The Derived Category of D-modules

I Derived categories

Motto: Complexes good, homology of complexes bad.

To justify, look a bit at algebraic topology:

Problem: Define invariant of simplicial complexes that decides when $X \sim Y$ (homotopically equivalent).

Recall: this means $\exists\, f: X \to Y$ continuous s.t. $g: Y \to X$

$$g \circ f \sim \mathrm{id}_X$$
$$f \circ g \sim \mathrm{id}_Y$$

Idea: use homology: we know already that
$$X \sim Y \Rightarrow H_i(X) \cong H_i(Y).$$

Bad, because $\exists\ X, Y$ with $H_i(X) \cong H_i(Y)$ s.t. $X \not\sim Y$.

$\Rightarrow$ homology only gives limited information about homotopy type.

Ex:

~~$X = S^2 \times \mathbb{RP}^3$~~
~~$Y = \mathbb{RP}^2 \times S^3$~~ ~~$\Rightarrow \pi_1(X) = \pi_1(Y) = \mathbb{Z}_2$~~
~~same universal cover $S^2 \times S^3$~~

$X = S^4 \vee (S^2 \times S^2)$ same homology, but different
$Y = \mathbb{CP}^2 \vee \mathbb{CP}^2$ cohomology rings

But what is homology? Recall it is defined ~~as~~ from

<u>simplicial chain complex</u>

$$C_\cdot(X) \qquad \cdots \to C_k(X) \xrightarrow{\partial_k} C_{k-1}(X) \to \cdots$$

$$H_k(X) = \ker \partial_k / \operatorname{im} \partial_{k+1}$$

Then motto suggests we should take as invariant the whole $C_\cdot(X)$.

<u>Theorem (Whitehead)</u> $X \sim Y$ iff $\exists$ Z simplicial complex &
simpl. cplexes

$$\begin{tikzcd} & Z \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & Y \end{tikzcd}$$

maps $f: Z \to X$, $g: Z \to Y$ s.t. the induced maps

$$\begin{tikzcd} & C_\cdot(Z) \arrow[dl, "f_*"'] \arrow[dr, "g_*"] & \\ C_\cdot(X) & & C_\cdot(Y) \end{tikzcd}$$

$f_*: C_\cdot(Z) \to C_\cdot(X)$
$g_*: C_\cdot(Z) \to C_\cdot(Y)$ are quasi-isomorphisms

Recall: Quasi-isomorphism: $f_*: H_i(Z) \to H_i(X)$ iso
$g_*: H_i(Z) \to H_i(Y)$ iso

<u>Note</u>: 1. We need to keep the info on $C_\cdot$ to get homotopic invariant
i.e not enough to have $H_i(X) \cong H_i(Y)$ as v.sp.,
we need a chain map inducing them.

2. We need a third space Z
Why? Homotopy ~~equivaleng~~ equivalence might mess up the simplicial structure, so need refinement.

~~3. Same statement also for cochains $C^\cdot(X)$~~
4. Same statement also for cochains $C^\cdot(X)$
and cohomology

5. Quasi-isomorphisms are not usually invertible.

⌜Ex:
$$\begin{tikzcd} \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r, "\cdot 2"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r] & 0 \arrow[r] & \cdots \end{tikzcd}$$
not invertible ⌟

Back to algebraic geometry

X top space

$\mathcal{F} \in \underline{Ab}(X)$ = sheaves of abelian groups on X

$\Gamma(X, \cdot) : \underline{Ab}(X) \to \underline{Ab}$ global section functor
left-exact

Want invariants on $\underline{Ab}(X)$ → obtained as coh of a complex

~~([illegible])~~

How? Replace $\mathcal{F}$ by resolution

$$0 \to \mathcal{F} \to I^0 \to I^1 \to \cdots$$

Not just any resolution, but an injective resolution

I^i = injective objects of $\underline{Ab}(X)$

I injective : $\mathrm{Hom}(\cdot, I)$ is exact
(usually just left-exact)

Can we even write this resolution?

Fact $\underline{Ab}(X)$ has enough injectives i.e. $\forall\, A \in \underline{Ab}(X)$
$\exists\, I \in \underline{Ab}(X)$ inj s.t. $0 \to A \to I$.

$\Rightarrow \forall$ object in $\underline{Ab}(X)$ has an injective resolution.

