathrm{trop}}$ is actually the skeleton of the Berkovich analytification $\overline{\mathcal{M}}_{g,n}^{\mathrm{an}}$.

1. Cones

Definition 1.1. A *rational polyhedral cone with integral structure*, or simply a *cone*, is a topological space σ together with a finitely generated subgroup $M \subseteq \mathrm{Hom}_{\mathrm{Top}}(\sigma, \mathbb{R})$ such that the image of $\sigma \to \mathrm{Hom}_{\mathrm{Grp}}(M, \mathbb{R})$ is a finite intersection of $\mathbb{Q}$-linear closed halfspaces through 0 that does not contain a line through 0. ♦

A cone (σ, M) satisfies $\dim \sigma = \mathrm{rk}\, M$, in other words, σ is not contained in a proper linear subspace of $\mathrm{Hom}_{\mathrm{Grp}}(M, \mathbb{R})$. We will usually suppress M from the notation.

Example 1.2. Let N be a finite rank free abelian group and $\sigma \subseteq N_{\mathbb{R}}$ a finite intersection of $\mathbb{Q}$-linear closed halfspaces through 0 that does not contain a line through 0. Suppose that σ is not contained in a proper linear subspace of $N_{\mathbb{R}}$. Then $(\sigma, N^\vee)$ is a cone; all cones arise in this manner. Our alternative approach emphasizes the space σ and not the ambient space $N_{\mathbb{R}}$. ♦

Definition 1.3. A *morphism of cones* $\sigma \to \sigma'$ is a continuous map such that the induced map $\mathrm{Hom}_{\mathrm{Top}}(\sigma', \mathbb{R}) \to \mathrm{Hom}_{\mathrm{Top}}(\sigma, \mathbb{R})$ sends M' to M. ♦

If $u \in M$ is non-negative on σ, then $\tau = \{v \in \sigma \colon u(v) = 0\}$ is a *face* of σ. It inherits the structure of a cone, and the inclusion $\tau \to \sigma$ is a morphism of cones. A cone morphism isomorphic to such a morphism is called a *face map*. In particular, automorphisms of cones are face maps.

2. Extended cones

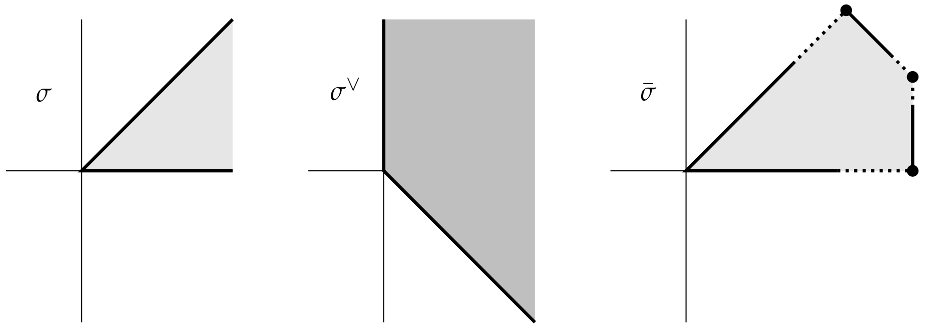
Let σ be a cone. We define the *dual cone*

$$\sigma^\vee = \{u \in M_{\mathbb{R}} \colon u(v) \geq 0 \text{ for all } v \in \sigma\}$$

and set $S_\sigma = \sigma^\vee \cap M$. Then S_σ is a monoid under addition and one can canonically identify $\sigma = \mathrm{Hom}_{\mathrm{Mon}}(S_\sigma, \mathbb{R}_{\geq 0})$.

Definition 2.1. The *extended cone* of σ is $\bar{\sigma} = \mathrm{Hom}_{\mathrm{Mon}}(S_\sigma, \overline{\mathbb{R}_{\geq 0}})$, where $\overline{\mathbb{R}_{\geq 0}} = \mathbb{R}_{\geq 0} \sqcup \{\infty\}$. ♦

The extended cone is compact and $\sigma \subseteq \bar{\sigma}$ is dense open. Here is a suggestive picture.



