The tannakian category of D-modules
Marcin Lara, 09/05/2016

Tensor categories. Let $\mathcal{C}$ a category, $\otimes : \mathcal{C}\times\mathcal{C} \to \mathcal{C}$. An associativity constraint is a natural isomorphism $\varphi_{XYZ}: X\otimes(Y\otimes Z) \xrightarrow{\sim} (X\otimes Y)\otimes Z$ satisfying the pentagon axiom; a commutativity constraint is a $\psi_{XY}: X\otimes Y \xrightarrow{\sim} Y\otimes X$ with $\psi_{YX}\psi_{XY} = \mathrm{id}$. They are compatible if they satisfy the hexagon axiom. Then there are natural $\otimes : \mathcal{C}^n \to \mathcal{C}$ "independent of brackets". An identity object is a $\mathbb{1}\in\mathcal{C}$ with an isom. $\mathbb{1} \xrightarrow{\sim} \mathbb{1}\otimes\mathbb{1}$ such that $\mathbb{1}\otimes - : \mathcal{C}\to\mathcal{C}$ is an equivalence. A tensor category is a compatible tuple $(\mathcal{C},\otimes,\varphi,\psi)$ as above that admits an identity object. (Identity objects are uniquely unique.) Ex: R-Mod. ~ fin. gen.

An internal hom object $\underline{\mathrm{Hom}}(X,Y)$ is one representing $T\mapsto \mathrm{Hom}(T\otimes X, Y)$. Write $X^\vee = \underline{\mathrm{Hom}}(X,\mathbb{1})$. There are evaluation maps $ev_{XY}: \underline{\mathrm{Hom}}(X,Y)\otimes X \to Y$. If the natural map $X\to X^{\vee\vee}$ is an iso, X is called reflective. There are maps $X^\vee\otimes Y \to \underline{\mathrm{Hom}}(X,Y)$. A tensor category $(\mathcal{C},\otimes,\varphi,\psi)$ is rigid if all internal hom's exist, all $X\in\mathcal{C}$ are reflexive, and all maps

$$\underline{\mathrm{Hom}}(X_1,Y_1)\otimes\underline{\mathrm{Hom}}(X_2,Y_2) \longrightarrow \underline{\mathrm{Hom}}(X_1\otimes X_2, Y_1\otimes Y_2)$$

are isomorphisms; in particular $X^\vee\otimes Y = \underline{\mathrm{Hom}}(X,Y)$. Now applying $\mathrm{Hom}(\mathbb{1},-)$ to $\underline{\mathrm{Hom}}(X,X)\cong X^\vee\otimes X \xrightarrow{ev} \mathbb{1}$ we get a trace map $\mathrm{End}(X)\to\mathrm{End}(\mathbb{1})$. The rank of X is the image of id_X in $\mathrm{End}(\mathbb{1})$.

A tensor functor $(\mathcal{C},\otimes,\varphi,\psi)\to(\mathcal{C}',\otimes',\varphi',\psi')$ is a functor $F:\mathcal{C}\to\mathcal{C}'$ plus a natural isom $F(X)\otimes' F(Y)\cong F(X\otimes Y)$ compatible with constraints. If $\mathcal{C},\mathcal{C}'$ are rigid, $F(\underline{\mathrm{Hom}}(X,Y)) \to \underline{\mathrm{Hom}}'(FX,FY)$ is an isom. We call it a tensor equivalence if F is an equivalence; then there is a tensor quasi-inverse. There are also morphisms of tensor functors. If $\mathcal{C},\mathcal{C}'$ are rigid and $\tau: F\to G:\mathcal{C}\to\mathcal{C}'$, then τ is an isomorphism.

An additive/abelian tensor cat is one with $\mathcal{C}$ additive/abelian and $\otimes$ biadditive. If $\mathcal{C}$ is rigid and abelian, $\otimes$ is automatically biadditive and commutes with all limits and colimits; in particular, $-\otimes X$ is exact. If $\mathcal{C}$ is rigid abelian, every subobject U of $\mathbb{1}$ has a "complement" $U^\perp$ with $\mathbb{1} = U \oplus U^\perp$. So if $\mathrm{End}(\mathbb{1})$ is a field, $\mathbb{1}$ is a simple object. Exact tensor functors among rigid abelian tensor categories $\mathcal{C} \to \mathcal{C}'$ with $\mathrm{End}(\mathbb{1})$ a field and $\mathrm{End}(\mathbb{1}') \neq 0$ are faithful. (Idea: $X \neq 0$ iff $X^\vee \otimes X \to \mathbb{1}$ is epi.)

