Berkovich analytification

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Oct 2017 — correction to example 1.3.

1. Berkovich spaces

Throughout, k is a non-archimedean field.

Let's recall some definitions. An *affinoid algebra* A is a quotient of $k\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n\}$ for some $r_1, \ldots, r_n > 0$. Its *Berkovich spectrum* is the topological space

$$X = \mathcal{M}(A) = \left\{ \begin{matrix} \text{bounded multiplicative} \\ \text{seminorms on } A \end{matrix} \right\} = \left\{ \begin{matrix} \text{(equivalence classes of) bounded} \\ \text{maps from } A \text{ to a valuation field} \end{matrix} \right\}.$$

A *domain* $D \subseteq X$ is a closed subset that admits a bounded homomorphism $A \to A_D$ universal for the property that $\mathcal{M}(A_D) \to \mathcal{M}(A)$ has image inside D. For such D one has $D = \mathcal{M}(A_D)$. The map $A \to A_D$ is flat. Domains form systems of neighborhoods at each point of X. In other words, domains behave exactly like the standard opens $\mathrm{D}(f) \subseteq \operatorname{Spec} A$ do for schemes.

A subset $Z \subseteq X$ is *special* if it is a finite union of domains. If $Z = D_1 \cup \ldots \cup D_n$ is such a presentation, then we write $A_Z = \ker\big(\prod_i A_{D_i} \rightrightarrows \prod_{i,j} A_{D_i \cap D_j}\big)$. Define

$$\mathcal{O}_X(U) = \lim_{Z \subseteq U \text{ special}} A_Z.$$

Now $(X, \mathcal{O}_X)$ is a locally ringed space, satisfying $\mathcal{O}_X(X) = A$.

Remark 1.1. The functor $\mathcal{M} \colon \{\text{affinoid algebras}\} \to \{\text{locally ringed spaces}\}$ is faithful, but not full. This problem, together with the lack of localizations, warrants the somewhat involved definition below. ♦

A *quasi-affinoid space* is a locally ringed space U equipped with the data of an affinoid algebra A and an open immersion $U \hookrightarrow \mathcal{M}(A)$. A *morphism* of quasi-affinoid spaces $f \colon (U, A) \to (V, B)$ is a map of locally ringed spaces such that for all domains $D \subseteq U$, $E \subseteq V$ with $f(D) \subseteq E$ the restriction $f \colon D \to E$ is affinoid, i.e. comes from a bounded map $B_E \to A_D$.

Definition 1.2. A *Berkovich space* is a locally ringed space X equipped with the equivalence class of an *analytic atlas* $\{(U_i, A_i) \colon i \in I\}$ where the U_i form an open covering of X, each (U_i, A_i) is a quasi-affinoid space, and the identity maps $(U_i \cap U_j, A_i) \to (U_i \cap U_j, A_j)$ are morphisms of quasi-affinoid spaces. A *morphism* of Berkovich spaces is a morphism of locally ringed spaces that is locally a morphism of quasi-affinoid spaces. ♦

Berkovich spaces are locally Hausdorff, locally compact, and locally path-connected [2].

Example 1.3. Consider the Berkovich spectrum of $k\{r^{-1}T\} = k\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n\}$. We assume that k is algebraically closed. Let $E = \mathrm{E}(a,s) \subseteq k$ be a closed disk contained in $\mathrm{E}(0,r)$. Then the rule $\left|\sum_n c_n (T-a)^n\right|_E = \max_n |c_n| s^n$ defines a point of $\mathcal{M}(k\{r^{-1}T\})$. It is of *type I* if $s = 0$, of *type II* if $s \in |k^\times|$, and of *type III* otherwise. The type II point $|\cdot|_{\mathrm{E}(0,r)}$ is called the *Gauss point*. When $E_0 \supseteq E_1 \supseteq \ldots$ is a decreasing sequence of disks with empty intersection, the limit $\lim_n |\cdot|_{E_n}$ is a *type IV* point.

