Talk: Leray spectral sequence

Note: we will use cohomology but there is a dual theory for homology. And the thy is more general

Conventions: For the 1st part of the talk we fix an abelian category $\mathcal{A}$ in which we work. We omit the expression "in $\mathcal{A}$" or "with values in $\mathcal{A}$".

Idea of ss: approximate $H^n(A)$ "difficult" by $\begin{cases} \cdot \text{ filtr } A \rightsquigarrow \text{filtr } H^n(A) \\ \cdot \text{ approx } gr^p H^n(A) \approx H^n(gr^p A) \end{cases}$

def A (co) complex $A^\cdot$ is a collection $\{A^i, d^i\}_{i \in \mathbb{Z}}$

- $A^i \in \operatorname{obj} \mathcal{A}$
- $d^i : A^i \longrightarrow A^{i+1}$

st $d^2 = 0$.

Rmk Hence $\operatorname{Im} d^{i-1} \subset \ker d^i \rightsquigarrow H^i(A^\cdot) = \ker d^i / \operatorname{Im} d^{i-1}$ is a subquotient of A^i.

def A double complex $A^{\cdot\cdot}$ is a collection $\{A^{ij}, d^{ij}, \delta^{ij}\}_{i,j \in \mathbb{Z}}$

i) $A^{ij} \in \operatorname{obj} \mathcal{A}$

ii) $d^{ij} : A^{ij} \longrightarrow A^{i\,j+1}$ vertical differential

iii) $\delta^{ij} : A^{ij} \longrightarrow A^{i+1\,j}$ horizontal differential

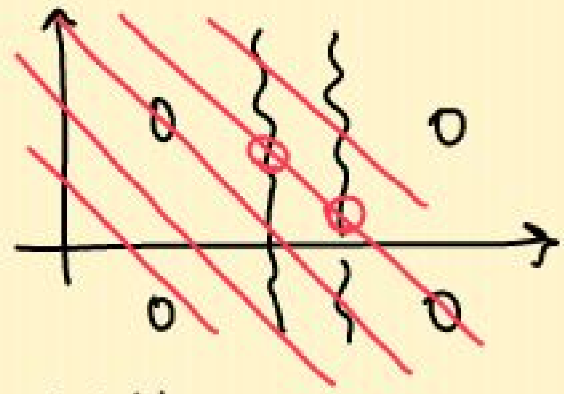
iv) $d^2 = \delta^2 = d\delta + \delta d = 0$
$\hookrightarrow$ anticommute

$$\begin{tikzcd}
 & \vdots & \vdots & & \\
\arrow[r] & A^{i-1\,j} \arrow[u] \arrow[r, "\delta"] & A^{ij} \arrow[u, "d"] \arrow[r, "\delta"] & A^{i+1\,j} \arrow[r] & \cdots \\
 & A^{i-1\,j-1} \arrow[u, "d"] \arrow[r, "\delta"] & A^{i} \arrow[u, "d"] \arrow[r] & \cdots &
\end{tikzcd}$$

The total complex associate to (A, d, δ) is

$$(\operatorname{Tot} A^\cdot, \partial) \qquad (\operatorname{Tot} A)^n = \bigoplus_{i+j=n} A^{ij} \qquad \partial = d + \delta$$

Rmk $\partial^2 = d^2 + \delta^2 + d\delta + \delta d = 0$. This is sometimes denoted as $\operatorname{Tot}^{\oplus} A$ to distinguish from $\operatorname{Tot}^{\pi} A$



<u>Ex</u>. Assume $A^{ij} = 0 \ \forall \ i \neq p-1, p$

then $(\mathrm{Tot}\, A)^n = A^{p-1, q+1} \oplus A^{p,q}$

with $p+q=n$.

Then $\partial : A^{p-1\, q+1} \oplus A^{p\, q} \to A^{p-1\, q+2} \oplus A^{p\, q+1}$

$$\begin{pmatrix} d & 0 \\ \delta & d \end{pmatrix}$$

Let $C^\bullet = \mathrm{Tot}\, A^\bullet$

If $C^\bullet = C_1^\bullet \oplus C_2^\bullet \rightsquigarrow H^n(C^\bullet) = H^n(C_1^\bullet) \oplus H^n(C_2^\bullet)$ but it is not the case.