Prop. Let $\mathcal{C}$ be an abelian k-linear category, $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, and φ_{XYZ}, ψ_{XY} natural isomorphisms (not nec. satisfying the axioms). Let $F : \mathcal{C} \to \mathrm{Vec}_k$ (fin. dim.) be faithful exact and k-linear. If $F \circ \otimes = \otimes \circ (F \times F)$, $F(\varphi_{XYZ})$ is the assoc constraint in Vec_k, $F(\psi_{XY})$ the assoc constraint in Vec_k, there is an identity object in $\mathcal{C}$, $\mathrm{End}(\mathbb{1}) = k$, $\dim F(\mathbb{1}) = 1$, and for all $L \in \mathcal{C}$ with $\dim F(L) = 1$ there is $L^{-1} \in \mathcal{C}$ with $L \otimes L^{-1} \cong \mathbb{1}$, then $(\mathcal{C}, \otimes, \varphi, \psi)$ is a rigid abelian tensor category.

Ex. Vec_k, R-Mod (not rigid in general), $\mathrm{Rep}_k G$; $\mathbb{Z}/2\mathbb{Z}$-graded vector spaces.
not tannakian

Affine group schemes. A bialgebra is the extra structure on an algebra R corresponding to a group structure on $\mathrm{Spec}\, R$. Recall that any affine k-group scheme G is of the form $G = \varprojlim G_i$ with G_i/k affine algebraic (i.e. finite type).

If G/k is an affine k-group scheme, write $\omega : \mathrm{Rep}_k G \to \mathrm{Vec}_k$ and $\varphi_R : \mathrm{Vec}_k \to \mathrm{Mod}_R$; then for a k-algebra R set $\underline{\mathrm{Aut}}^\otimes(\omega)(R) = \mathrm{End}^\otimes(\varphi_R \circ \omega)$. The natural morphism $G \longrightarrow \underline{\mathrm{Aut}}^\otimes(\omega)$ is an isomorphism; so we can recover G from ω! (Idea: for $X \in \mathrm{Rep}_k G$ consider $\underbrace{G_X}_{= \mathrm{im}(G \to GL(X))} \subseteq \underline{\mathrm{Aut}}^\otimes(\omega|_{\mathcal{C}_X}) \subseteq GL(X \otimes -)$.)

$\mathcal{C}_X$: full subcat of subquotients of sums of X and $X^\vee$

It follows that functors $\mathrm{Rep}_k G' \longrightarrow \mathrm{Rep}_k G$ commuting with the ω's come from group homomorphisms $G \to G'$.

Thm. Let $(\mathcal{C}, \otimes)$ be a rigid abelian k-linear tensor category with $k = \mathrm{End}(\mathbb{1})$ a field. Let $\omega: \mathcal{C} \to \mathrm{Vec}_k$ be an exact (faithful) k-linear tensor functor. Then there exists an affine group scheme G/k such that $(\mathcal{C}, \otimes)$ is tensor equivalent to $\mathrm{Rep}_k G$. More precisely, G represents $\underline{\mathrm{Aut}}^{\otimes}(\omega)$.

Such a tuple $(\mathcal{C}, \otimes, \omega)$ is called a neutral tannakian category.