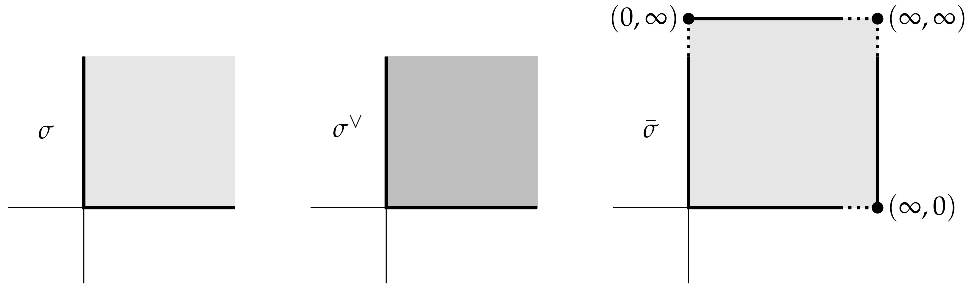
The picture shows that $\bar{\sigma}$ should have more faces than σ. Let's make that precise. For faces $\tau' \subseteq \tau$ of σ we define

$$F(\tau, \tau') = \left\{ v \in \bar{\sigma} \colon \begin{array}{l} u \text{ vanishes on } \tau' \Longleftrightarrow u(v) \neq \infty \\ u \text{ vanishes on } \tau \;\; \Longrightarrow u(v) = 0 \end{array} \text{ for all } u \in S_\sigma \right\}.$$

It may be identified with the quotient cone τ/τ'. Its closure $\bar{F}(\tau, \tau')$ in $\bar{\sigma}$ is the extended cone $\overline{\tau/\tau'}$. We call $\bar{F}(\tau, \tau')$ a *face* of $\bar{\sigma}$. There are two kinds of faces:

- if $\tau' = 0$, then $F(\tau, 0) = \tau$ and we call $\bar{F}(\tau, 0) = \bar{\tau}$ an *extended face*;
- if $\tau' \neq 0$, then $F(\tau, \tau')$ does not intersect σ and we call $\bar{F}(\tau, \tau')$ a *face at infinity*.

Example 2.2. We compute a small example. Let x, y be the coordinate functions on $\mathbb{R}^2$. Then $\sigma = \mathbb{R}^2_{\geq 0}$ with $M = \mathbb{Z}x \oplus \mathbb{Z}y$ forms a cone. The dual cone is $\sigma^\vee = \mathbb{R}_{\geq 0}x \oplus \mathbb{R}_{\geq 0}y$ and thus $S_\sigma = \mathbb{N}x \oplus \mathbb{N}y$. A monoid homomorphism $v \colon S_\sigma \to \mathbb{R}_{\geq 0}$ is uniquely defined by the images $v(x), v(y) \in \mathbb{R}_{\geq 0}$. The identification $\sigma = \mathrm{Hom}_{\mathrm{Mon}}(S_\sigma, \mathbb{R}_{\geq 0})$ has $(a, b) \in \sigma$ corresponding to the unique monoid homomorphism v with $v(x) = a$ and $v(y) = b$. An element $v \in \bar{\sigma}$ is again given by the images $v(x), v(y)$, but this time they take values in $\overline{\mathbb{R}_{\geq 0}}$. We conclude that $\bar{\sigma} = \overline{\mathbb{R}_{\geq 0}}^2$.



If $\tau \subseteq \sigma$ is the face given by $y = 0$, then $\bar{F}(\tau, \tau)$ is the point $(\infty, 0)$ and $\bar{F}(\sigma, \tau)$ is the one-dimensional face at infinity where $x = \infty$. ♦

Definition 2.3. A *morphism of extended cones* $\bar{\sigma} \to \bar{\sigma}'$ is a continuous map that restricts to a cone morphism $\sigma \to F(\tau, \tau')$ for some faces $\tau' \subseteq \tau$ of σ'. ♦

Each morphism of cones $\sigma \to \sigma'$ induces a morphism of extended cones $\bar{\sigma} \to \bar{\sigma}'$. For instance, if $\tau \to \sigma$ is a face map, we call $\bar{\tau} \to \bar{\sigma}$ an *extended face map*. On the other hand, the inclusion of a face at infinity is a morphism of extended cones that does not come from a morphism of cones.

