$\mathcal{D}$-modules — Introduction

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1. Overview

Let's first say what this seminar is actually about. A *(coherent) $\mathcal{D}$-module* or *stratified bundle* is a vector bundle endowed with an infinitesimal descent datum. Informally such a descent datum is an algebraic differential equation. More concretely: let k be an algebraically closed field and X/k a smooth (integral and separated) variety. On X we will define a sheaf $\mathcal{D}_{X/k}$ of *differential operators* $\mathcal{O}_X \to \mathcal{O}_X$. Various examples of sections of $\mathcal{D}_{X/k}$ are

- (degree 0) sections $f \in \mathcal{O}_X$,
- (degree 1) derivations $\theta \in \mathcal{T}_{X/k}$,
- compositions of such.

Composition defines a ring structure on $\mathcal{D}_{X/k}$. It is not commutative! For instance, for $f \in \mathcal{O}_X$ and $\theta \in \mathcal{T}_{X/k}$ one has $\theta \circ f = f \circ \theta + \theta(f)$.

Example 1.1. Take $X = \mathbb{A}^n$ with coordinates $x_1, \ldots, x_n$.

- If $\operatorname{char} k = 0$, then $\mathcal{D}_{X/k}$ is the free $\mathcal{O}_X$-module on generators $\partial_{x_1}^{e_1} \cdots \partial_{x_n}^{e_n}$.
- If $\operatorname{char} k = p$ is positive, then we have $\partial_{x_i}^p = 0$. Nevertheless, there exists a degree p operator that acts like $\frac{1}{p!}\partial_{x_i}^p$ and does not come from derivations. ♦

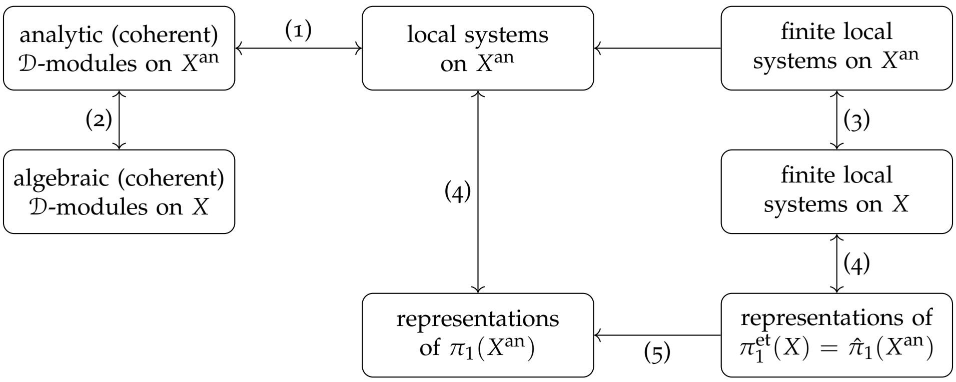
A *$\mathcal{D}$-module* on X/k is simply a sheaf $\mathcal{E}$ on X with a $\mathcal{D}_{X/k}$-module structure. A first example is the sheaf $\mathcal{E} = \mathcal{O}_X$ with action $\theta \cdot f = \theta(f)$.

Example 1.2. Take global sections $\theta_1, \cdots, \theta_r \in \mathcal{D}_{X/k}(X)$. They determine a subsheaf $\mathcal{S}ol \subseteq \mathcal{O}_X$ of *solutions*, i.e. sections $f \in \mathcal{O}_X$ satisfying $\theta_1(f) = \ldots = \theta_r(f) = 0$. Let $\mathcal{M}$ be the $\mathcal{D}$-module $\mathcal{D}_{X/k}/(\mathcal{D}_{X/k}\theta_1 + \ldots + \mathcal{D}_{X/k}\theta_r)$. Then

$$\mathcal{S}ol \longleftrightarrow \mathcal{H}om_{\mathcal{D}_{X/k}}(\mathcal{M}, \mathcal{O}_X), \qquad \begin{aligned} f &\mapsto [\bar\theta \mapsto \theta(f)] \\ \varphi(\bar 1) &\leftarrow\!\shortmid \varphi \end{aligned}$$

are inverse isomorphisms. ♦

One of our main goals is to derive the *Riemann–Hilbert correspondence*. This is a result from complex geometry that relates $\mathcal{D}$-modules to local systems. The latter are purely topological! It is best expressed in terms of the analytic space X^{an} associated to a complex variety $X/\mathbb{C}$. Diagrammatically:



Here (1) is the classical, analytic Riemann–Hilbert correspondence. The equivalence (2) holds by the standard GAGA theorems if X is proper. On the other side, however, local systems on X^{an} do not have a good algebraic analogue. But *finite* local systems do. Equivalence (3) is called the Riemann existence theorem. In another direction, local systems on X^{an} correspond to representations of the fundamental group $\pi_1(X^{\mathrm{an}})$. Similarly finite local systems correspond to representations of the profinite completion $\hat{\pi}_1(X^{\mathrm{an}})$. These are the equivalences (4). The profinite completion $\hat{\pi}_1(X^{\mathrm{an}})$ has an algebraic description as the *étale fundamental group* $\pi_1^{\mathrm{et}}(X)$. Finally, the induced inclusion (6) is simply given by restriction along $\pi_1(X^{\mathrm{an}}) \to \pi_1^{\mathrm{et}}(X)$.

