Efsthathia — Gauss-Manin Connection 27.06.16

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I - (Higher) - Direct Image of left $\mathcal{D}$-modules

<u>Recall</u>: • Direct image of left $\mathcal{D}$-modules

— we have seen the correspondence

$$\omega_X \otimes - : \mathrm{Mod}^{left}(\mathcal{D}_X) \to \mathrm{Mod}^{Right}(\mathcal{D}_X)$$

that has a quasi-inverse

$$\omega_X^\vee \otimes - = \mathcal{H}om(\omega_X, \cdot)\ \mathrm{Mod}^{right}(\mathcal{D}_X) \to \mathrm{Mod}^{left}(\mathcal{D}_X)$$

Let $f: X \to Y$ be a morph b/w sm. k-schemes.

$$\begin{tikzcd}
\mathrm{Mod}^{left}(\mathcal{D}_X) \arrow[r] \arrow[d, "\omega_X \otimes -"'] \arrow[dr, phantom, "\circlearrowright"] & \mathrm{Mod}^{left}(\mathcal{D}_Y) \\
\mathrm{Mod}^{right}(\mathcal{D}_X) \arrow[r, "f_*(- \otimes_{\mathcal{D}_X} \mathcal{D}_{X\to Y})"'] & \mathrm{Mod}^{right}(\mathcal{D}_Y) \arrow[u, "\omega_Y^\vee \otimes_{\mathcal{O}_Y} -"']
\end{tikzcd}$$

$$\mathcal{D}_{X\to Y} = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}(\mathcal{D}_Y)$$

is a $(\mathcal{D}_X, f^{-1}\mathcal{D}_Y)$-bimodule.

To a left $\mathcal{D}_X$-module M, we can associate a left $\mathcal{D}_Y$-module

$$\omega_Y^\vee \otimes_{\mathcal{O}_Y} f_*\{(\omega_X \otimes_{\mathcal{O}_X} M)\} \otimes_{\mathcal{D}_X} \mathcal{D}_{X\to Y})$$

$$\mathcal{D}_{Y\leftarrow X} := \omega_X \otimes_{\mathcal{O}_X} \mathcal{D}_{X\to Y} \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}(\omega_Y^\vee)$$

$$\Rightarrow f_+ : \mathrm{Mod}^{left}(\mathcal{D}_X) \longrightarrow \mathrm{Mod}^{left}(\mathcal{D}_Y)$$

$$M \longmapsto f_*(\mathcal{D}_{Y\leftarrow X} \otimes_{\mathcal{D}_X} M)$$

$\mathrm{Mod}^{right}(\mathcal{D}_X) = \mathrm{mod}(\mathcal{D}_X^{op})$

$$\mathcal{D}_X^{op} \cong \omega_X \otimes \mathcal{D}_X \otimes \omega_X^\vee$$

Lemma: We have the following locally free resolutions of the left $\mathcal{D}_X$-module $\mathcal{O}_X$ and the right $\mathcal{D}_X$-module ω_X:

(1) $0 \to \mathcal{D}_X \otimes_{\mathcal{O}_X} \Lambda^n T_X \xrightarrow{\partial} \dots \to \mathcal{D}_X \otimes_{\mathcal{O}_X} \overset{0}{\Lambda} T_X \to \mathcal{O}_X \to 0.$

2) $0 \to \Omega_X^0 \otimes_{\mathcal{O}_X} \mathcal{D}_X \xrightarrow{\partial} \dots \to \Omega_X^n \otimes \mathcal{D}_X \to \omega_X \to 0$

$n = \dim X$

For (1) $\partial : \mathcal{D}_X \otimes \Lambda^0 T_X \to \mathcal{O}_X$

$$P \mapsto P(1)$$

$$\partial : \mathcal{D}_X \otimes_{\mathcal{O}_X} \overset{k}{\Lambda} T_X \to \mathcal{D}_X \otimes_{\mathcal{O}_X} \overset{k-1}{\Lambda} T_X$$

$$\partial : (P \otimes \theta_1 \otimes \dots \otimes \theta_k) = \sum (-1)^i P\theta_i \otimes \theta_1 \otimes \dots \otimes \hat{\theta}_i \dots \otimes \theta_k + \sum_{i<j} (-1)^{i+j} P \otimes [\theta_i, \theta_j] \wedge \theta_1 \wedge \dots \wedge \hat{\theta}_i \wedge \dots \wedge \hat{\theta}_j \wedge \dots \wedge \theta_k$$

For (2) $\partial : \Omega_X^n \otimes_{\mathcal{O}_X} \mathcal{D}_X \to \omega_X$

$$\omega \otimes P \to \omega(P)$$

$$\partial : \Omega_X^k \otimes_{\mathcal{O}_X} \mathcal{D}_X \to \Omega_X^{k+1} \otimes_{\mathcal{O}_X} \mathcal{D}_X, \quad k \neq n.$$

$$\omega \otimes P \mapsto \partial\omega \otimes P + \sum_i \partial z_i \wedge \omega \otimes \partial_i P,$$

$\{z_i, \partial_i\}$ local coord.

