$\mathcal{D}$-modules — *F*-divided bundles

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1. The Gieseker conjecture

From this lecture onwards, we will study $\mathcal{D}$-modules in positive characteristic. But for the sake of motivation, we first recall some facts from characteristic zero. Let $X/\mathbb{C}$ be a proper smooth variety. We have seen equivalences

$$\boxed{\begin{matrix}\text{stratified}\\ \text{bundles on } X\end{matrix}} \overset{(1)}{\longleftrightarrow} \boxed{\begin{matrix}\text{stratified}\\ \text{bundles on } X^{\text{an}}\end{matrix}} \overset{(2)}{\longleftrightarrow} \boxed{\begin{matrix}\text{representations}\\ \text{of } \pi_1(X^{\text{an}})\end{matrix}}$$

where (1) comes from the GAGA theorems and (2) is the Riemann–Hilbert correspondence. We had trouble in making this fully algebraic since, algebraically, only the étale fundamental group $\pi_1^{\text{et}}(X) = \hat{\pi}_1(X^{\text{an}})$ is available. Nevertheless we can say something, due to the following forgotten-and-rediscovered fact.

Theorem 1.1 (Malčev–Grothendieck). *Let $G \to H$ be a homomorphism of finitely generated groups. If $\hat{G} \to \hat{H}$ is an isomorphism, then the pullback functor $\operatorname{Rep}_{\mathbb{C}} H \to \operatorname{Rep}_{\mathbb{C}} G$ is an equivalence.* ⧫

Theorem 1.2. *Let $X/\mathbb{C}$ be a proper smooth variety. Then*
- $\pi_1^{\text{et}}(X)$ *is abelian if and only if every irreducible stratified bundle on X is one-dimensional, and*
- $\pi_1^{\text{et}}(X) = 1$ *if and only if every stratified bundle on X is trivial.*

Proof. Suppose $\pi_1^{\text{et}}(X)$ is abelian. The groups $\pi = \pi_1(X^{\text{an}})$ and $\pi^{\text{ab}} = \pi/[\pi, \pi]$ are finitely generated and $\hat{\pi} \to (\pi^{\text{ab}})^\wedge$ is an isomorphism by assumption, so all representations of π come from π^{ab}. Due to Schur's lemma, all irreducible representations of an abelian group are one-dimensional.

If all irreducible representations of $\pi_1(X^{\text{an}})$ are one-dimensional, all representations have an eigenvector, hence all representations are conjugate to one on upper-triangular matrices. Let G be a finite quotient of $\pi_1(X^{\text{an}})$ and ρ the regular representation of G, which we may assume upper-triangular. For all $g, h \in G$ the matrix $\rho([g, h])$ is unipotent, i.e. has 1s on the diagonal. As G is finite, $\rho([g, h])$ has finite order, implying $\rho([g, h]) = 1$. The regular representation is faithful, so $[g, h] = 1$ and G is abelian. Then also $\pi_1^{\text{et}}(X) = \hat{\pi}_1(X^{\text{an}})$ is abelian.

Suppose $\pi_1^{\text{et}}(X) = 1$. Then $\pi_1(X^{\text{an}}) \to 1$ becomes an isomorphism after passing to profinite completions, so all representations of $\pi_1(X^{\text{an}})$ are trivial. Conversely, if all representations of $\pi_1(X^{\text{an}})$ are trivial, then in particular the regular representations of all finite quotients of $\pi_1(X^{\text{an}})$ are trivial. Regular representations are faithful, hence all those finite quotients are trivial and $\pi_1^{\text{et}}(X) = \hat{\pi}_1(X^{\text{an}}) = 1$. ■

This leads to the following conjecture in positive characteristic.

Conjecture 1.3 (Gieseker). *Let k be an algebraically closed field of characteristic $p > 0$ and X/k a proper smooth variety. Then $\pi_1^{\text{et}}(X) = 1$ if and only if every stratified bundle on X is trivial.* ⧫

The implication $\Leftarrow$ is easy and will be proved in the next lecture. The converse was proved only recently by Esnault–Mehta.

2. *F*-divided bundles

Let k be an algebraically closed field of characteristic $p > 0$ and X/k a smooth variety. In characteristic zero stratified bundles have a simpler description as integrable connections. In positive characteristic there is a different description.

Definition 2.1. The *absolute Frobenius* of X is the map $F\colon X \to X$ that is the identity on topological spaces and the p-power map on sheaves. It fits into a commutative square

$$\begin{array}{ccc} X & \xrightarrow{F} & X \\ \downarrow & & \downarrow \\ \operatorname{Spec} k & \xrightarrow[F_k]{} & \operatorname{Spec} k. \end{array}$$

If $\mathcal{E}$ is an $\mathcal{O}_X$-module, its *Frobenius twist* is the $\mathcal{O}_X$-module $F^*\mathcal{E}$. ♦

Observe that the natural map $\lambda\colon \mathcal{E} \to F^*\mathcal{E}$, $s \mapsto 1 \otimes s$ is *p-linear*, i.e. for $f \in \mathcal{O}_X$ and $s \in \mathcal{E}$ one has $\lambda(fs) = f^p\lambda(s)$.