Idea of proof. Let P_X the largest subobject of $\underline{\mathrm{Hom}}(\omega(X), X)$ such that for all $n \in \mathbb{N}$, $Y \subset X^n$, the image of P_X in $\underline{\mathrm{Hom}}(\omega(X^n), X^n)$ lies in $\ker(\underline{\mathrm{Hom}}(\omega(X^n), X^n) \longrightarrow$ $\longrightarrow \underline{\mathrm{Hom}}(\omega(Y), X^n/Y))$. Set $A_X = \omega(P_X)$; it is a k-algebra, in fact $A_X = \mathrm{End}^{\otimes}(\omega|_{\langle X \rangle})$. Then take the limit. There is a bialgebra structure. $\square$

Ex. · $\mathbb{Z}$-graded vector spaces $\rightsquigarrow$ representations of $\mathbb{G}_m$.
· X/k smooth variety, $x \in X$, then the category of stratified bundles on X with ω_x the fiber at x is neutral tannakian; here use

$$\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}, \quad \theta \cdot (e \otimes f) = \theta e \otimes f + e \otimes \theta f$$

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F}), \quad (\theta \cdot \varphi)(e) = \theta \cdot \varphi(e) - \varphi(\theta e).$$

The Riemann–Hilbert correspondence
Pedro Ángel Castillejo. 23/05/2016

Let $X/\mathbb{C}$ a smooth variety. Recall that we have $\{\mathcal{O}_X\text{-coherent } D\text{-modules}\} = \{\text{stratified bundles}\} \approx \{\text{vector bundles with flat connection}\}$. If $x \in X$ is a closed point, these categories with the fiber functor F_x are tannakian. Hence they are equivalent to the category of representations of some affine group scheme $\underline{\mathrm{Aut}}^{\otimes} F_x$.

Today we see that, analytically, these categories are equivalent to that of local systems on X^{an}.

euclidean open in $\mathbb{C}^n$

Analytic spaces. An analytic space is a loc ringed space $(X,\mathcal{O}_X)$ that is locally of the form $(U,\mathcal{O}_U)$ where $U = Z(f_1,\dots,f_r) \subset V$, f_i analytic, and $\mathcal{O}_U$ the sheaf of analytic functions. There is analytification $X \rightsquigarrow X^{an}$; locally if $X \subseteq \mathbb{A}^n$ is affine, let $X^{an} = X(\mathbb{C})$ with the subspace top from $\mathbb{C}^n$, and $\mathcal{O}_{X^{an}}$ = analytic functions on X^{an}. In general, glue. One has $\hat{\mathcal{O}}_{X_x} = \hat{\mathcal{O}}_{X^{an},x}$, $\dim \mathcal{O}_X = \dim \mathcal{O}_{X^{an}}$, and $\dim X = \dim X^{an}$. Also for an $\mathcal{O}_X$-module $\mathcal{F}$ there is an $\mathcal{O}_{X^{an}}$-module $\mathcal{F}^{an}$.

Thm (GAGA). Let $X/\mathbb{C}$ be proper.
- $H^i(X,\mathcal{F}) = H^i(X^{an},\mathcal{F}^{an})$
- $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F},\mathcal{G}) = \mathrm{Hom}_{\mathcal{O}_{X^{an}}}(\mathcal{F}^{an},\mathcal{G}^{an})$
- There is an equivalence $\{\text{coherent } \mathcal{O}_X\text{-modules}\} \approx \{\text{coherent } \mathcal{O}_{X^{an}}\text{-modules}\}$.

The Riemann-Hilbert correspondence. Let X be a smooth sep. analytic space.

Thm (Riemann-Hilbert). For every local system V on X there is a vector bundle $\mathcal{V} = \mathcal{O}_X \otimes V$ with a unique connection ∇, called canonical, such that for all $v \in \mathcal{V}$ one has $\nabla v = 0$ iff $v \in V$. In particular, for $f \in \mathcal{O}_X$ and $v \in V$ we get $\nabla(fv) = df \otimes v$. The connection ∇ is flat. Moreover, there is an equivalence of tannakian categories

$$\{\text{local systems}\} \underset{\psi}{\overset{\varphi}{\rightleftarrows}} \{\text{flat connections}\}$$

where $\varphi(V) = \mathcal{O}_X \otimes V$ and $\psi(\mathcal{V}) = \mathcal{V}^{\nabla} := \{v \in \mathcal{V} : \nabla v = 0\}$.

In particular, for $X/\mathbb{C}$ a proper variety, we get with GAGA an equivalence between (algebraic) stratified bundles on X and local systems on X^{an}.