Proof: (sketch)

(2) follows from (1) by using the operation $\omega_X \otimes_{\mathcal{O}_X} -$

$$\mathcal{D}_X \otimes \Lambda^k T_X \xmapsto{\omega_X \otimes} \Lambda^n \Omega^1_X \otimes \mathcal{D}_X \otimes \Lambda^k T_X =$$

$$\Lambda^n \Omega^1 \otimes \mathcal{D}_X \otimes \Lambda^k (\omega_X^\vee) =$$

$$= \underline{\qquad} \otimes \Lambda^{n-k} \Omega^1 \otimes \omega_X^\vee$$

$$= \Lambda^{n-k} \Omega^1 \otimes \mathcal{D}_X^{op}$$

let $\Lambda^n E = L$

perf. pairing $\quad \Lambda^k E \otimes \Lambda^{n-k} E \longrightarrow L$

$$\Lambda^k E \cong Hom(\Lambda^{n-k} E, L) \cong \Lambda^{n-k} E^\vee \otimes L$$

- $N^\cdot = [\mathcal{D}_X \otimes \Lambda^n T_X \to \dots \to \mathcal{D}_X \to \mathcal{O}_X]$

is acyclic. we consider its filtration

$\{F_p N^\cdot\}_p$

$$F_p N^\cdot = [F_{p-n} \mathcal{D}_X \otimes_{\mathcal{O}_X} \Lambda^n T_X \to \dots \to F_p(\mathcal{D}_X) \to F_p \mathcal{O}_X]$$

with
- $F_p \mathcal{O}_X = \mathcal{O}_X$ for $p \geq 0$, $F_p \mathcal{O}_X = 0$ for $p < 0$
- $F_0 \mathcal{D}_X = \mathcal{O}_X$
- $F_\ell \mathcal{D}_X = \{ p \in End_k(\mathcal{O}_X) \mid [p, f] \in F_{\ell-1} \mathcal{D}_X, \ \forall f \in \mathcal{O}_X \}$

Proof: We show that the fil is acyclic.

Let $\pi : T^*X \to X$

$i : X \hookrightarrow T^*X$ by the zerosection

$\operatorname{gr} N^\cdot \cong \pi_* L^\cdot$

$$L^\cdot = [\mathcal{O}_{T^*X} \otimes_{\pi^{-1}\mathcal{O}_X} \Lambda^n \pi^{-1} T_X \xrightarrow{\partial} \cdots \to \underbrace{\mathcal{O}_{T^*X} \otimes_{\pi^{-1}\mathcal{O}_X} \Lambda^0 \pi^{-1} T_X \to i_* \mathcal{O}_X}]$$

$$\partial : \mathcal{O}_{T^*X} \otimes \Lambda^k \pi^{-1} T_X \to \mathcal{O}_{T^*X} \to \Lambda^{k-1}_{\pi^{-1}} T_X \quad \varphi \mapsto i_*(\varphi \circ i)$$

$$\varphi \otimes \theta_1 \wedge \cdots \wedge \theta_k \longmapsto \sum_i (-1)^{i+1} \varphi(\theta_i) \wedge \theta_1 \wedge \cdots \wedge \hat{\theta}_i \wedge \cdots \wedge \theta_k$$

Fact: $L^\cdot$ is the Koszul resolution of the $\mathcal{O}_{T^*X}$-module $i_* \mathcal{O}_X$ & since π is an affine map

$\Rightarrow$ $\pi_* L$ is acyclic.

(2) $0 \to \Omega_X^0 \otimes_{\mathcal{O}_X} \mathcal{D}_X \to \cdots \to \Omega_X^n \otimes_{\mathcal{O}_X} \mathcal{D}_X \to \mathcal{O}_X \to 0.$

Defn: For a left $\mathcal{D}_X$-module M, we define the deRham complex of M as

$$\Omega^{n+\cdot}_{X/k}(M)$$

- $\Omega^{n+k}(M) = \Omega_X^{n+k} \otimes_{\mathcal{O}_X} M$ for $-n \le k \le 0$

$\Omega^{n+k}(M) = 0$, otherwise

Rmk: Lemma above says that $\Omega_X^{n+\cdot}$ is a resolution of $\mathcal{O}_X$ by locally free modules.