<u>def</u> Let $A \in \mathrm{obj}\, \mathcal{A}$, a <u>decreasing filtration</u> is a chain of subobjects $F^i A \subset A$ st $F^i A \supset F^{i+1} A$. Then $\boxed{gr^i_F A} \doteq F^i A / F^{i+1} A$ and $\boxed{gr_F A} \doteq \bigoplus gr^i_F A$.

Note that $gr^i_F A \rightsquigarrow A$ up to choice of extensions (if $\cap F^i A = 0$ and $\cup F^i A = A$)

(in our case we will filter $\mathrm{Tot}\, A^\bullet$ in two ways)

if $d F^i C^j \subseteq F^i C^{j+1}$ then $F^i C^\bullet$ is a subcomplex and induces $F^\bullet H^n(C^\bullet)$ by $F^i C^\bullet \hookrightarrow C^\bullet \rightsquigarrow H^n(F^i C^\bullet) \to H^n(C^\bullet)$

$F^i H^n(C^\bullet) \doteq \mathrm{Im}\,(H^n(F^i C^\bullet))$.

<u>Aim</u>: $\boxed{\text{Relate } H^n(gr^i_F C^\bullet) \text{ with } gr^i_F H^n(C^\bullet).}$

<u>def</u> A <u>spectral sequence</u> (of cohomological type) <u>starting at a</u> for $a \in \mathbb{N}$ is a collection $\{E_r^{pq}, d_r^{pq}\}$ for $r \geq a$, $p, q \in \mathbb{Z}$

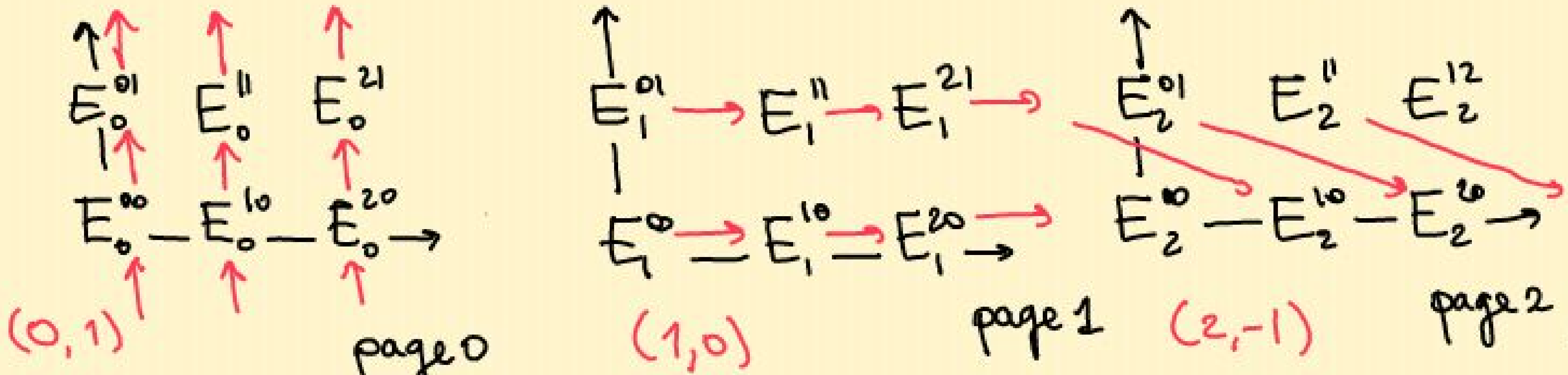
- $E_r^{pq} \in \mathrm{obj}\, \mathcal{A}$
- $d_r^{pq} : E_r^{pq} \longrightarrow E^{p+r, q-r+1}$

st. $\cdot\ d_r^2 = 0$

- $E_{r+1}^{pq} \cong \ker d_r^{pq} / \mathrm{Im}\, d_r^{p-r, q+r-1}$

(p,q) is the <u>bidegree</u>, $p+q=n$ is the <u>total degree</u> and we can say that d_r has bidegree $(r, -r+1)$.