<u>Cohomology functor</u> $H^i(X,\cdot) := R^i\Gamma(X,\cdot)$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ ith cohomology obj h^i

$$H^i(X,\mathcal{F}) = R^i(X,\mathcal{F}) = h^i(\Gamma(X,I^\bullet))$$
$$= \ker d^i / \operatorname{im} d^{i-1}$$

$$I^0 \to I^1 \to \dots \to I^i \xrightarrow{d^i} I^{i+1} \to \dots$$

So far X just top sp; $\mathcal{F}$ sheaf of groups.

<u>Extra structure</u> $(X,\mathcal{O}_X)$ ringed space

$\mathcal{F}$ $\mathcal{O}_X$-module

$$\Gamma(X,\cdot) : \underline{Mod}_{\mathcal{O}_X}(X) \to \underline{Ab}$$

<u>Fact</u> $\underline{Mod}_{\mathcal{O}_X}(X)$ has enough injectives

So again take inj resolutions etc

One can check that calculating

$R^i\Gamma(X,\cdot)$ with inj resol in $\underline{Mod}_{\mathcal{O}_X}(X)$

yields the same cohomology functors

$H^i(X,\cdot)$.

Motto: perhaps better to keep the complexes instead of the coh groups.

Main objects of study are complexes $\Rightarrow$ want a category

The homotopy category

Base abelian category $\mathcal{A}$:

Construct new category with

objects = complexes of objects of $\mathcal{A}$

maps = chain maps

$$A^\bullet = \cdots \to A^{i-1} \xrightarrow{d^{i-1}} A^i \xrightarrow{d^i} A^{i+1} \to \cdots$$

$f : A^\bullet \to B^\bullet$ = $\{ f^i : A^i \to B^i \}$ s.t. the obvious square commutes

$$\begin{tikzcd} A^i \arrow[r, "d_A^i"] \arrow[d, "f^i"'] & A^{i+1} \arrow[d, "f^{i+1}"] \\ B^i \arrow[r, "d_B^i"] & B^{i+1} \end{tikzcd}$$

Want to do smth similar to homotopy theory.

Back to our example: X, Y simplicial complexes

$f, g : Y \to X$ homotopic maps

$\Rightarrow$ get induced maps $f^*, g^* : C^\bullet(X) \to C^\bullet(Y)$ &

$h^i : C^i(X) \to C^{i-1}(Y)$ s.t.

$$f^i - g^i = d_Y^{i-1} \circ h^i + h^{i+1} \circ d_X^i .$$

Want smth like: homotopic maps are equal

In $\mathcal{A}$ arbitrary ab cat, set

~~de~~ $f^\bullet, g^\bullet : A^\bullet \to B^\bullet$ homotopic iff $\exists$ $h^\bullet$ as above.

<u>Fact</u> : $f^\bullet \sim g^\bullet$ is an equivalence relation &

$$\left.\begin{array}{l} f_1^\bullet \sim g_1^\bullet \\ f_2^\bullet \sim g_2^\bullet \end{array}\right\} \Rightarrow f_1^\bullet \circ f_2^\bullet \sim g_1^\bullet \circ g_2^\bullet .$$

<u>Def</u> $\mathcal{A}$ abelian category

$K(\mathcal{A})$ homotopy category
- objects : $A^\bullet$, $A \in Obj(\mathcal{A})$
- morphisms : chain maps $/ \sim$

<u>Properties</u>

1. Another great reason to treat homotopic maps as equal: injective resolutions

$A, B \in Ob(\mathcal{A})$ $f : A \to B$ morphism

$$\begin{array}{l} 0 \to A \to I_A^\bullet \\ 0 \to B \to I_B^\bullet \end{array} \Rightarrow f \text{ can be lifted to } \bar{f} : I_A^\bullet \to I_B^\bullet$$
($\bar{f}$ not unique, but $\bar{f}_1 \sim \bar{f}_2$)

$$\Rightarrow \mathrm{Hom}_{\mathcal{A}}(A, B) \cong \mathrm{Hom}_{K(\mathcal{A})}(I_A^\bullet, I_B^\bullet)$$

2. $K(\mathcal{A})$ not abelian; have some other notion for exactness (triangulated structure)

Not quite done. Want also to express a Whitehead-theorem type result in a natural way.