Proposition: There is a functorial isom

$$Sp_X^{\cdot}(\omega_X \otimes_{\mathcal{O}_X} \mathcal{M})^{\cdot} \xrightarrow{\sim} \Omega_X^{n+\cdot}(M),$$ for any

left - $\mathcal{D}_X$-module M, which is termwise $\mathcal{O}_X$-linear

Steps ~~of~~ for the proof:

(a) Let N be a right $\mathcal{D}_X$-module

the natural isom

$$N \otimes_{\mathcal{D}_X} (\mathcal{D}_X \otimes_{\mathcal{O}_X} \wedge^k T_X) \longrightarrow N \otimes_{\mathcal{O}_X} \wedge^k T_X \qquad \xi$$

$$m \otimes P \otimes \xi \longmapsto mP \otimes \xi$$

induces an isom of complexes

$$N \otimes_{\mathcal{D}_X} Sp_X^{\cdot}(\mathcal{D}_X) \xrightarrow{\sim} Sp^{\cdot}(N)$$

for M a left $\mathcal{D}_X$-module

$$\Omega_X^{n+\cdot}(\mathcal{D}_X) \otimes_{\mathcal{D}_X} M \xrightarrow{\sim} \Omega_X^{n+\cdot}(M)$$

(b) $\omega_X \otimes_{\mathcal{O}_X} (Sp^{\cdot}(\mathcal{D}_X)^{\cdot}, \delta) \xrightarrow{\sim} (\Omega_X^{n+\cdot} \otimes_{\mathcal{O}_X} \mathcal{D}_X, \nabla)$

(c) $(\omega_X \otimes_{\mathcal{O}_X} M) \otimes_{\mathcal{D}_X} (Sp_X^{\cdot}(\mathcal{D}_X)^{\cdot}, \delta) \simeq$

$$(\omega_X \otimes_{\mathcal{O}_X} M \otimes \wedge^{\cdot} T_X, \delta) \xrightarrow{\sim} (\Omega_X^{n+\cdot} \otimes_{\mathcal{O}_X} M, \nabla)$$

$$\cong (\Omega_X^{n+\cdot} \otimes_{\mathcal{O}} \mathcal{D}_X, \nabla) \otimes_{\mathcal{D}_X} \mathcal{M}$$

- Alternative construction of the higher direct image of a left $\mathcal{D}_X$-mod M

Let $f: X \to Y$

Recall, $M \otimes_{\mathcal{O}_X} \mathcal{D}_{X \to Y}$ has a structure of a left $\mathcal{D}_X$-module and a ~~compa~~ compatible structure of a right $f^{-1}\mathcal{D}_Y$-module

Prop^n (1): The deRham complex $\Omega_X^{n+\cdot}(M \otimes_{\mathcal{O}_X} \mathcal{D}_{X\to Y})$ is isomorphic to $(\omega_X \otimes_{\mathcal{O}_X} M) \otimes_{\mathcal{D}_X} Sp^{\cdot}_{X\to Y}(\mathcal{D}_X)$

$$Sp^{\cdot}_{X\to Y}(\mathcal{D}_X) = Sp^{\cdot}(\mathcal{D}_X) \otimes_{\mathcal{O}_X} \mathcal{D}_{X\to Y}$$

(2) $Rf_* M$ is the complex associated to
$$\mathbb{R} f_*(\Omega_X^{n+\cdot}(M \otimes_{\mathcal{O}_X} \mathcal{D}_{X\to Y})).$$

II The Gauß-Manin connection:

1. Definition via Spectral Sequences

$\pi : X \longrightarrow S \quad g : S \longrightarrow T$

□ $\Omega^{\cdot}_{X/S}$ is filtered by the locally free subsheaves
$$F^i(\Omega^{\cdot}_{X/S}) = \text{image of } (\pi^*(\Omega^i_{S/T}) \otimes_{\mathcal{O}_X} \Omega^{\cdot - i}_{X/S} \to \Omega^{\cdot}_{X/S})$$

& $\Omega^{\cdot}_{X/S} = F^0(\Omega^{\cdot}_{X/S}) \supset F^1(\Omega^{\cdot}_{X/S}) \supset \dots$