3. Cone complexes

Definition 3.1. A *cone complex* is a topological space Σ, together with finitely many closed subsets $\sigma_1, \ldots, \sigma_n \subseteq \Sigma$ each equipped with the structure of a cone, such that

- $\Sigma = \sigma_1 \cup \ldots \cup \sigma_n$,
- each face of σ_i occurs as a unique σ_j, and
- the intersection $\sigma_i \cap \sigma_j$ is a union of faces of both σ_i and σ_j.

A *morphism of cone complexes* $\Sigma \to \Sigma'$ is a continuous map such that for each cone σ of Σ there is a cone σ' of Σ' such that the map restricts to a morphism of cones $\sigma \to \sigma'$. ◆

Definition 3.2. An *extended cone complex* is a topological space $\bar{\Sigma}$, together with finitely many closed subsets $\bar{\sigma}_1, \ldots, \bar{\sigma}_n \subseteq \bar{\Sigma}$ each equipped with the structure of an extended cone, such that

- $\bar{\Sigma} = \bar{\sigma}_1 \cup \ldots \cup \bar{\sigma}_n$,
- each extended face of $\bar{\sigma}_i$ occurs as a unique $\bar{\sigma}_j$, and
- the intersection $\bar{\sigma}_i \cap \bar{\sigma}_j$ is a union of extended faces of both $\bar{\sigma}_i$ and $\bar{\sigma}_j$.

A *morphism of extended cone complexes* $\bar{\Sigma} \to \bar{\Sigma}'$ is a continuous map such that for each extended cone $\bar{\sigma}$ of $\bar{\Sigma}$ there is an extended cone $\bar{\sigma}'$ of $\bar{\Sigma}'$ such that the map restricts to a morphism of extended cones $\bar{\sigma} \to \bar{\sigma}'$. ◆

Informally, a cone complex is the result of gluing cones along faces. Note that in the extended case we only glue along extended faces, not along faces at infinity.

Cone complexes differ from the fans of toric geometry in two important aspects. Firstly, fans come with an embedding into an ambient space; cone complexes do not. Moreover, two cones in a fan intersect in precisely one face, whereas we allow the cones in a cone complex to intersect in a union of faces.

Example 3.3. Let Σ be the cone complex obtained by gluing two copies of $\mathbb{R}^2_{\geq 0}$ along the boundary $\mathbb{R}_{\geq 0} \times \{0\} \cup \{0\} \times \mathbb{R}_{\geq 0}$. Then Σ cannot be realized as a fan. ◆

The *relative interior* σ° of a cone σ is its interior as a subset of $\mathrm{Hom}_{\mathrm{Grp}}(M, \mathbb{R})$. Equivalently, $\sigma^\circ = \sigma \setminus \bigcup_{\tau \subsetneq \sigma} \tau$ is the complement in σ of its proper faces. For extended cones we similarly define $\bar{\sigma}^\circ = \bar{\sigma} \setminus \bigcup_{\tau \subsetneq \sigma} \bar{\tau}$ as the complement of its proper extended faces.

Proposition 3.4. *If $\Sigma = \sigma_1 \cup \ldots \cup \sigma_n$ is a cone complex, then $\Sigma = \sigma_1^\circ \sqcup \ldots \sqcup \sigma_n^\circ$. If $\bar{\Sigma} = \bar{\sigma}_1 \cup \ldots \cup \bar{\sigma}_n$ is an extended cone complex, then $\bar{\Sigma} = \bar{\sigma}_1^\circ \sqcup \ldots \sqcup \bar{\sigma}_n^\circ$.* ◆

4. Generalized cone complexes

A more categorical point of view says that, equivalently, a cone complex is a topological space presented as a colimit $\Sigma = \mathrm{colim}_{i \in I} \sigma_i$ over cones and face maps, where I is a finite partially ordered set. A natural generalization is the following.