Theorem 1.3 (Grothendieck–Malčev). *Let π be a finitely generated group. If $\hat{\pi} = 1$, then all representations of π are trivial.* ♦

Corollary 1.4. *All coherent $\mathcal{D}$-modules on a proper smooth variety $X/\mathbb{C}$ are trivial iff $\pi_1^{\mathrm{et}}(X) = 1$.* ♦

Gieseker conjectured in 1975 that the same should be true in positive characteristic, and proved the implication $\Rightarrow$. The converse was proven only in 2010 by Esnault–Mehta. We will not go into that proof. Instead, towards the end of the seminar we will verify the conjecture in several specific examples.

2. Preliminaries

In this section we recall a few basic notions that we'll need. The results below can be found in Hartshorne's *Algebraic geometry* or Liu's *Algebraic geometry and arithmetic curves*. All schemes are tacitly assumed to be locally noetherian.

Let X be a scheme. An $\mathcal{O}_X$*-module* is an abelian sheaf $\mathcal{E}$ endowed with a module structure under the structure sheaf $\mathcal{O}_X$. We recall some constructions:

$$\bigoplus_{i\in I} \mathcal{E}_i = \Big[U \mapsto \bigoplus_{i\in I} \mathcal{E}_i(U)\Big]^{\#}, \qquad \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{F}) = \Big[U \mapsto \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{E}|_U, \mathcal{F}|_U)\Big],$$

$$\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F} = \Big[U \mapsto \mathcal{E}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{F}(U)\Big]^{\#}, \qquad \mathcal{E}^{\vee} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X).$$

An $\mathcal{O}_X$-module $\mathcal{E}$ is *free* if it is isomorphic to a direct sum $\bigoplus_{i\in I} \mathcal{O}_X$. It is *locally free* if locally on X it is free. An important class of $\mathcal{O}_X$-modules are *vector bundles*, locally free $\mathcal{O}_X$-modules of finite rank. If $f\colon X \to Y$ is a morphism, there is a *direct image* functor $f_*\mathcal{E}$ and an *inverse image* functor $f^*\mathcal{F} = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{F}$. They satisfy $\mathrm{Hom}_{\mathcal{O}_X}(f^*\mathcal{F}, \mathcal{E}) = \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{F}, f_*\mathcal{E})$.

Let A be a ring and M an A-module. Associated to M is an $\mathcal{O}_X$-module $\widetilde{M}$ on $X = \operatorname{Spec} A$ defined by $\widetilde{M}(\mathrm{D}(f)) = M_f$. An $\mathcal{O}_X$-module is called *quasi-coherent* if locally on X it is of this form. It is called *coherent* if moreover the module M is finitely generated. Being (quasi-)coherent is stable under taking kernels, cokernels, images, extensions, direct sums, and tensor products. However, the dual of a quasi-coherent $\mathcal{O}_X$-module is not necessarily quasi-coherent.

Theorem 2.1. *Let $f\colon X \to Y$ be a morphism, $\mathcal{E}$ an $\mathcal{O}_X$-module, and $\mathcal{F}$ and $\mathcal{O}_Y$-module.*

- *If $\mathcal{F}$ is (quasi-)coherent, then so is $f^*\mathcal{F}$.*
- *Suppose f is quasi-compact and quasi-separated. If $\mathcal{E}$ is quasi-coherent, then so is $f_*\mathcal{E}$.*
- *Suppose f is proper. If $\mathcal{E}$ is coherent, then so is $f_*\mathcal{E}$.* ♦

Example 2.2. Let $f\colon \operatorname{Spec} A \to \operatorname{Spec} B$ be a morphism of affine schemes, M an A-module, and N a B-module. Then

$$f^*\widetilde{N} = \widetilde{A \otimes_B N}, \qquad f_*\widetilde{M} = \widetilde{{}_BM}$$

where ${}_BM$ is the B-module obtained from M by restriction of scalars. ♦

As next topic we recall derivations. Let A be a ring, B an A-algebra, and M a B-module. An A-*derivation* $\theta\colon B \to M$ is an A-linear map satisfying the Leibniz rule $\theta(bc) = b\theta(c) + c\theta(b)$. Each A-algebra B has a *universal* A-*derivation* denoted $\mathrm{d}\colon B \to \Omega^1_{B/A}$. The elements of $\Omega^1_{B/A}$ are called *differential forms*. The construction of $\Omega^1_{B/A}$ commutes with base change, in particular with localization. Hence for a scheme X/S we obtain the *sheaf of differential forms* $\Omega^1_{X/S}$. It is quasi-coherent; if X/S is locally of finite type, it is coherent.