To a filtration like that we can associate a spectral sequence

$\rightsquigarrow (E_r^{p,q}, \partial_r^{p,q})$ that converges to $\mathbb{R}^q \pi_*(\Omega^{\cdot}_{X/S})$

$\Rightarrow$ we define the deRham cohomology sheaves
$$H^q_{dR}(X/S) := \mathbb{R}^q \pi_*(\Omega^{\cdot}_{X/S})$$

For $(\mathcal{E}, \nabla)$ module w/ integrable connection on X/T

$$MIC(X/T) \longrightarrow MIC(X/S)$$
$$(\mathcal{E}, \nabla) \longmapsto (\mathcal{E}, \nabla|_{Der(X,S)})$$

$\rightsquigarrow$ associated deRham complex

$\Omega^{\bullet}_{X/S} \otimes \mathcal{E}$

$\hookrightarrow$ filtration:

$$F^i(\Omega^{\bullet}_{X/S} \otimes \mathcal{E}) = F^i(\Omega^{\bullet}_{X/S}) \otimes \mathcal{E}$$

$\rightsquigarrow$ spectral sequence. $E_r^{p,q}$ s.t.

$$E_1^{p,q} = \Omega^p_{S/T} \otimes_{\mathcal{O}_S} H^q_{dR}(X/S, (\mathcal{E}, \nabla))$$

so the deRham complex of $H^q_{dR}(X/S, (\mathcal{E}, \nabla))$ is just $(E_1^{\bullet,q}, d_1^{\bullet,q})$

<u>def^n</u>: The Gauß-Manin connection on the relative deRham cohomology sheaf of $H^q_{dR}(X/S, (\mathcal{E}, \nabla))$ is the differential map $d_1^{0,q}$ of the associated spectral sequence.

<u>Rmk</u>: we get the "classical" def^n of GM connection for $\mathcal{E} = \mathcal{O}_X$, $\nabla = \partial : \mathcal{O}_X \to \Omega^1_{X/S}$

$$\underline{H^q_{dR}(X/S, (\mathcal{E}, \nabla))} = \mathbb{R}^q \pi_* \Omega^{\bullet}_{X/S}$$

- <u>curves</u>

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S/k : smooth curve

$f : X \longrightarrow S$ smooth

- computing is a local question

$\Rightarrow$ Assume S is affine.

We take the filtration of $\Omega^{\cdot}_{X/S}$

$$F^i(\Omega^{\cdot}_{X/S}) = \mathrm{Im}\left(f^*(\Omega^i_{S/k}) \otimes_{\mathcal{O}_X} \Omega^{\cdot - i}_{X/S} \to \Omega^{\cdot}_{X/S}\right)$$

and we obtain an exact sequence

$$0 \to F^1/F^2 \to F^0/F^2 \to F^0/F^1 \to 0$$

- S/k curve $F^2 = 0$

$$\Rightarrow \; 0 \to f^*\Omega^1_{S/k} \otimes_{\mathcal{O}_X} \Omega^{\cdot -1}_{X/S} \otimes_{\mathcal{O}_X} \Omega^{\cdot -1}_{X/S} \to \Omega^{\cdot}_{X/k} \to \Omega^{\cdot}_{X/S} \to 0 \quad (\mathrm{I})$$

$\underline{f_*}$ left exact & hypercohomology $\Rightarrow$

$$\mathbb{H}^i(U, \Omega^{\cdot}_{X/S}|_U) := H^i_{dR}(X/S) = (\mathbb{R}^i f_* \Omega^{\cdot}_{X/S})(U), \; U \text{ open}$$

$\leadsto$ the Gauß-Manin conn ∇_i is equal to the boundary map of the long exact seq. ass to (I)

$$\nabla_i \; H^i_{dR}(X/S) = \mathbb{H}^i(\Omega^{\cdot}_{X/S}) \to \mathbb{H}^{i+1}(f^{-1}\Omega_{S/k} \otimes f^{-1}\mathcal{O}_S \Omega^{\cdot -1}_{X/S})$$

$\| \; b/c \; f^{-1}(\Omega^{\cdot}_{S/k})$ loc. free & diff is $f^{-1}(\mathcal{O}_S)$-linear

$$\Omega^{\cdot}_{S/k} \otimes_{\mathcal{O}_S} H^{i+1}(\Omega^{\cdot -i}_{X/S})$$

$$\|$$

$$\Omega^{\cdot}_{S/k} \otimes_{\mathcal{O}_S} H^i_{dR}(X/S$$

In general,

- Find $\{U_i\}_{i \in I}$ open cover of X, I well ordered

$$C^{p,q} = \prod_{i_0 < \dots < i_q} \Gamma(U_{i_0, \dots i_q}, \Omega^p_{X/S})$$