$\{E_r^{pq}, d_r^{pq}\}$ for a fixed r is called the r-th page



— Some additional definitions —

def: A morphism of spectral sequences $f : \{E_r^{pq}\} \to \{\tilde{E}_r^{pq}\}$ is a collection $f_r^{pq} : E_r^{pq} \to \tilde{E}_r^{pq}$ st

i) $f_r d_r = \tilde{d}_r f_r$

ii) f_{r+1}^{pq} is induced by f_r^{pq} on cohomology

Lemma (mapping lemma) If $f_r^{pq} : E_r^{pq} \xrightarrow{\sim} \tilde{E}_r^{pq}$ is an iso $\forall\, p,q$ and fixed $r \Rightarrow$ is an iso $\forall\, s \geq r$ $\forall\, p,q$.

def A spectral sequence starting at a is bounded if for every n $\exists$ only fin. many $E_a^{pq} \neq 0$ for $p+q=n$

It is uniformly bounded if $E_a^{pq} = 0$ $\forall$ $p < N_1$ $p > N_2$
$q < M_1$ $q > M_2$

Rmk If $\{E_r^{pq}\}_{r \geq a}$ is bounded then for any p,q $\exists\, \bar{r}$ st $\forall\, r \geq \bar{r}$

$$E_r^{p-r,q+r-1} = E_r^{p+r,q-r+1} = 0$$ hence $\operatorname{Im} d^{p-r,q+r-1} = 0$, $\ker d^{pq} = E_r^{pq}$

Fix p,q, if $\exists\, \bar{r}$ st $E_r^{pq} = E_{\bar{r}}^{pq}$ $\forall\, r \geq \bar{r}$, we denote $E_{\bar{r}}^{pq} = E_\infty^{pq}$ and call it the stable value at the bidegree (p,q).

$\{E_r^{pq}\}_{r \geq a}$ collapses at level $\bar{r}$ if $E_r = E_{\bar{r}}$ $\forall\, r \geq \bar{r}$

i.e. $E_{\bar{r}} = E_\infty$. e.g. unif. bounded $\Rightarrow$ collapsing at a finite level.

Ex (stupid ss) $E_r^{pq} = A$ $\quad d_r^{pq} = 0$ $\forall\, p,q,\ r \geq 0$.

def E_a^{pq} converges to $H^\cdot$ if $\exists$ family $\{H^n\}$ in obj $\mathcal{A}$
st i) each H^n has a finite decreasing filtration

$$0 = F^s H^n \subseteq \dots \subseteq F^p H^n \subseteq \dots \subseteq F^t H^n = H^n$$

ii) $E_\infty^{pq} \simeq F^p H^{p+q} / F^{p+1} H^{p+q} = gr_F^p H^{p+q}$

we denote it by $E_a^{pq} \Rightarrow H^{p+q}$.

Note: Hence a converging spectral sequence allows at ∞ to obtain $gr_F^i H^n$. If we know H^n to be extension free (eg a k-vector space) we actually have reconstructed H^n (but not Aut). In general:

Lemma (Comparison Lemma) $E_r^{pq} \Rightarrow H^*$ $\tilde{E}_r^{pq} \Rightarrow \tilde{H}^*$
$h: H^* \to \tilde{H}^*$ compatible w/ $f: E_r \to \tilde{E}_r$. If $f_r: E_r \to \tilde{E}_r$ is an iso for some r $\Rightarrow$ h is an iso.

— SS associated to graded filtered obj —

$A = \oplus A^n$ graded object w/ differential of degree 1 d.
$F^p A$ decreasing filtration st • $F^p A$ is a subgraded obj $\forall p$
• $dF^i A^n \subseteq F^i A^{n+1}$

define $E_0^{pq}(A,F) \doteq F^p A^{p+q} / F^{p+1} A^{p+q}$, then d induces
$d_0^{pq}: E_0^{pq}(A,F) \to E_0^{p\,q+1}(A,F)$.

thm with the previous conditions on A, F, d then $\exists \{E_r^{pq}\}_{r\geq 0}$
st $E_0^{pq} = E_0^{pq}(A,F)$, $E_1^{pq} = H^{p+q}(F^p A / F^{p+1} A)$. Moreover:

if i) E_r^{pq} is bounded
ii) F bounded above ($F^n A = 0$ $n \gg 0$) and $F^n A^i = A^i$
$n = n(i) \ll 0$ (or, more generally, $\cup F^n A = A$).

then $E_0^{pq} \Rightarrow H^{p+q}(A)$. Pf: last page.