Definition 4.1. A *generalized cone complex* is a topological space Σ together with a presentation $\Sigma = \operatorname{colim}_{i \in I} \sigma_i$ as a colimit over cones and face maps, where I is a finite diagram. A *morphism of generalized cone complexes* $\Sigma \to \Sigma'$ is a continuous map such that for every cone σ of Σ there is a cone σ' of Σ' and a morphism of cones $\sigma \to \sigma'$ that makes the diagram

$$\begin{array}{ccc} \sigma & \longrightarrow & \sigma' \\ \downarrow & & \downarrow \\ \Sigma & \longrightarrow & \Sigma' \end{array}$$

commute. ♦

Definition 4.2. A *generalized extended cone complex* is a topological space $\bar{\Sigma}$ together with a presentation $\bar{\Sigma} = \operatorname{colim}_{i \in I} \bar{\sigma}_i$ as a colimit over extended cones and extended face maps, where I is a finite diagram. A *morphism of generalized extended cone complexes* $\bar{\Sigma} \to \bar{\Sigma}'$ is a continuous map such that for every extended cone $\bar{\sigma}$ of $\bar{\Sigma}$ there is an extended cone $\bar{\sigma}'$ of $\bar{\Sigma}'$ and a morphism of extended cones $\bar{\sigma} \to \bar{\sigma}'$ that makes the diagram

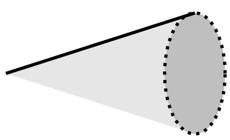
$$\begin{array}{ccc} \bar{\sigma} & \longrightarrow & \bar{\sigma}' \\ \downarrow & & \downarrow \\ \bar{\Sigma} & \longrightarrow & \bar{\Sigma}' \end{array}$$

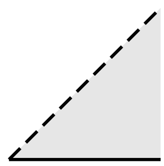
commute. ♦

We emphasize that the transition maps in the colimit as required to be face maps. As mentioned before, automorphisms of cones are face maps.

Remark 4.3. There is an obvious functor from (generalized) cone complexes to (generalized) extended cone complexes, sending a cone σ to the extended cone $\bar{\sigma}$. This functor is faithful and essentially surjective. However, it is not an equivalence: not all morphisms of extended cones come from morphisms of cones. ♦

The diagram I in the presentation of a generalized cone complex may fail to be a partially ordered set in essentially two ways. On the one hand, I can contain parallel arrows. This means that one is allowed to identify multiple faces of one cone. For instance, we can glue the faces $\mathbb{R}_{\geq 0} \times \{0\}$ and $\{0\} \times \mathbb{R}_{\geq 0}$ of the cone $\mathbb{R}^2_{\geq 0}$ along each other (see the picture on the left). It follows that the map $\sigma \to \Sigma$ is not injective in general.





On the other hand, I can contain loops. That means we may take quotients by automorphisms. For instance, consider $\mathbb{R}^2_{\geq 0}$ modulo the automorphism $(a, b) \mapsto (b, a)$. Visually this is the effect of folding $\mathbb{R}^2_{\geq 0}$ onto itself along the line $x = y$ (see the picture on the right). This example shows that $\sigma \to \Sigma$ is not even injective on the relative interior σ°.

We would nevertheless like to formulate an analogon of proposition 3.4. Clearly such a statement has to be more subtle! Say a presentation $\Sigma = \operatorname{colim}_{i \in I} \sigma_i$ is *reduced* if

- every face map $\tau \to \sigma_i$ is isomorphic to a transition map $\sigma_j \to \sigma_i$,
- finite compositions of transition maps are transition maps, and
- if two transition maps $\sigma_j \to \sigma_i$ and $\sigma_k \to \sigma_i$ are isomorphic as maps to σ_i, then $j = k$ and the (unique) isomorphism $\sigma_j \to \sigma_k$ over σ_i is a transition map.

The second condition includes the existence of identity maps, hence the third condition implies that all transition isomorphisms are automorphisms. In the first condition we cannot require unicity, or all automorphisms would be trivial.