Definition 2.2. An *F-divided bundle* on X is a collection of coherent $\mathcal{O}_X$-modules $\mathcal{E}_n$, $n \in \mathbb{N}$, together with $\mathcal{O}_X$-linear isomorphisms $\sigma_n\colon F^*\mathcal{E}_{n+1} \to \mathcal{E}_n$. A *morphism of F-divided bundles* $(\mathcal{E}_n, \sigma_n)_{n\in\mathbb{N}} \to (\mathcal{F}_n, \tau_n)_{n\in\mathbb{N}}$ is a collection of $\mathcal{O}_X$-linear maps $\alpha_n\colon \mathcal{E}_n \to \mathcal{F}_n$ such that all squares

$$\begin{array}{ccc} F^*\mathcal{E}_{n+1} & \xrightarrow{\sigma_n} & \mathcal{E}_n \\ {\scriptstyle F^*\alpha_{n+1}}\downarrow & & \downarrow{\scriptstyle \alpha_n} \\ F^*\mathcal{F}_{n+1} & \xrightarrow[\tau_n]{} & \mathcal{F}_n \end{array}$$

commute. ♦

In fact the sheaves $\mathcal{E}_n$ are automatically locally free. We omit the proof.

Theorem 2.3 (Katz). *There is an equivalence of tensor categories*

$$\{\textit{stratified bundles on } X\} \longleftrightarrow \{\textit{F-divided bundles on } X\}.$$

Proof. We just sketch the constructions and omit the verification that they are functorial and quasi-inverse to each other. In one direction, let $\mathcal{E}$ be a stratified bundle. We define subsheaves

$$\mathcal{E}_n = \{s \in \mathcal{E}\colon \theta s = 0 \text{ for all } \theta \in \mathcal{D}^{<p^n} \text{ with } \theta(1) = 0\},$$

yielding $\ldots \subseteq \mathcal{E}_1 \subseteq \mathcal{E}_0 = \mathcal{E}$. For $f \in \mathcal{O}_X$ and $s \in \mathcal{E}_n$ one has $f^{p^n}s \in \mathcal{E}_n$. Indeed, for $\theta \in \mathcal{D}^{<p^n}$ with $\theta(1) = 0$ we compute $\theta \cdot f^{p^n}s = f^{p^n}\theta s + \theta(f^{p^n})s = f^{p^n} \cdot 0 + 0 \cdot s = 0$. Therefore we can make $\mathcal{E}_n$ into a coherent $\mathcal{O}_X$-module by defining $f * s = f^{p^n}s$. Moreover, the maps

$$\sigma_n\colon F^*\mathcal{E}_{n+1} = \mathcal{O}_X \otimes_{F^{-1}\mathcal{O}_X} F^{-1}\mathcal{E}_{n+1} \longrightarrow \mathcal{E}_n, \quad f \otimes s \mapsto f^{p^n}s$$

are well-defined and $\mathcal{O}_X$-linear. Let's prove they are isomorphisms. We define a connection ∇_n on $\mathcal{E}_n$ as follows: if θ is a derivation, choose a differential operator $\theta' \in \mathcal{D}^{\leq p^n}$ satisfying

$\theta'(f^{p^n}) = \theta(f)^{p^n}$ for all $f \in \mathcal{O}_X$ and set $\nabla_{n,\theta}(s) = \theta' s$. This is independent of the choice of θ' and preserves $\mathcal{E}_n$. Observe that at least locally such θ' really exists: in coordinates,

$$\theta = f_1 \partial_{x_1} + \ldots + f_d \partial_{x_d} \quad \rightsquigarrow \quad \theta' = f_1^{p^n} \frac{\partial_{x_1}^{p^n}}{p^n!} + \ldots + f_d^{p^n} \frac{\partial_{x_d}^{p^n}}{p^n!}.$$

Now ∇_n is an integrable connection on $\mathcal{E}_n$ whose *p-curvature* is zero, i.e. $\nabla_{n,\theta^p} = (\nabla_{n,\theta})^p$. Moreover, $\mathcal{E}_n^{\nabla_n} = \mathcal{E}_{n+1}$. We are done by the following result.