Recall from the simplicial complexes example

$$X \sim Y \Rightarrow \begin{tikzcd} & C^\bullet(Z) \arrow[dl, "\text{quasi-iso}"'] \arrow[dr, "\text{quasi-iso}"] & \\ C^\bullet(X) \arrow[rr, dashed, shift left] & & C^\bullet(Y) \arrow[ll, dashed, shift left] \end{tikzcd}$$

Want to be able to invert them.
So pretend that q-i are iso!
This is <u>localisation.</u>

<u>Def</u> $\mathcal{A}$ abelian category
$D(\mathcal{A})$ = derived category
= pretend q-i in $K(\mathcal{A})$ are isomorphisms

Morphisms in $D(\mathcal{A})$: $A^\bullet \to B^\bullet$ is the <u>roof</u>

$$\begin{tikzcd} & C^\bullet \arrow[dl, "f"'] \arrow[dr, "g"] & \\ A^\bullet \arrow[rr, dashed] & & B^\bullet \end{tikzcd}$$

$f, g \in \mathrm{Mor}_{K(\mathcal{A})}$
f q-i

Roof represents $g \circ f^{-1}$ even though f^{-1} doesn't exist just like $\frac{3}{4}$ represents $3 \cdot 4^{-1} \in \mathbb{Q}$ even though $4^{-1} \notin \mathbb{Z}$

No worries, the construction is sound!

<u>Gabriel-Zisman Thm</u> $D(\mathcal{A})$ exists.

We have functor $Q : K(\mathcal{A}) \to D(\mathcal{A})$

It is universal!

$$Q(A^\circ) = A^\circ$$

$$Q(f : A^\circ \to B^\circ) = \begin{tikzcd} & A^\circ \arrow[dl, "id"'] \arrow[dr, "f"] & \\ A^\circ \arrow[rr, dashed] & & B^\circ \end{tikzcd}$$

if q-i $\Rightarrow$

$$\begin{tikzcd} & A^\circ \arrow[dl, "id"'] \arrow[dr, "f"] & \\ A^\circ \arrow[rr, dashed, shift left] & & B \arrow[ll, dashed, shift left] \end{tikzcd}$$

Why bother? Now we have a way of representing obj in $\mathcal{A}$ as inj resolutions and ~~all~~ we have a nice way of dealing with their morphisms too

<u>Advantage</u>: The right derived functors can be expressed in a unified way:

More precisely: For simplicity, work only w/ bounded complexes i.e. $A^\cdot$ s.t. $H^i(A^\cdot)=0$ for $|i|>>0$.

~~Spse $\mathcal{A}$ has enough injectives.~~

Denote by $K^b(\mathcal{A})$, $D^b(\mathcal{A})$ the bounded categories.

Spse $\mathcal{A}$ has enough injectives.
Then every bounded below $A^\cdot \in K^b(\mathcal{A})$ is ~~[illegible]~~ quasi-isomorphic to complex of injectives.

Denote by $K^b(inj(\mathcal{A}))$ complexes of injectives.

Then we have an equivalence

$$\boxed{K^b(inj(\mathcal{A})) \xrightarrow{Q} D^b(\mathcal{A})}$$

Left-exact functor $F: \mathcal{A} \longrightarrow \mathcal{B}$ $\forall \mathcal{A}, \mathcal{B}$

$\Rightarrow$ have exact functor $\tilde{F}: K^b(inj(\mathcal{A})) \longrightarrow K^b(\mathcal{B})$

If $\mathcal{A}$ has enough injectives, ~~then~~ F left-exact,

$$RF: D^b(\mathcal{A}) \xrightarrow{Q^{-1}} K^b(inj(\mathcal{A})) \xrightarrow{\tilde{F}} K^b(\mathcal{B}) \xrightarrow{Q} D^b(\mathcal{B})$$

What is then $RF(A)$?, $A \in \mathcal{A}$.

$I^\cdot = Q^{-1}(A)$ q-i to $0 \to A \to 0 \to 0 \to \dots$

So $I^\cdot$ is ~~an resolu~~ inj resolution of A

Apply F to $I^\cdot$, get $RF(A)$.

~~$\bar{f}$ not unique but $\bar{f}_1 \sim \bar{f}_2$~~

~~$\Rightarrow$ morphisms between inj resolutions are the same up to homotopy~~

~~So $\mathrm{Hom}_{K(\mathcal{A})}(I_A^\cdot, I_B^\cdot) \simeq \mathrm{Hom}_{\mathcal{A}}(A, B)$~~

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Take cohomology:

$$H^i(RF(A)) = H^i(F(I^\cdot)) \overset{\text{usual def}}{=} R^iF(A)$$

$\Rightarrow$ now can do this for complexes &

$RF(A^\cdot)$ is a complex not a bunch of objects.

