Introduction and preliminaries

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1. Introduction

Theorem 1.1 (Serre duality). *Let k be a field, X a smooth projective scheme over k of relative dimension n, and $\mathcal{F}$ a locally free $\mathcal{O}_X$-module of finite rank. Then for $i \in \mathbb{Z}$ there is a canonical isomorphism*

$$\mathrm{H}^i(X, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \Omega^n_{X/k})) \cong \mathrm{H}^{n-i}(X, \mathcal{F})^\vee. \qquad \blacklozenge$$

One goal of the coherent cohomology seminar is to state and prove a generalization of this theorem. First, we drop the smoothness condition. We need to replace $\Omega^n_{X/k}$ by the more abstract *dualizing sheaf* $\omega_{X/k}$.

Also we make the situation relative and consider arbitrary proper morphisms $X \to Y$. In general there will no longer be a suitable notion of dualizing sheaf. To remedy the situation we resort to cochain complexes and derived categories.

Theorem 1.2 (Grothendieck–Serre duality). *Let $f\colon X \to Y$ be a proper morphism of locally noetherian schemes. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module and $\mathcal{G}$ a coherent $\mathcal{O}_Y$-module. Under suitable (weak) conditions, there is a canonical isomorphism*

$$\mathrm{R}f_*\mathrm{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, f^!\mathcal{G}) \cong \mathrm{R}\mathcal{H}om_{\mathcal{O}_Y}(\mathrm{R}f_*\mathcal{F}), \mathcal{G}). \qquad \blacklozenge$$

The main difficulty is the construction of $f^!$. It will turn out to be a question of representability. Compare Serre duality: for $i = 0$ the theorem states that $\mathrm{Vec}\, X \to \mathrm{Set}, \mathcal{F} \mapsto \mathrm{H}^n(X, \mathcal{F})^\vee$ is represented by the sheaf of differentials $\Omega^n_{X/k}$.

2. Sheaves of modules

Let X be a ringed space. Recall the notions of
- *$\mathcal{O}_X$-modules* and their *morphisms*;
- *free* and *locally free* $\mathcal{O}_X$-modules, their *rank*;
- *vector bundle*: locally free $\mathcal{O}_X$-module of finite rank;
- *line bundle*: locally free $\mathcal{O}_X$-module of rank 1.

Some constructions:

tensor product	$\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} := \Big(U \mapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)\Big)^\#,$
direct sum	$\bigoplus_{i\in I} \mathcal{F}_i := \Big(U \mapsto \bigoplus_{i \in I} \mathcal{F}_i(U)\Big)^\#,$
sheaf hom	$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) := \Big(U \mapsto \mathrm{Hom}_{\mathcal{O}_X\|_U}(\mathcal{F}\|_U, \mathcal{G}\|_U)\Big),$
dual	$\mathcal{F}^\vee := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X).$

The category $\mathcal{O}_X$-Mod is abelian.

Definition 2.1. Let $f\colon X \to Y$ be a morphism of ringed spaces, $\mathcal{F}$ an $\mathcal{O}_X$-module and $\mathcal{G}$ an $\mathcal{O}_Y$-module. The *direct image* or *pushforward* of $\mathcal{F}$ is the sheaf of abelian groups

$$f_*\mathcal{F} := \Big(U \mapsto \mathcal{F}(f^{-1}U)\Big)$$

with the $\mathcal{O}_Y$-module structure from restriction of scalars $\mathcal{O}_Y \to f_*\mathcal{O}_X$. The *inverse image* or *pullback* of $\mathcal{G}$ is the sheaf of abelian groups

$$f^*\mathcal{G} := \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{G}$$

with the $\mathcal{O}_X$-module structure by multiplication on the left. ♦

Despite the somewhat complicated definition, pullback is very well-behaved: for instance, we have $f^*\mathcal{O}_Y \cong \mathcal{O}_X$, and the stalk at $x \in X$ is given by

$$(f^*\mathcal{G})_x \cong \mathcal{O}_{X,x} \otimes_{\mathcal{O}_{Y,f(x)}} \mathcal{G}_{f(x)}.$$

Proposition 2.2. *Let $f\colon X \to Y$ be a morphism of ringed spaces.*

- *Pullback and pushforward constitute adjoint functors*

$$f^*\colon \mathcal{O}_Y\text{-Mod} \to \mathcal{O}_X\text{-Mod}, \quad f_*\colon \mathcal{O}_X\text{-Mod} \to \mathcal{O}_Y\text{-Mod}.$$

- *Pullback is right exact and pushforward is left exact.*
- *If $g\colon Y \to Z$ is another morphism of ringed spaces, then $(gf)_* = g_*f_*$ and $(gf)^* \cong f^*g^*$.* ♦

See [Stacks `0094`, `01AF`] for more details.