> **Lemma 2.4 (Cartier).** *Let $(\mathcal{F}, \nabla)$ be an integrable connection with p-curvature zero. Then the natural map $F^*\mathcal{F}^\nabla \to \mathcal{F}$ is an isomorphism.* ♦

The proof is a straightforward but nasty computation in local coordinates, which we omit.

Conversely, let $(\mathcal{E}_n, \sigma_n)_{n\in\mathbb{N}}$ be an F-divided bundle. We make $\mathcal{E}_0$ into a stratified bundle. If θ is a differential operator of order less than p^n, locally choose a basis $u_1, \ldots, u_r$ of $\mathcal{E}_n$. Via the p-linear inclusions $\mathcal{E}_{m+1} \to F^*\mathcal{E}_{m+1} \to \mathcal{E}_m$ this maps to a basis $\tilde{u}_1, \ldots, \tilde{u}_r$ of $\mathcal{E}_0$. Define

$$\theta(f_1 \tilde{u}_1 + \ldots + f_r \tilde{u}_r) = \theta(f_1)\tilde{u}_1 + \ldots + \theta(f_r)\tilde{u}_r.$$

We claim that this is a well-defined stratification. Let's just verify that it does not depend on the choice of basis of $\mathcal{E}_n$. Take a second basis $v_1, \ldots, v_r$ with base change matrix $A = (a_{ij})$. The corresponding base change matrix between $\tilde{u}_1, \ldots, \tilde{u}_r$ and $\tilde{v}_1, \ldots, \tilde{v}_r$ is $A' = ({a_{ij}}^{p^n})$. Hence the claim follows by the identity $\theta(f a^{p^n}) = \theta(f) a^{p^n}$ for $a, f \in \mathcal{O}_X$. ■

Proposition 2.5. *Suppose X/k is proper. Two F-divided bundles $(\mathcal{E}_n, \sigma_n)_{n\in\mathbb{N}}$ and $(\mathcal{F}_n, \tau_n)_{n\in\mathbb{N}}$ are isomorphic if and only if $\mathcal{E}_n$ is isomorphic to $\mathcal{F}_n$ for all $n \in \mathbb{N}$.*

Proof. Suppose $\mathcal{E}_n \cong \mathcal{F}_n$ for all $n \in \mathbb{N}$. The maps $\mathrm{End}(\mathcal{E}_{n+1}) \to \mathrm{End}(F^*\mathcal{E}_{n+1}) \to \mathrm{End}(\mathcal{E}_n)$ are p-linear and injective. Since X is proper, $\mathrm{End}(\mathcal{E}_0)$ is a finite-dimensional k-vector space. Thus there exists $m \in \mathbb{N}$ such that for all $n \geq m$ the inclusions $\mathrm{End}(\mathcal{E}_{n+1}) \to \mathrm{End}(\mathcal{E}_n)$ are bijections. Fix an isomorphism $\alpha\colon \mathcal{E}_m \to \mathcal{F}_m$ satisfying $\alpha(\mathcal{E}_{m+1}) = \mathcal{F}_{m+1}$. (For instance take $\alpha = \beta^p$, where β is an isomorphism $\mathcal{E}_{m+1} \to \mathcal{F}_{m+1}$.) Because every automorphism of $\mathcal{E}_m$ preserves $\mathcal{E}_{m+1}$, we conclude that in fact every isomorphism $\mathcal{E}_m \to \mathcal{F}_m$ sends $\mathcal{E}_{m+1}$ to $\mathcal{F}_{m+1}$. By induction we see that $\alpha(\mathcal{E}_n) = \mathcal{F}_n$ for all $n \geq m$. So α induces an isomorphism of F-divided bundles. ■

Corollary 2.6 (Katz). *Suppose X/k is projective. Then a line bundle $\mathcal{L}$ admits a stratification if and only if it has finite and p-prime order in $\mathrm{NS}(X)$. The group of stratifications on $\mathcal{O}_X$ is isomorphic to the Tate group $\mathrm{T}_p X = \mathrm{T}_p \mathrm{Pic}(X) = \lim_n \mathrm{Pic}(X)[p^n]$.*

Proof. The Frobenius twist of a line bundle is $F^*\mathcal{L} = \mathcal{L}^{\otimes p}$. So the first statement follows from the facts that $\mathrm{NS}(X) = \mathrm{Pic}(X)/\mathrm{Pic}^\circ(X)$ is finitely generated and that $\mathrm{Pic}^\circ(X)$ is p-divisible.

As for the second statement, by the preceding proposition the group of stratifications on $\mathcal{O}_X$ is precisely the group of systems $(\mathcal{L}_n)_{n\in\mathbb{N}}$ with $\mathcal{L}_0 \cong \mathcal{O}_X$ and $F^*\mathcal{L}_{n+1} = {\mathcal{L}_{n+1}}^{\otimes p} \cong \mathcal{L}_n$. ■