Here is an abstract construction. Let I be the kernel of the multiplication map $B \otimes_A B \to B$. Then $\Omega^1_{B/A} = I/I^2$ with $\mathrm{d}b = 1 \otimes b - b \otimes 1$. Similarly, for a scheme X/S let $\Delta\colon X \to X \times_S X$ be the diagonal and $\mathcal{I}$ the kernel of $\Delta^\#\colon \mathcal{O}_{X\times_S X} \to \Delta_*\mathcal{O}_X$. Then $\Omega^1_{X/S} = \Delta^*(\mathcal{I}/\mathcal{I}^2)$.

Example 2.3. Let $X = \mathrm{Z}(f_1, \ldots, f_r) \subseteq \mathbb{A}^n$ over $S = \operatorname{Spec} k$. Then

$$\Omega^1_{X/k} = \frac{\bigoplus_{i=1}^n \mathcal{O}_X \mathrm{d}x_i}{\mathcal{O}_X \mathrm{d}f_1 + \ldots + \mathcal{O}_X \mathrm{d}f_r}.$$

Here $\mathrm{d}x_1, \ldots, \mathrm{d}x_n$ are symbols and $\mathrm{d}f = \frac{\partial f}{\partial x_1}\mathrm{d}x_1 + \ldots + \frac{\partial f}{\partial x_n}\mathrm{d}x_n$. ♦

Example 2.4. There is a short exact sequence

$$0 \longrightarrow \Omega^1_{\mathbb{P}^n/k} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^{n+1} \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow 0.$$

In particular $\Omega^1_{\mathbb{P}^1/k} = \mathcal{O}_{\mathbb{P}^1}(-2)$. ♦

The *tangent sheaf* on X/S is $\mathcal{T}_{X/S} = (\Omega^1_{X/S})^\vee$. By the universal property of $\Omega^1_{X/S}$ we immediately see that $\mathcal{T}_{X/S}$ is the sheaf of $\mathcal{O}_S$-derivations $\mathcal{O}_X \to \mathcal{O}_X$.

To conclude we recall some facts about smoothness. A morphism $X \to S$ is *smooth* if it is locally of finite type, flat, and has regular geometric fibers. It is *smooth of relative dimension n* if moreover all fibers are pure of dimension n. It is *étale* if it is smooth of relative dimension zero. Recall that X/k is regular if at all (closed) points $x \in X$ one has $\dim_{\kappa(x)} \mathfrak{m}_x/\mathfrak{m}_x^2 = \dim \mathcal{O}_{X,x}$.

Theorem 2.5. *Let k be an algebraically closed field and X/k a variety. The following are equivalent:*
- *X is smooth of dimension n,*
- *X is regular of dimension n,*
- *$\Omega^1_{X/k}$ is locally free of rank n,*
- *$\mathcal{T}_{X/k}$ is locally free of rank n,*
- *locally X is of the form $\mathrm{Z}(f_1, \ldots, f_r) \subseteq \mathbb{A}^{n+r}$ such that $J = \left(\frac{\partial f_i}{\partial x_j}\right)_{i,j}$ has rank r.* ♦

Theorem 2.6. *Let $f\colon X \to Y$ be a morphism of smooth k-varieties.*
- *f is smooth iff $\Omega^1_{X/Y}$ is locally free iff the tangent map $\mathcal{T}_{X/k} \to f^*\mathcal{T}_{Y/k}$ is surjective.*
- *f is étale iff $\Omega^1_{X/Y} = 0$ iff the tangent map $\mathcal{T}_{X/k} \to f^*\mathcal{T}_{Y/k}$ is bijective.* ♦

Theorem 2.7. *Let X/k be a smooth variety and $x \in X$. There exists an open neighborhood $U \subseteq X$ of x and regular functions $f_1, \ldots, f_n \in \mathcal{O}_X(U)$ such that $\mathrm{d}f_1, \ldots, \mathrm{d}f_n$ form a basis of $\Omega^1_{X/k}$ on U. The induced k-morphism $f\colon U \to \mathbb{A}^n$ is étale.* ♦

Combining the last two theorems yields the following fact. On any smooth variety X/k locally one has $\Omega^1_{X/k} = f^*\Omega^1_{\mathbb{A}^n/k}$ for some étale map f. Thus we know the local structure of $\Omega^1_{X/k}$ on any smooth variety. We will use the same idea to compute the local structure of $\mathcal{D}_{X/k}$.