Proposition 4.4. *Every generalized (extended) cone complex admits a reduced presentation. Furthermore, if $\Sigma = \operatorname{colim}_{i \in I} \sigma_i$ is in reduced presentation, then $\Sigma = \bigsqcup_{i \in I} G_i \backslash \sigma_i^\circ$ and $\overline{\Sigma} = \bigsqcup_{i \in I} G_i \backslash \bar{\sigma}_i^\circ$, where $G_i \subseteq \operatorname{Aut} \sigma_i$ is the subgroup of automorphisms in the diagram.*

Proof. We prove the second part.[1] As a colimit of topological spaces (or sets), we have

$$\Sigma = \Big(\bigsqcup_{i \in I} \sigma_i \Big) / {\sim}$$

where $\sim$ is the equivalence relation generated by $(j, y) \sim (i, t(y))$ for transition maps $t \colon \sigma_j \to \sigma_i$ and $y \in \sigma_j$. The canonical map $\bigsqcup_{i \in I} \sigma_i^\circ \to \Sigma$ is surjective; we claim that if (i, x) and (j, y) have the same image in Σ, then $i = j$ and there is a diagram automorphism of σ_i that sends x to y.

Indeed, in this situation there are indices $i = i_0, \ldots, i_n = j$, elements $x = x_0, \ldots, x_n = y$ with $x_k \in \sigma_{i_k}$, and pointed transition maps

$$\sigma_{i_0} \xrightarrow{t_1} \sigma_{i_1} \xleftarrow{t_2} \sigma_{i_2} \xrightarrow{t_3} \sigma_{i_3} \quad \cdots \quad \sigma_{i_{n-1}} \xrightarrow{t_n} \sigma_{i_n}$$

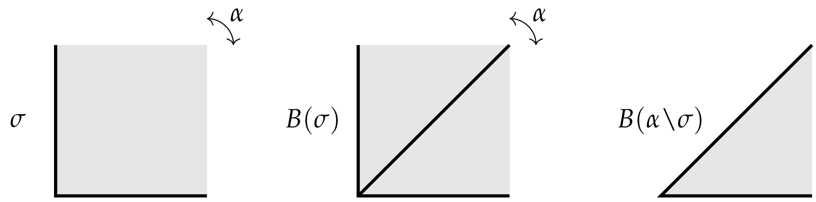
(Because the diagram has identities, the order of up and down arrows is irrelevant.) Since x, y lie in the interiors of σ_i, σ_j, we are done if $n \leq 1$. Otherwise we can make a shorter diagram as follows. Let $\sigma_m \to \sigma_{i_2}$ be a diagram face map with $x_2 \in \sigma_m^\circ$. As $\sigma_m \to \sigma_{i_1}$ is isomorphic to t_1, we have $m = i_0$ and there is a transition map $\sigma_{i_0} \to \sigma_{i_2}$ over σ_{i_1} sending x_0 to x_2. Now we can shorten the diagram by removing indices i_1 and i_2. ■

The above discussion explains that generalized cone complexes are not cone complexes in the obvious way. Yet there is a way to view them as such; we sketch it here. Associated with a cone σ is its canonical *barycentric subdivision* $B(\sigma)$. It is a cone complex on the topological space σ, with the property that automorphisms of σ permute the cones in $B(\sigma)$, but do not act on the cones themselves.

Therefore, if $\Sigma = \operatorname{colim}_{i \in I} \sigma_i$, then $B(\Sigma) = \operatorname{colim}_{i \in I} B(\sigma_i)$ is actually a (non-generalized) cone complex with underlying topological space Σ. The identity map $\Sigma \to B(\Sigma)$ is a morphism of generalized cone complexes. The construction extends to generalized extended cone complexes; details can be found in [2]. We conclude with just an example.

[1] The literature contains various versions of this statement. Unfortunately, I have found none with a proof, and even worse, none that are correct. Therefore I felt compelled to include a proof in these notes.

Example 4.5. Consider $\sigma = \mathbb{R}^2_{\geq 0}$. Its barycentric subdivision $B(\sigma)$ adds a one-dimensional face $\{(a,a) \colon a \in \mathbb{R}_{\geq 0}\}$ and correspondingly splits the two-dimensional face of σ into two parts. There is a unique non-trivial automorphism α of σ, given by $(a,b) \mapsto (b,a)$. We see that α interchanges the two cones of $B(\sigma)$, and $B(\alpha\backslash\sigma)$ is a single cone as depicted.



♦

5. Graphs

In this section we collect some formal definitions concerning graphs.