In the one-dimensional case $n = 1$, each point of $\mathcal{M}(k\{r^{-1}T\})$ is one of type I–IV. See figure (1) for a picture. The quasi-affinoid subspace $\mathrm{D}(0,r) = \{x \in \mathcal{M}(k\{r^{-1}T\}) \colon |T|_x < r\}$ is indicated as well. Observe also that for $s \leq r$ one has $\mathcal{M}(k\{s^{-1}T\}) \subseteq \mathcal{M}(k\{r^{-1}T\})$. ♦

Example 1.4. The n-dimensional *Berkovich affine space* is

$$\mathbb{A}^{n,\mathrm{an}} = \left\{ \begin{matrix} \text{multiplicative seminorms on } k[T_1, \ldots, T_n] \\ \text{extending the given norm on } k \end{matrix} \right\}$$

with the weakest topology that renders all functions $\mathbb{A}^{n,\mathrm{an}} \to \mathbb{R}$, $x \mapsto |f|_x$ continuous. Let $\mathcal{O}_{\mathbb{A}^{n,\mathrm{an}}}$ be the sheaf of *analytic functions* on $\mathbb{A}^{n,\mathrm{an}}$, i.e. the sheaf of functions that are locally limits of rational functions. Then $\mathbb{A}^{n,\mathrm{an}}$ is a Berkovich space. Indeed, $\mathbb{A}^{n,\mathrm{an}} = \bigcup_{r \in \mathbb{R}^n_{>0}} \mathrm{D}(0, r)$ where $\mathrm{D}(0, r)$ is the quasi-affinoid space from example 1.3. ♦

2. Géometrie algébrique et géométrie analytique

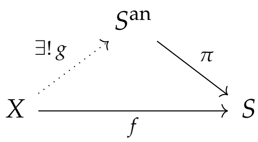
Here is the main result of this talk.

Theorem 2.1. *Let S be a scheme locally of finite type over k. The functor*

$$\mathrm{Berk}/k \to \mathrm{Set}, \qquad X \mapsto \mathrm{Hom}_{\mathrm{LRS}/k}(X, S)$$

is representable by a Berkovich space S^{an}.

Proof. In other words, there should be a Berkovich space S^{an} and a canonical map $\pi\colon S^{\mathrm{an}} \to S$ such that every map of locally ringed spaces $f\colon X \to S$ factors uniquely as



where g is a morphism of Berkovich spaces. We proceed in several steps.

Case 1: $S = \mathbb{A}^n$. Then $\pi\colon \mathbb{A}^{n,\mathrm{an}} \to \mathbb{A}^n$, $x \mapsto \ker |\cdot|_x$ does the job, with $\mathbb{A}^{n,\mathrm{an}}$ the Berkovich affine space from example 1.4. Indeed, $\mathrm{Hom}_{\mathrm{LRS}/k}(X, \mathbb{A}^n) = \mathcal{O}_X(X)^n = \mathrm{Hom}_{\mathrm{Berk}/k}(X, \mathbb{A}^{n,\mathrm{an}})$. The last identity holds by reduction to X affinoid, since a map $\mathcal{M}(A) \to \mathbb{A}^{n,\mathrm{an}}$ corresponds to a bounded map $k\{r^{-1}T\} \to A$ for some $r \in \mathbb{R}^n_{>0}$.

Case 2: $T \subseteq S$ open. Suppose that $\pi\colon S^{\mathrm{an}} \to S$ is a Berkovich analytification, i.e. satisfies the statement of the theorem. Then $T^{\mathrm{an}} = \pi^{-1}(T)$ is a Berkovich space and $T^{\mathrm{an}} \to T$ is a Berkovich analytification.

Case 3: $T \subseteq S$ closed. Suppose that $\pi\colon S^{\mathrm{an}} \to S$ is a Berkovich analytification. Let $\mathcal{I}$ be the quasi-coherent sheaf of $\mathcal{O}_S$-ideals defining T. Then $\pi^*\mathcal{I}$ is a quasi-coherent sheaf of $\mathcal{O}_{S^{\mathrm{an}}}$-ideals. If $T^{\mathrm{an}} \subseteq S^{\mathrm{an}}$ is the corresponding closed subspace, then $T^{\mathrm{an}} \to T$ is a Berkovich analytification.

Case 4: S general. If S is affine, then S may be realized as a closed subscheme of affine space, so the theorem holds by combining cases 1 and 3. The general situation follows by glueing, using case 2. ■

Due to the universal property, the construction $S \mapsto S^{\mathrm{an}}$ is functorial.

