Representations of the fundamental group

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<u>Outline.</u> Let $X/\mathbb{C}$ a smooth variety. We look at

$$\boxed{\begin{tikzcd}
\{\text{local systems on } X^{an}\} \arrow[r, leftrightarrow, "\sim"] & \{\mathbb{C}\text{-repr of } \pi_1^{top}(X^{an})\} \\
\{\text{finite local systems on } X^{an}\} \arrow[u, hook] \arrow[r, leftrightarrow, "\sim"] & \{\mathbb{C}\text{-repr of } \hat{\pi}_1^{top}(X^{an})\} \arrow[u, hook]
\end{tikzcd}}$$

and first

$$\boxed{\begin{tikzcd}
\{\text{fin covers of } X^{an}\} \arrow[r, leftrightarrow, "\sim"] & \{\text{finite } \pi_1^{top}(X^{an})\text{-sets}\} \\
\{\text{finite étale schemes}/X\} \arrow[u, leftrightarrow, "\wr"] \arrow[r, leftrightarrow, "\sim"] & \{\text{finite } \pi_1^{et}(X)\text{-sets}\} \arrow[u, leftrightarrow, "\wr"]
\end{tikzcd}}$$

<u>Covering spaces.</u> If L/K is finite Galois with group G, then intermediate fields $L/M/K$ correspond to subgroups $1 \subseteq H \subseteq G$, via $M \mapsto \mathrm{Aut}(L/M)$, $H \mapsto L^H$, and M/K is Galois iff $H \subseteq G$ is normal; then $\mathrm{Gal}(M/K) = G/H$.

If $Y \xrightarrow{p} X$ is a topological Galois cover, with group G, i.e. Y is connected and $G \backslash Y = X$ where $G = \mathrm{Aut}(Y/X)$, then intermediate covers $Y/Z/X$ correspond to subgroups $1 \subseteq H \subseteq G$ via $Z \mapsto \mathrm{Aut}(Y/Z)$, $H \mapsto H\backslash Y$, and Z/X is Galois iff $H \subseteq G$ is normal; then $\mathrm{Gal}(Z/X) = G/H$.

<u>Thm.</u> Let X a connected locally simply connected top space and $x \in X$. There are equivalences

$$\{\text{locally constant sheaves on } X\} \xleftrightarrow{\sim} \{\text{covers of } X\} \xleftrightarrow{\sim} \{\pi_1^{top}(X)\text{-sets}\}$$

$$\mathcal{F} \longmapsto X_{\mathcal{F}} \qquad\qquad Y \longmapsto p^{-1}(x)$$

$$\text{sheaf of sections} \longleftarrow\!\shortmid\; Y \qquad\qquad H\backslash\tilde{X}_x \longleftarrow\!\shortmid\; S$$

espace étalé

if $S = \pi_1/H$ is transitive

Corollary. $\{\text{finite covers of } X\} \xleftrightarrow{\sim} \{\text{finite (cts) } \hat{\pi}_1^{top}(X)\text{-sets}\}$
where $\hat{\pi} = \varprojlim_{\substack{N \triangleleft \pi \\ \text{finite index}}} \pi/N$ is the profinite completion of π.

Recall that a $\mathbb{C}$-local system on X is a locally constant sheaf V of finite dimensional $\mathbb{C}$-vs; it is finite if there is a finite Galois cover $Y \xrightarrow{s} X$ such that $s^{-1}V$ is constant.

Corollary. $\{\mathbb{C}\text{-local systems on } X\} \xleftrightarrow{\sim} \{\text{fin. dim } \mathbb{C}\text{-repr of } \pi_1^{top}(X)\}$
$\{\text{finite } \mathbb{C}\text{-local systems on } X\} \xleftrightarrow{\sim} \{\text{fin. dim } \mathbb{C}\text{-repr of } \hat{\pi}_1^{top}(X)\}$.

Finite étale morphisms. Recall: a scheme map $Y \xrightarrow{f} X$ is finite locally free if there is an affine open cover of X by $U_i = \operatorname{Spec} A_i$ such that $f^{-1}(U_i) = \operatorname{Spec} B_i$ with B_i a free finite rank A_i-module.