Definition 5.1. A *weighted graph with legs*, or simply a *graph*, is a tuple $G = (V, E, L, h)$ where
- V is a finite set of *nodes*,
- E is a finite set of *edges* equipped with incidence functions $E \rightrightarrows V$,
- L is a finite ordered list $l_1, \dots, l_n$ of nodes, called *legs* or *ends*, and
- $h \colon V \to \mathbb{N}$ is a *weight function* on the nodes.

We call (V, E) the *underlying graph* of G. ♦

Edges are undirected; parallel edges and loops are allowed. The legs are usually interpreted as edges attached to only one node and trailing off to infinity.

We omit the definition of *isomorphisms of graphs* except for one remark: edges should be considered as coupled pairs of half-edges, so that for each loop in G there exists a non-trivial automorphism of G that changes the 'direction' of the loop.

Definition 5.2. A graph G is *stable* if the underlying graph (V, E) is connected, each vertex of weight 0 has valence at least 3, and each vertex of weight 1 has valence at least 1. ♦

Definition 5.3. The *genus* of a graph G is $g(G) = h^1(V, E) + \sum_{v \in V} h(v)$. ♦

Here $h^1(V, E)$ is the *first Betti number* of the underlying graph (V, E), equal to $\#E - \#V + C$ with C the number of connected components.

Definition 5.4. Let G be a graph and $e \in E$ an edge between vertices v and w. The *edge contraction* of e in G is the graph obtained by removing the edge e, replacing v and w by a single new vertex, and giving the new vertex weight either $h(v) + h(w)$ if $v \neq w$, or $h(v) + 1$ if $v = w$. A *contraction* of G is a graph isomorphic to the result of a sequence of edge contractions. ♦

Each automorphism of G is a way to view G as a contraction of itself. More generally, contractions of stable graphs are stable, and all contractions of G have the same genus $g(G)$.

6. Moduli of algebraic curves

This section serves as reference only, so details are omitted. They can be found for instance in chapter XII of [3]. Throughout, we work over a fixed algebraically closed field k.

Definition 6.1. A *pointed nodal curve* is a proper curve C/k with at most ordinary nodal singularities, plus an ordered list of distinct smooth points $p_1, \dots, p_n \in C(k)$. A *stable curve* is a pointed nodal curve that is connected and has finite automorphism group. ♦

Associated with a pointed nodal curve $(C, p_1, \dots, p_n)$ is a graph $G = (V, E, L, h)$, called the *dual graph*, as follows:

- V is the set of irreducible components of C,
- E is the set of nodes of C, with each node of C incident to the components it lies on,
- L is the list of components on which the marked points lie, and
- $h(v)$ is the geometric genus of the component v.

It is well-known that G encodes many properties of $(C, p_1, \dots, p_n)$.

Theorem 6.2. *Let $(C, p_1, \dots, p_n)$ be a pointed nodal curve with dual graph G. Then $(C, p_1, \dots, p_n)$ is stable if and only if G is stable. The arithmetic genus $g(C)$ of C is equal to $g(G)$.* ♦

Fix natural numbers $g, n \geq 0$ such that $2g - 2 + n$ is positive.

Let $\overline{\mathcal{M}}_{g,n}$ be the moduli stack of n-pointed genus g stable curves. It is proper and smooth. The substack $\mathcal{M}_{g,n}$ of smooth curves is open. (The condition on (g, n) ensures that $\mathcal{M}_{g,n}$ is not empty.) The embedding $\mathcal{M}_{g,n} \subseteq \overline{\mathcal{M}}_{g,n}$ is toroidal, hence defines a stratification of $\overline{\mathcal{M}}_{g,n}$.

The strata are as follows. Let G be any stable graph with n legs and genus g. Then the substack of curves with dual graph isomorphic to G is a locally closed stratum $\mathcal{M}_G$. We find $\overline{\mathcal{M}}_{g,n} = \bigsqcup_G \mathcal{M}_G$. The codimension of $\mathcal{M}_G$ in $\overline{\mathcal{M}}_{g,n}$ is the number of edges in G. In particular, $\mathcal{M}_{g,n}$ is the unique stratum of maximal dimension $3g - 3 + n$. The stratum $\mathcal{M}_G$ is contained in the closure of $\mathcal{M}_{G'}$ if and only if G' is a contraction of G. In order words, by contracting the dual graph we pass to larger strata of $\overline{\mathcal{M}}_{g,n}$.