Example 2.2. Let A be a finitely generated k-algebra and $S = \mathrm{Spec}\, A$. Then

$$S^{\mathrm{an}} = \left\{ \begin{matrix} \text{multiplicative seminorms on } A \\ \text{extending the given norm on } k \end{matrix} \right\}$$

with the weakest topology that renders all functions $S^{\mathrm{an}} \to \mathbb{R}$, $x \mapsto |f|_x$ continuous. The canonical map $\pi\colon S^{\mathrm{an}} \to S$ is given by $x \mapsto \mathfrak{p}_x = \ker |\cdot|_x$. ♦

Here is an alternative description of S^{an}. In the example, $x \in S^{\mathrm{an}}$ extends to a multiplicative norm on the residue field $\kappa(\pi(x)) = \mathrm{Frac}(A/\mathfrak{p}_x)$. Conversely, if $|\cdot|\colon \kappa(s) \to \mathbb{R}$ is a multiplicative norm, then $|\cdot|$ induces a multiplicative seminorm on A. Hence, as a set,

$$S^{\mathrm{an}} = \left\{ (s, |\cdot|)\colon \begin{array}{l} s \in S,\ |\cdot|\colon \kappa(s) \to \mathbb{R} \text{ absolute} \\ \text{value extending the norm on } k \end{array} \right\}, \tag{2.3}$$

with $\pi\colon S^{\mathrm{an}} \to S$ the forgetful map. The topology is the weakest for which π as well as all maps $\pi^{-1}(U) \to \mathbb{R}$, $(s, |\cdot|) \mapsto |f(s)|$ for $U \subseteq S$ open and $f \in \mathcal{O}_X(U)$ are continuous.

Proposition 2.4. *Let S be a scheme locally of finite type over k.*

- $\pi\colon S^{\mathrm{an}} \to S$ *is faithfully flat.*
- *Set $S_0 = \{s \in S\colon [\kappa(s) : k] < \infty\}$ and, for X a Berkovich space, $X_0 = \{x \in X\colon [\mathcal{H}(x) : k] < \infty\}$. Then π restricts to a bijection $(S^{\mathrm{an}})_0 \to S_0$.*
- *For $x \in (S^{\mathrm{an}})_0$, the map $\mathcal{O}_{S^{\mathrm{an}},x} \to \mathcal{O}_{S,\pi(x)}$ becomes an isomorphism after completion.*

Proof. A morphism of locally ringed spaces is *faithfully flat* if has flat stalk maps and is surjective. Flatness of π can be deduced from the proof of theorem 2.1. Surjectivity and the second statement are an immediate consequence of the description of S^{an} in (2.3). The last item is again a consequence of the proof of theorem 2.1. ■

The Berkovich analytification S^{an} shares many properties of S. We omit the proofs. Some more details may be found in section 3.4 of [2].

Proposition 2.5. *Let S be a scheme locally of finite type over k. Then S is reduced, Cohen–Macaulay, regular, or smooth if and only if S^{an} has the same property.* ♦

Proposition 2.6. *Let f a morphism of schemes locally of finite type over k. Then f is flat, unramified, étale, smooth, separated, injective, surjective, an open immersion, an isomorphism, or a monomorphism if and only if f^{an} has the same property. If f is quasi-compact, then f is dominant, a closed immersion, proper, or finite if and only if f^{an} has the same property.* ♦

The *dimension* of a Berkovich space X is the smallest natural number n such that every finite open covering of X has a finite refinement $(U_i)_{i\in I}$ such that if $i_0, \ldots, i_m \in I$ are distinct and the intersection $U_{i_0} \cap \ldots \cap U_{i_m}$ is non-empty, then $m \le n$.

Proposition 2.7. *Let S be a scheme locally of finite type over k. Then*

- *S is separated if and only if S^{an} is Hausdorff,*
- *S is proper if and only if S^{an} is compact Hausdorff,*
- *S is connected if and only if S^{an} is connected, and*
- $\dim S = \dim S^{\mathrm{an}}$. ♦

As in complex geometry, there are comparison theorems for coherent sheaves.

Theorem 2.8. *Let S be a scheme locally of finite type over k. The functor*

$$\mathcal{O}_S\text{-Mod} \to \mathcal{O}_{S^{\mathrm{an}}}\text{-Mod}, \qquad \mathcal{F} \mapsto \mathcal{F}^{\mathrm{an}} = \pi^*\mathcal{F}$$

is faithful, exact, and preserves coherence. If S is proper, the restriction $(-)^{\mathrm{an}}\colon \mathrm{Coh}\, S \to \mathrm{Coh}\, S^{\mathrm{an}}$ is an equivalence. ♦