3. Quasi-coherent modules

Let A be a ring and M an A-module. There is a unique sheaf of modules $M^\sim$ on $\operatorname{Spec} A$ such that for all $f \in A$ we have $M^\sim(\mathrm{D}(f)) = M_f$ as A_f-module, with the obvious restriction maps. The construction $M \mapsto M^\sim$ is a functor $A\text{-Mod} \to \mathcal{O}_{\operatorname{Spec} A}\text{-Mod}$, left adjoint to the global sections functor $\Gamma(\operatorname{Spec} A, \cdot)$.

Definition 3.1. Let X be a scheme. An $\mathcal{O}_X$-module $\mathcal{F}$ is *quasi-coherent* if for every affine open $U \subseteq X$ we have $\mathcal{F}|_U = \mathcal{F}(U)^\sim$. ♦

Definition 3.2. Let X be a locally noetherian scheme. An $\mathcal{O}_X$-module $\mathcal{F}$ is *coherent* if it is quasi-coherent and for every affine open $U \subseteq X$ the $\mathcal{O}_X(U)$-module $\mathcal{F}(U)$ is finitely generated. ♦

The category $\operatorname{QCoh} X$ contains all kernels, cokernels, extensions, direct sums, and tensor products. If X is locally noetherian, the same is true for $\operatorname{Coh} X$ (only finite direct sums). The categories $\operatorname{QCoh} X$ and $\operatorname{Coh} X$ are abelian.

Pullbacks of quasi-coherent modules are again quasi-coherent. In the locally noetherian case the same is true for coherent modules. However, pushforwards of a quasi-coherent module are not necessarily quasi-coherent.

Proposition 3.3. *Let $f\colon X \to Y$ be a quasi-compact quasi-separated morphism of schemes and $\mathcal{F}$ a quasi-coherent $\mathcal{O}_X$-module. Then $f_*\mathcal{F}$ is also quasi-coherent.* ♦

We will see a similar statement for coherent modules later. See [Stacks `01I6`, `01LA`, `01XY`] for more details.

4. Derived functors

Let $\mathcal{A}$ be an abelian category. An object $I \in \mathcal{A}$ is *injective* if the functor $\mathrm{Hom}(\cdot, I)$ is exact. If every object of $\mathcal{A}$ is a subobject of an injective object, then $\mathcal{A}$ has *enough injectives*. An *injective resolution* of an object $A \in \mathcal{A}$ is a complex $I^\bullet$ with a morphism $A \to I^0$, such that all I^i are injective, $I^i = 0$ for $i < 0$, and

$$0 \longrightarrow A \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \dots$$

is exact. If $\mathcal{A}$ has enough injectives, then every object has an injective resolution.

Lemma 4.1. *Let $\mathcal{A}$ be an abelian category and $f\colon A \to B$ a morphism in $\mathcal{A}$. Let $I^\bullet, J^\bullet$ be injective resolutions of A, B. Then there exists a morphism of complexes $I^\bullet \to J^\bullet$ that induces f on cohomology, and such a morphism is unique up to homotopy.* ♦

Definition 4.2. Let $\mathcal{A}, \mathcal{B}$ be abelian categories, $F\colon \mathcal{A} \to \mathcal{B}$ a left exact functor, and suppose $\mathcal{A}$ has enough injectives. The *i-th right derived functor* of F is $\mathrm{R}^i F\colon \mathcal{A} \to \mathcal{B}, A \mapsto \mathrm{H}^i(F(I^\bullet))$ where $I^\bullet$ is an injective resolution of A. ♦

Dually there are *projective resolutions* and *left derived functors*. In a certain sense, derived functors are 'exact approximations'. This will be made precise in the language of *derived categories*.