Proof. It is clear that $\mathcal{V}$ is locally free and that $\nabla(fs) = df \otimes v$. Then set $\nabla = d \otimes id : \mathcal{O}_X \otimes V \to \Omega^1_X \otimes V$. We have $\nabla\nabla(fv) = \nabla(df \otimes v) = ddf \otimes v = 0$ so ∇ is a flat connection. Note that for $v \in V$ one has $\nabla(1 \otimes v) = d1 \otimes v = 0$. Conversely, we show that $\Omega^\bullet_X(\mathcal{V})$ is a resolution of V, as in $0 \to V \to \mathcal{V} \to \Omega^1(\mathcal{V}) \to \dots$
- $\mathrm{rk}\, V = 1$; we may assume $X = D^m$ and $V = \mathbb{C}$, and show that the sequence $0 \to \mathbb{C} \to \Gamma(\mathcal{O}_X) \to \Gamma(\Omega^1_X) \to \dots$ is acyclic. But there is the homotopy operator

$H: \Gamma(\mathcal{O}_X) \to \mathbb{C}$, given by evaluation at 0, and $H: \Gamma(\Omega^1_X) \to \Gamma(\mathcal{O}_X)$ sending $\omega = \sum_{j=1}^{m} \sum_{\alpha \in \mathbb{N}^n} a^j_\alpha x^\alpha dx_j$ to $\sum_{j=1}^{m} \sum_{\alpha \in \mathbb{N}^n} a^j_\alpha x^\alpha \cdot \frac{x_j}{\alpha_j + 1}$, and similar in higher degrees; this works.

- $X = D^m$, V free: follows because exact sequences respect finite direct sums.

For the equivalence we know already that $\psi \circ \varphi \cong \mathrm{id}$ for objects, and for morphisms it is clear. Also $\mathrm{Hom}(V_1, V_2) \to \mathrm{Hom}(\mathcal{V}_1, \mathcal{V}_2)$ is injective, hence surjective by fin dim. It remains to prove that every flat connection comes from a local system.

- $X = D$, $(\mathcal{V}, \nabla)$ free. The difference between ∇ and $d^n: \mathcal{O}_X^n \to \Omega^1_X \otimes \mathcal{O}_X^n$ is an $\mathcal{O}_X$-linear map $\mathcal{O}_X^n \to (\Omega^1_X)^n$. Also Ω^1_X is free of rank 1 so ∇ is given by a 'connection matrix' $\Omega \in \mathrm{Mat}_{n\times n}(\Omega^1_X(X))$, $\Omega = A\,dz$, with $A = (a_{ij})_{i,j} \in \mathrm{Mat}_{n \times n}(\mathcal{O}_X(X))$. Now $\nabla(f) = 0$ iff $df = -Af$, i.e. iff f is a solution of $y' = -Ay$. Hence $\mathcal{V}$ comes from the corresponding sheaf of solutions V, which is locally constant by the Cauchy theorem.
- $X = D$, $(\mathcal{V}, \nabla)$ locally free. Then we can make a presentation $\mathcal{V}_1 \xrightarrow{h} \mathcal{V}_0 \to \mathcal{V}$ with $\mathcal{V}_1, \mathcal{V}_0$ free; and then $(\mathcal{V}, \nabla) = \mathcal{O}_X \otimes (V_0/hV_1)$, where V_0/hV_1 is locally constant.
- In higher dimension, use induction on a relative version of the above.

$\square$

Rmk. What if $U/\mathbb{C}$ is not proper? Choose a smooth compactification $X/\mathbb{C}$ with sncd $D = X \setminus U$, i.e. $D = \bigcup_i D_i$ with D_i smooth crossing transversally. Choose local coordinates $x_1, \dots, x_n$ with $D_i = Z(x_i)$, $i = 1, \dots, r$, so that $\mathrm{Der}_D(X/\mathbb{C})$ is free on $x_i \frac{\partial}{\partial x_i}$, $i = 1, \dots, r$, and $\frac{\partial}{\partial x_j}$, $j = r+1, \dots, n$. Dually we obtain $\Omega^1_X(\log D)$ with basis $\frac{dx_i}{x_i}$, $i = 1, \dots, r$, dx_j, $j = r+1, \dots, n$. We find a correspondence between local systems on U and flat connections with regular singularities on D.