Theorem 2.9. *Let $f\colon S \to T$ be a proper morphism of schemes locally of finite type over k. Let $\mathcal{F}$ be a coherent $\mathcal{O}_S$-module. There are canonical isomorphisms $(\mathrm{R}^i f_*\mathcal{F})^{\mathrm{an}} \cong \mathrm{R}^i f^{\mathrm{an}}_*(\mathcal{F}^{\mathrm{an}})$. In particular, if S is a proper k-scheme and $\mathcal{F}$ a coherent $\mathcal{O}_S$-module, then $\mathrm{H}^p(S, \mathcal{F}) \cong \mathrm{H}^p(S^{\mathrm{an}}, \mathcal{F}^{\mathrm{an}})$.* ♦

Corollary 2.10. *Let S be a proper scheme over k. The functor $T \mapsto T^{\mathrm{an}}$ from finite S-schemes to finite S^{an}-Berkovich spaces is an equivalence. In particular, every proper Berkovich curve is algebraic.* ♦

3. Tate elliptic curves

Recall that every elliptic curve over $\mathbb{C}$ has a uniformization $E = \mathbb{C}/\Lambda$ where $\Lambda = \mathbb{Z} \oplus \mathbb{Z}\tau$ with $\operatorname{Im} \tau > 0$. A similar approach over $\mathbb{Q}_p$ or $\mathbb{C}_p$ cannot work, for instance because p-adic fields have no non-trivial discrete subgroups. There is, however, another perspective in complex geometry. Write $q = e^{2\pi i \tau}$. The convention $\operatorname{Im} \tau > 0$ implies $0 < |q| < 1$. There is an isomorphism of complex spaces

$$\mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\tau) \to \mathbb{C}^\times/q^{\mathbb{Z}}, \qquad z \mapsto e^{2\pi i z}.$$

The latter description has a non-archimedean analogue due to Tate.

As usual, let k be a non-archimedean field. Fix an element $q \in k$ with $0 < |q| < 1$. We define $\sigma_m(n) = \sum_{d|n} d^m$, the series $s_m(q) = \sum_{n \geq 1} \sigma_m(n) q^n$, and

$$a_4(q) = -5s_3(q), \qquad a_6(q) = -\frac{5s_3(q) + 7s_5(q)}{12}.$$

Both $a_4(q)$ and $a_6(q)$ lie in $\mathbb{Z}[\![q]\!]$, so converge to elements of k.

Theorem 3.1. *Let $E_q \subseteq \mathbb{P}^2$ be the elliptic curve with affine equation $y^2 + xy = x^3 + a_4(q)x + a_6(q)$. It has discriminant $\Delta(q) = q \prod_{n \geq 1} (1 - q^n)^{24}$ and j-invariant $j(q) = \frac{1}{q} + 744 + \ldots$. For any algebraic extension K of k, there is a Galois-equivariant isomorphism of rigid spaces $K^\times/q^{\mathbb{Z}} \cong E_q(K)$.* ♦

Note that $|\Delta(q)| = |q|$, hence $\Delta(q)$ is non-zero and E_q is truly an elliptic curve. On the other hand, $|\Delta(q)| < 1$ implies that E_q has bad reduction over the residue field $\tilde{k}$ of k. Therefore its reduction over $\tilde{k}$ is either a nodal curve or a cusp; equivalently, its connected Néron model is either a form of $\mathbb{G}_\mathrm{m}$ or a form of $\mathbb{G}_\mathrm{a}$. In the former case we speak of *multiplicative reduction*, in the latter of *additive reduction*. It turns out that the inequality $|j(q)| = |q|^{-1} > 1$ implies that E_q always has multiplicative reduction.

Theorem 3.2. *Let E/k be an elliptic curve with $|j(E)| > 1$. There is a unique element $q \in k$ satisfying $0 < |q| < 1$ such that the elliptic curves E and E_q become isomorphic over the algebraic closure of k. There is an isomorphism $E \cong E_q$ over k if and only if E has split multiplicative reduction.* ♦

See chapter V of [3] for proofs.

Let's now construct the Berkovich space E_q^{an}. Consider first the complex case. If q is a complex number satisfying $0 < |q| < 1$, the quotient $\mathbb{C}^\times/q^{\mathbb{Z}}$ may be constructed as follows: take in $\mathbb{C}$ the closed annulus centered at 0 with outer radius 1 and inner radius $|q|$, then identify the outer and inner boundaries. See figure (2).