We have a canonical isomorphism $F \cong \mathrm{R}^0 F$. Each short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

in $\mathcal{A}$ gives rise to a long exact sequence

$$0 \longrightarrow \mathrm{R}^0F(A) \longrightarrow \mathrm{R}^0F(B) \longrightarrow \mathrm{R}^0F(C) \longrightarrow \mathrm{R}^1F(A) \longrightarrow \mathrm{R}^1F(B) \longrightarrow \mathrm{R}^1F(C) \longrightarrow \dots .$$

Proposition 4.3 (Leray acyclicity). *Let $\mathcal{A}, \mathcal{B}$ be abelian categories, $F\colon \mathcal{A} \to \mathcal{B}$ a left exact functor, and suppose $\mathcal{A}$ has enough injectives. An object $J \in \mathcal{A}$ is* acyclic *for F if $\mathrm{R}^i F(J) = 0$ for all $i > 0$. Let $A \in \mathcal{A}$ be an object and $J^\bullet$ an acyclic resolution of A. Then $\mathrm{R}^i F(A) \cong \mathrm{H}^i(F(J^\bullet))$ for all $i \in \mathbb{Z}$.* ♦

Acyclic resolutions tend to be more available than injective ones, so they are useful for computations. See [Stacks `0134`, `0156`, `05TB`] for more details.

5. Sheaf cohomology

The category $\mathcal{O}_X$-Mod on a ringed space X has enough injectives.

Definition 5.1. Let X be a ringed space. The *i-th cohomology functor* of X is the right derived functor $\mathrm{H}^i(X, \cdot) := \mathrm{R}^i(\Gamma(X, \cdot))\colon \mathcal{O}_X\text{-Mod} \to \mathcal{O}_X(X)\text{-Mod}$. ♦

Definition 5.2. Let $f\colon X \to Y$ be a morphism of ringed spaces. The *i-th higher direct image functor* of f is the right derived functor $\mathrm{R}^i f_*\colon \mathcal{O}_X\text{-Mod} \to \mathcal{O}_Y\text{-Mod}$. ♦

Both versions of cohomology can also be computed on the level of abelian sheaves and abelian groups; the result is the same.

The higher direct image functors are relative versions of the absolute cohomology functors $\mathrm{H}^i(X, \cdot)$. If $f\colon X \to Y$ is a morphism of ringed spaces and $\mathcal{F}$ an $\mathcal{O}_X$-module, then

$$\mathrm{R}^i f_* \mathcal{F} \cong \Big(V \mapsto \mathrm{H}^i(f^{-1}V, \mathcal{F})\Big)^{\#}.$$

For schemes we have the following nice relation. Let $f: X \to Y$ be a quasi-compact quasi-separated morphism of schemes with Y affine. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then $\mathrm{R}^i f_*\mathcal{F} \cong \mathrm{H}^i(X, \mathcal{F})^\sim$.

Proposition 5.3. *Let $f: X \to Y$ be an affine morphism of schemes and $\mathcal{F}$ a quasi-coherent $\mathcal{O}_X$-module. Then for all $i > 0$ we have $\mathrm{R}^i f_*\mathcal{F} = 0$, and for all $i \in \mathbb{Z}$ we have $\mathrm{H}^i(X, \mathcal{F}) = \mathrm{H}^i(Y, f_*\mathcal{F})$.* ♦

Theorem 5.4 (Grothendieck vanishing). *Let X be a noetherian ringed space and $\mathcal{F}$ an $\mathcal{O}_X$-module. Then $\mathrm{H}^i(X, \mathcal{F}) = 0$ for all $i > \dim X$.* ♦

See [Stacks `01DH`, `01DZ`, `01E0`, `01X8`, `01XH`, `02UU`] for more details.