Then, $H^i_{dR}(X/S) = H^i(\mathrm{Tot}(C^{\cdot,\cdot}))$

$-H^0_{dR}(X/S) = \mathrm{Ker}(d \cdot \mathcal{O}_X \longrightarrow \Omega^1_{X/S}(X))$

$$H^1_{dR}(X/S) = Z^1/B^1$$

We want to compute $\nabla_1 : H^1_{dR}(X/S) \to \Omega^1_{S/k} \otimes_{\mathcal{O}_S} H^1_{dR}(X/S)$

Let $[\omega] \in H^1_{dR}(X/S)$

To calculate $\nabla_1([\omega])$

$$\begin{tikzcd}
 & & Z^1(\Omega^\cdot_{X/S}) \arrow[d] \arrow[dddll, bend right] \\
 & T^1(\Omega_{X/k}) \arrow[r] \arrow[d, "D"] & T^1(\Omega^\cdot_{X/S}) = \text{total cohomology} \\
T^2(dt \otimes \Omega^{\cdot -1}_{X/S}) \arrow[r] \arrow[d] & T^2(\Omega^\cdot_{X/k}) & \\
K^2(dt \otimes \Omega^{\cdot -1}_{X/S}) & &
\end{tikzcd}$$

For $S = \mathrm{Spec}\ R$

$\mathcal{U}_1 = \mathrm{Spec}\ B_i$

$\Omega^1_{S/k} = R\,dt,\ t \in R$

D : k-derivation on the coord. ring of $\mathcal{O}_S$.

Example: $k = \mathbb{C}$, $A = \mathbb{C}[t]$

$$B = \mathbb{C}[x, y, t] / (F(x, y, t))$$

$$X = \mathrm{Spec}\ B \longrightarrow S = \mathrm{Spec}\ A \longrightarrow \mathrm{Spec}\ k$$

- We can consider X/S as a family of non-singular curves parameteized by an indeterminant t.
- such that $\forall\ a \in \mathbb{C}$, the fiber X_a corresponds to the curve given by $F(X, Y, a)$

$$- \Omega_{X/k} = \frac{\mathcal{O}_X\{dx, dy, dt\}}{\left(\frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy + \frac{\partial F}{\partial t}dt\right)}$$

$$- \Omega^1_{X/S} = \frac{\mathcal{O}_X\{dx, dy\}}{\left(\frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy\right)}, \quad \Omega^1_{S/k} = \mathcal{O}_S\{dt\}$$

$C^{p,q} = \Gamma(\mathcal{X}, \Omega^p_{X/S})$ — cover X by $\mathcal{X}$.

For $q > 0$, $C^{p,q} = 0$.

$$Z^1_{(X/S)} = \{\omega \in \Omega^1_{B/A} : d\omega = 0\}$$

$$B^1(X/S) = \{df_i, f_i \in A\}$$

$$T^i(\Omega^i_{X/S}) = \Omega^i_{X/S}$$

$$T^i(\Omega^i_{X/k}) = \Omega^i_{X/k}$$

Let $[\omega] \in H^1$ is repr. by $\omega \in Z^1$

$$\| \quad Z^1/B^1$$

$\omega = f_1\, dx + f_2\, dy \,,\ f_1, f_2 \in \mathbb{C}[x,y]$

lift this to $\Omega^1_{X/k}$

$\bar{\omega} = \bar{f_1}\, dx + \bar{f_2}\, dy \,,\ \bar{f_1}, \bar{f_2} \in B$

$d : \Omega^1_{X/k} \longrightarrow \Omega^2_{X/k}$

$$d(\bar{\omega}) = \frac{\partial \bar{f_1}}{\partial y}\, dy \wedge dx + \frac{\partial \bar{f_1}}{\partial t}\, dt \wedge dx$$

$$+ \frac{\partial \bar{f_2}}{\partial x}\, dx \wedge dy + \frac{\partial \bar{f_2}}{\partial t}\, dt \wedge dy \,.$$

$d(\omega) = 0$

$\rightsquigarrow$ GM is given by

$$\nabla_1([\omega]) = dt \otimes \left(\frac{\partial \bar{f_1}}{\partial t}\, dx + \frac{\partial \bar{f_2}}{\partial t}\, dy \right)$$