In the setting of Berkovich spaces we take the same approach. The complex closed unit disk corresponds to the affinoid Berkovich space $\mathcal{M}(k\{T\})$. It naturally contains the quasi-affinoid $\mathrm{D}(0, |q|)$. Now E_q^{an} arises by identifying the outer and inner boundaries of $\mathcal{M}(k\{T\}) \setminus \mathrm{D}(0, |q|)$. In $\mathcal{M}(k\{T\})$ a canonical path connects the Gauss points $|\cdot|_{\mathrm{E}(0,1)}$ and $|\cdot|_{\mathrm{E}(0,|q|)}$. This path maps to a loop in E_q^{an}. This loop will later be called the *skeleton* of E_q^{an} and is a deformation retract of the whole Berkovich space. See figure (3) and section 5.2 from [1].

Remark 3.3. We can now point out another reason why lattice uniformization of elliptic curves does not work for Berkovich spaces. Recall that a complex elliptic curve E is a torus, hence has fundamental group $\mathbb{Z} \times \mathbb{Z}$. The map $\mathbb{C}^\times \to \mathbb{C}^\times/q^{\mathbb{Z}} = E$ is a covering space with group $\mathbb{Z}$. The exponential $\mathbb{C} \to \mathbb{C}^\times$ is also a covering space with group $\mathbb{Z}$. Indeed, $\mathbb{C}$ is simply connected and $\mathbb{C} \to E$ is a universal covering space.

On the other hand, the Berkovich space E_q^{an} deformation restracts to a circle, hence has fundamental group $\mathbb{Z}$. The first uniformization step $\mathbb{G}_\mathrm{m}^{\mathrm{an}} \to E_q^{\mathrm{an}}$ still applies. But the Berkovich space $\mathbb{G}_\mathrm{m}^{\mathrm{an}} = \mathbb{A}^{n,\mathrm{an}} \setminus \{0\}$ is already simply connected, so $\mathbb{C} \to \mathbb{C}^\times$ cannot have an analogue. ♦

References

[1] M. Baker, *An introduction to Berkovich analytic spaces and non-archimedean potential theory on curves*. In: *p-Adic Geometry, Lectures from the 2007 Arizona Winter School*, University Lecture Series 45, American Mathematical Society, 2008, pp. 123–174.

[2] V.G. Berkovich, *Spectral Theory and Analytic Geometry over Non-Archimedean Fields*. Mathematical Surveys and Monographs 33, American Mathematical Society, 1990.

[3] J.H. Silverman, *Advanced Topics in the Arithmetic of Elliptic Curves*. Graduate Texts in Mathematics 151, Springer-Verlag, 1994.

Figure (1)

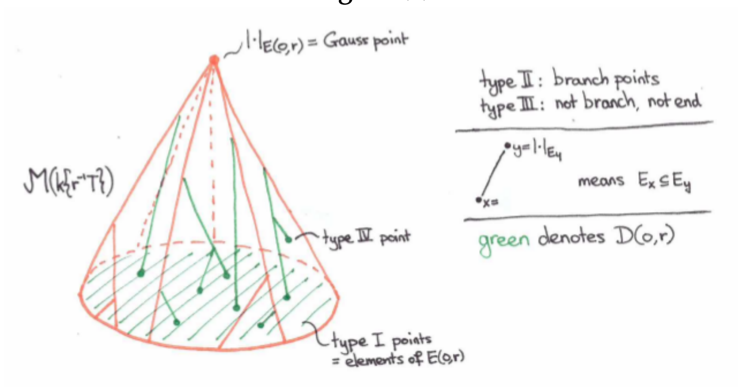


Figure (2)

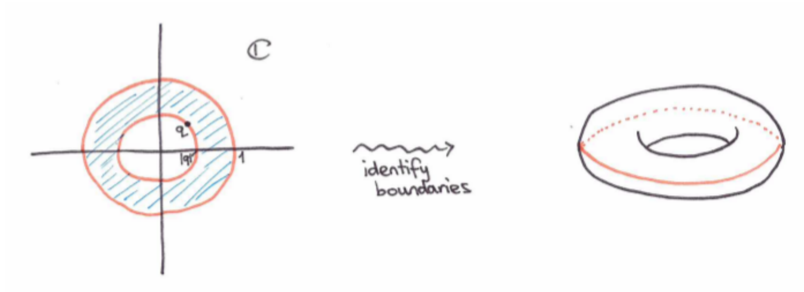


Figure (3)