6. Čech cohomology

Definition 6.1. Let X be a ringed space and $\mathcal{F}$ an $\mathcal{O}_X$-module. Let $\mathcal{U} = (U_i)_{i \in I}$ be an open cover of X. Put

$$C^r(\mathcal{U}, \mathcal{F}) := \prod_{i_0, \ldots, i_r \in I} \mathcal{F}(U_{i_0} \cap \ldots \cap U_{i_r})$$

and define maps

$$C^r(\mathcal{U}, \mathcal{F}) \to C^{r+1}(\mathcal{U}, \mathcal{F}), \quad (a_{i_0 \ldots i_r})_{i_0, \ldots, i_r} \mapsto \Big(\sum_{j=0}^{r+1} (-1)^j a_{i_0 \ldots \hat{\imath}_j \ldots i_{r+1}}|_{U_{i_0} \cap \ldots \cap U_{i_{r+1}}} \Big)_{i_0, \ldots, i_{r+1}}.$$

The *r-th Čech cohomology group* of $\mathcal{F}$ relative to $\mathcal{U}$, denoted $\check{\mathrm{H}}^r(\mathcal{U}, \mathcal{F})$, is the r-th cohomology group of the cochain complex $C^\bullet(\mathcal{U}, \mathcal{F})$. ♦

The purpose of Čech cohomology is to compute the 'true' cohomology. For simplification one may endow I with a total ordering $<$ and consider the *ordered complex*: define

$$C^r_<(\mathcal{U}, \mathcal{F}) := \prod_{i_0 < \ldots < i_r \in I} \mathcal{F}(U_{i_0} \cap \ldots \cap U_{i_r})$$

and maps $C^r_<(\mathcal{U}, \mathcal{F}) \to C^r_<(\mathcal{U}, \mathcal{F})$ as before. The cohomology of $C^\bullet_<(\mathcal{U}, \mathcal{F})$ is canonically isomorphic to the usual Čech cohomology.

Theorem 6.2. *Let X be a scheme and $\mathcal{U} = (U_i)_{i \in I}$ an open cover of X such that $U_{i_0} \cap \ldots \cap U_{i_r}$ is affine for all $r \geq 0$. Then for all quasi-coherent $\mathcal{O}_X$-modules $\mathcal{F}$ and all $r \in \mathbb{Z}$ we have $\check{\mathrm{H}}^r(\mathcal{U}, \mathcal{F}) \cong \mathrm{H}^r(X, \mathcal{F})$ as $\mathcal{O}_X(X)$-modules.* ♦

An important application is the computation of the cohomology of projective space.

Theorem 6.3. *Let A be a ring, $n \geq 0$ and $d \in \mathbb{Z}$. Then*

$$\mathrm{H}^i(\mathbb{P}^n_A, \mathcal{O}(d)) = \begin{cases} A[x_0, \ldots, x_n]_d & \text{if } i = 0, \\ \left(\frac{1}{x_0 \cdots x_n} A[\frac{1}{x_0}, \ldots, \frac{1}{x_n}]\right)_d & \text{if } i = n, \\ 0 & \text{otherwise.} \end{cases}$$ ♦

This computation is a main ingredient in the proof of the following theorem. The remainder of the proof will be given next week.

Theorem 6.4. *Let $f: X \to Y$ be a proper morphism of locally noetherian schemes. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then $\mathrm{R}^i f_*\mathcal{F}$ is a coherent $\mathcal{O}_Y$-module for all $i \in \mathbb{Z}$.* ♦

Yet another approach to cohomology computations is by a resolution in sheaves with known cohomology. For instance, let A be a ring, $n \geq 2$, and $f \in A[x_0, \ldots, x_n]$ a non-zero homogeneous polynomial of degree $d \geq 1$. Let $j\colon X \to \mathbb{P}^n_A$ be the closed subscheme defined by f. We have an exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n_A}(-d) \longrightarrow \mathcal{O}_{\mathbb{P}^n_A} \longrightarrow j_*\mathcal{O}_X \longrightarrow 0.$$

The long exact sequence of higher direct images gives

$$\mathrm{H}^i(X, \mathcal{O}_X) = \begin{cases} A & \text{if } i = 0, \\ \left(\frac{1}{x_0 \cdots x_n} A[\frac{1}{x_0}, \ldots, \frac{1}{x_n}]\right)_{-d} & \text{if } i = n-1, \\ 0 & \text{otherwise.} \end{cases}$$

See [Stacks `01ED`, `01FG`, `01X8`, `01XS`, `0203`] for more details.