Note For left exact need injectives

For right-exact need projectives.

Luckily, for our D-modules, we have both!

$\mathcal{O}_X$-quasi-coherent $\mathcal{D}_X$-modules

Denote by $\mu(\mathcal{D}_X)$ = category of $\mathcal{O}_X$-quasi-coherent $\mathcal{D}_X$-modules.

$\mu(\mathcal{D}_X)$ has enough injectives:

Any $\mathcal{M} \in \mu(\mathcal{D}_X)$ can be embedded in an inj $\mathcal{O}_X$-quasi-coh $\mathcal{J}$

Proof (Sketch)

$\{X_i\}$ finite open affine cover of X

$j_i : X_i \to X$

$\mathcal{M}_i := j_i^* \mathcal{M}$ $\qquad M_i = \mathcal{M}(X_i)$

Can find injective $D(X_i)$-module J_i containing M_i as $D(X_i)$-submodule

$\downarrow$ b/c cat of R-modules has enough inj (R not nec comm)

Take $\mathcal{J}_i$ = associated $\mathcal{D}_{X_i}$-module i.e. a sheaf on X_i s.t. $\mathcal{J}_i(U_j)$ is a $D(U_j)$-module $\forall\ U_j \subset X_i$ open

(X_i affine so ok)

$\mathcal{J}_i$ injective

Then we have $0 \to \mathcal{M}_i \to \mathcal{J}_i$

$\downarrow j_{i*}$ preserves q.c & is exact (X_i affine)

$$0 \to j_{i*}(\mathcal{M}_i) \xrightarrow{s_i} j_{i*}(\mathcal{J}_i)$$

- These direct images are still D_X-modules:

$U \subseteq X$ open $\quad j_{i*}(\mathcal{M}_i)(U) = \mathcal{M}_i(j_i^{-1}(U)) = \mathcal{M}_i(U \cap X_i)$

$j_{i*}(\mathcal{J}_i)(U) = \mathcal{J}_i(j_i^{-1}(U)) = \mathcal{J}_i(U \cap X_i)$

Now $D_X(U \cap X_i)$ acts on $\mathcal{M}_i(U \cap X_i)$ via $D_{X_i}(U \cap X_i)$.

- $j_{i*}(\mathcal{J}_i)$ injective (b/c j_{i*} left-adj to j_i^*
  j_i^* exact
  $\mathcal{J}_i$ inj)

iso b/c j_i inclusion

So $\quad \mathcal{M} \xrightarrow{r_i} j_{i*} j_i^* \mathcal{M} = j_{i*}\mathcal{M}_i$ &

$\mathcal{J} = \bigoplus_i j_{i*}\mathcal{J}_i$ injective

$s_i \circ r_i : \mathcal{M} \to j_{i*}\mathcal{M}_i \to j_{i*}\mathcal{J}_i$

$\downarrow$ still inj $\qquad \downarrow \oplus$

$\mathcal{M} \to \mathcal{J}$ $\square$

<u>$\mu(D_X)$ has enough projectives</u>

X quasi-proj, $F \in \mu(D_X)$
Then F = quotient of locally free D_X-module.

So we have $i: X \to P$ locally closed proj embedding

ETS: $i_* F$ = quotient of locally free $\mathcal{O}_P$-module Q
b/c then $Q|_X$ is locally free over $\mathcal{O}_X$
& F quotient of $D_X \otimes_{\mathcal{O}_X} (Q|_X)$

$$Q \to i_* F \to 0 \qquad i^* \text{ exact \& } -\otimes \text{ right exact}$$
$$\Downarrow$$
$$D_X \otimes_{\mathcal{O}_X} (Q|_X) \to F \to 0$$

Now $i_* F$ qc over $\mathcal{O}_P$ $\Rightarrow$ $i_* F$ = inductive limit of $\mathcal{O}_P$-coherent submodules F_i

Then $\forall i\ \exists m_i$ s.t. $\bigoplus \mathcal{O}_P(m_i) \to F_i \to 0$

$\Rightarrow Q := \bigoplus_s \mathcal{O}_P(m_s)$.

Is Q locally free?
Take $\{P_j\}$ open cover of P (jth hom coord is $\neq 0$)
Then $\mathcal{O}_P(m)|_{P_j}$ is free $\Rightarrow$ $Q|_{P_j}$ free $\Rightarrow$ Q locally free
$m \in \mathbb{Z}$