Lemma. An A-algebra B is finite projective iff there is an open cover of $\operatorname{Spec} A$ by $D(f_i)$ such that B_{f_i} is a free finite rank A_{f_i}-module.

Prop. $Y \xrightarrow{f} X$ is finite loc free iff for every affine open $U = \operatorname{Spec} A$ of X, $f^{-1}(U) = \operatorname{Spec} B$ with B a finite projective A-algebra.

For $Y \xrightarrow{f} X$ finite loc free there is a degree map $[Y:X] : X \to \mathbb{Z}$, given on affines $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$ by $[Y:X](\mathfrak{p}) = \operatorname{rk}_{A_\mathfrak{p}} B_\mathfrak{p}$. It is locally constant. Note that $[Y:X] = 0$ iff $Y = \emptyset$; $[Y:X] = 1$ iff $Y \to X$ is an iso; and $[Y:X] \geq 1$ iff $Y \to X$ is surjective.

A map $Y \xrightarrow{f} X$ is finite étale if there is an affine open cover of X by $U_i = \operatorname{Spec} A_i$ such that $f^{-1}(U_i) = \operatorname{Spec} B_i$ with B_i a free separable A_i-algebra, i.e. it is finite free and $B_i \to \operatorname{Hom}_{A_i}(B_i, A_i)$, $b \mapsto (b' \mapsto \operatorname{Tr}(bb'))$ is an isomorphism. Being finite étale is stable under base change.

Prop. Suppose $g: W \to X$ is surjective finite locally free. Then $Y \to X$ is finite étale iff $W \times_X Y \to W$ is finite étale.

(Proof: use that $Y \xrightarrow{f} X$ is finite étale iff for every affine open $U = \operatorname{Spec} A$ on X, $f^{-1}(U) = \operatorname{Spec} B$ with B a projective separable A-algebra. Also use faithfully flat descent of projective separability.)

Say $Y \xrightarrow{f} X$ is totally split if $X = \bigsqcup_{n \geq 0} X_n$ with $f^{-1}(X_n) = \bigsqcup_n X_n$. This implies f is locally étale.

Thm. A map $Y \to X$ is finite étale iff there exists a surj finite locally free $g: W \to X$ such that $W \times_X Y \to W$ is totally split.

(Proof: by induction to $n = [Y:X]$. Use that $Y \to Y \times_X Y$ is open and closed; writing $Y \times_X Y = Y \sqcup Z$, apply induction to $Z \to Y$ which has degree $n-1$.)

The étale fundamental group. Let X a conn scheme and $\bar{x}: \operatorname{Spec} \Omega \to X$ a geometric point. Let FEt_X be the cat of finite étale X-schemes. We define the fiber functor $\mathrm{Fib}_{\bar{x}} : \mathrm{FEt}_X \to \mathrm{FSet}$, $Y \mapsto |Y_{\bar{x}}|$. The étale fundamental group of $(X, \bar{x})$ is $\pi_1^{et}(X, \bar{x}) := \operatorname{Aut}(\mathrm{Fib}_{\bar{x}})$.

Thm (Grothendieck).
- $\pi_1^{et}(X, \bar{x})$ is profinite;
- for each $Y \in \mathrm{FEt}_X$ the action of $\pi_1^{et}(X, \bar{x})$ on $\mathrm{Fib}_{\bar{x}}(Y)$ is continuous;
- we get an equivalence of categories $\mathrm{FEt}_X \xrightarrow{\sim} \pi_1^{et}(X, \bar{x})\text{-}\mathrm{FSet}$, under which connected covers correspond to transitive sets and Galois covers correspond to finite quotient groups of $\pi_1^{et}(X, \bar{x})$.

Thm. Suppose $X/\mathbb{C}$ is conn of finite type and $\bar{x} \in X(\mathbb{C})$. There is an equiv of categories $\mathrm{FEt}_X \longrightarrow \{\text{finite covers of } X^{an}\}$, $Y \mapsto Y^{an}$. It follows that $\pi_1^{et}(X, \bar{x}) = \hat{\pi}_1^{top}(X^{an}, x)$.