7. Moduli of tropical curves

Definition 7.1. A *tropical curve* is a graph $G = (V, E, L, h)$ equipped with a *length function* $\ell \colon E \to \mathbb{R}_{>0}$. Similarly, an *extended tropical curve* is a graph $G = (V, E, L, h)$ equipped with a *length function* $\ell \colon E \to \overline{\mathbb{R}_{>0}}$, where $\overline{\mathbb{R}_{>0}} = \mathbb{R}_{>0} \sqcup \{\infty\}$. ♦

We leave *isomorphisms of (extended) tropical curves* to be defined by the reader.

As before, we fix natural numbers $g, n \geq 0$ such that the quantity $2g - 2 + n$ is positive. Let $M_{g,n}^{\text{trop}}$ and $\overline{M}_{g,n}^{\text{trop}}$ be the sets of isomorphism classes of n-pointed genus g stable tropical curves, respectively n-pointed genus g stable extended tropical curves. They come with natural structures of generalized (extended) cone complexes, to be introduced below.

Definition 7.2. The complexes $M_{g,n}^{\text{trop}}$ and $\overline{M}_{g,n}^{\text{trop}}$ are called the *coarse moduli spaces of n-pointed genus g stable (extended) tropical curves*. ♦

Remark 7.3. The name 'coarse moduli space' is justified as follows. With some care one defines *families* of tropical curves over cones. This leads to the notion of a moduli stack $\mathcal{M}_{g,n}^{\text{trop}}$. It admits a coarse moduli space, which indeed is $M_{g,n}^{\text{trop}}$. Details are in [4]. For us it suffices to view $M_{g,n}^{\text{trop}}$ as a set with some extra structure. ♦

Let G be a stable graph with n legs and genus g. Set $\sigma_G = \mathbb{R}_{\geq 0}^E$, where E is the set of edges in G. The relative interior σ_G° parametrizes tropical curves endowed with an isomorphism from their graph to G. The extended relative interior $\bar{\sigma}_G^\circ$ parametrizes extended tropical curves with such an identification. This identification is unique up to automorphisms of G, hence the quotients

$$M_G^{\text{trop}} = \text{Aut}(G)\backslash\sigma_G^\circ \qquad \text{and} \qquad \overline{M}_G^{\text{trop}} = \text{Aut}(G)\backslash\bar{\sigma}_G^\circ$$

parametrize (extended) tropical curves whose graph is isomorphic to G.

Now let G' be a contraction of G. Passing from G to G' corresponds to letting some edge lengths go to 0. As such we can identify $\sigma_{G'}$ with a face of σ_G. Recall that graph automorphisms are contractions as well. We conclude that

$$M_{g,n}^{\text{trop}} = \operatorname*{colim}_G \sigma_G \qquad \text{and} \qquad \overline{M}_{g,n}^{\text{trop}} = \operatorname*{colim}_G \bar{\sigma}_G$$

where the colimits are over the diagram of graphs and contractions. The obvious partitions $M_{g,n}^{\text{trop}} = \bigsqcup_G M_G^{\text{trop}}$ and $\overline{M}_{g,n}^{\text{trop}} = \bigsqcup_G \overline{M}_G^{\text{trop}}$ are a special case of proposition 4.4.

Remark 7.4. The stratification of $\overline{M}_{g,n}^{\text{trop}}$ is dual to that of $\overline{\mathcal{M}}_{g,n}$, in the following sense. While the stratum $\overline{M}_G^{\text{trop}} \subseteq \overline{M}_{g,n}^{\text{trop}}$ has dimension $\#E$, the corresponding stratum $\mathcal{M}_G \subseteq \overline{\mathcal{M}}_{g,n}$ has codimension $\#E$. Moreover, when G' is a contraction of G, the stratum $\overline{M}_{G'}^{\text{trop}}$ lies in the closure of $\overline{M}_G^{\text{trop}}$, whereas $\mathcal{M}_G$ is in the closure of $\mathcal{M}_{G'}$. ♦

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