Let now $(A^{\cdot\cdot}, d, \delta)$ be a bigraded object, then Tot A has two filtrations

by columns $F^p \mathrm{Tot} A^n = \bigoplus_{\substack{p'+q=n \\ p' \geq p}} A^{p'q}$ $G^q \mathrm{Tot} A^n = \bigoplus_{\substack{p+q'=n \\ q' \geq q}} A^{pq'}$ by rows

$E_0^{pq} = A^{pq}$ $\tilde{E}_0^{pq} = A^{qp}$ and if they satisfy thm hyp.
(eg. 1st quadrant, i.e $A^{pq} = 0 \ \forall p<0 \ q<0$) then

$$E_0^{pq}, \tilde{E}_0^{pq} \Rightarrow H^{p+q}(\mathrm{Tot}\, A^{\cdot})$$

— Grothendieck and Leray ss —

$$\begin{tikzcd} \mathcal{A} \arrow[r, "G"] \arrow[dr, "F\circ G"'] & \mathcal{B} \arrow[d, "F"] \\ & \mathcal{C} \end{tikzcd}$$

$\mathcal{A}, \mathcal{B}, \mathcal{C}$ abelian
$\mathcal{A}, \mathcal{B}$ w/ enough injectives

Thm. Suppose that $\forall$ I injective $G(I)$ is F-acyclic then $\forall$ $A \in \mathcal{A}$ there exist

$$\boxed{E_2^{pq} = (R^pF)(R^qG)(A) \Rightarrow R^{p+q}(FG)(A)}$$

pf (sketch):

1. fully injective resolutions: A^n complex, then $A^{\cdot} \to I^{\cdot,0} \to I^{\cdot,1} \to \dots$ is a fully injective resolution if

$$\begin{tikzcd} A^0 \arrow[r, "d"] \arrow[d] & A^1 \arrow[r] \arrow[d] & A^2 \arrow[d] \\ I^{0,0} \arrow[r, "d"] \arrow[d] & I^{1,0} \arrow[r] \arrow[d] & I^{2,0} \arrow[d] \\ I^{0,1} \arrow[r, "d"] & I^{1,1} \arrow[r] & I^{2,1} \end{tikzcd}$$

$\rightsquigarrow$ $Z^i(A^{\cdot}) \to Z^i(I^{\cdot,0}) \to Z^i(I^{\cdot,1})$

$B^i(A^{\cdot}) \to B^i(I^{\cdot,0}) \to B^i(I^{\cdot,1})$

$H^i(A^{\cdot}) \to H^i(I^{\cdot,0}) \to H^i(I^{\cdot,1})$

are injective resolutions of $Z^i(A^{\cdot}), B^i(A^{\cdot}), H^i(A^{\cdot})$

Fact If $\mathcal{B}$ has enough injectives then every cpx bound. below in $\mathcal{B}$ admits a fully injective resolution.

2. $A \in \mathcal{A} \rightsquigarrow A \to C^{\cdot}$ injective resolution

$\rightsquigarrow GC^{\cdot} \rightsquigarrow GC^{\cdot} \to I^{\cdot\cdot}$ fully inj. resolution

$\rightsquigarrow FI^{\cdot\cdot} \rightsquigarrow E_0^{pq}, \tilde{E}_0^{pq}$

$\longrightarrow$ 1st quadrant

③ $$E_1^{pq} = H^q(FI^{p,\cdot}) = R^qF(GC^{p,\cdot}) = \begin{cases} q=0 & FGC^{p,0} \\ q\neq 0 & 0 \end{cases}$$
↪ GC^p F-acyclic

$$E_2^{pq} = \begin{cases} H^p(FGC^\cdot) = R^p(FG)(A) & q=0 \\ 0 & q\neq 0 \end{cases}$$
have to go to E_2 as d_1 is horizontal

In particular E_0^{pq} collapses at 2 and thus

$$E_0^{pq} \Rightarrow R^pFG(A) = H^{p+q}(\mathrm{Tot}\, FI^{\cdot,\cdot})$$
↪ no ext involved as $\exists!$ nonzero

but then $\tilde{E}_0^{pq} \Rightarrow R^pFG(A)$

④ $\tilde{E}_1^{pq} = H^q(FI^{\cdot,p})$ using fully injectivity (Z,B)

we get $H^q(FI^{\cdot,\cdot}) = FH^q(I^{\cdot,\cdot})$

$$\rightsquigarrow \tilde{E}_2^{pq} = H^p(F\underbrace{H^q(I^{\cdot,\cdot})}_{\text{inj. res. of } H^q(GC^\cdot) = R^qG(A)}) =$$
↗ f.i. H.

$$=(R^pF)(R^qG)(A).$$ □

Corollary (Leray ss) X,Y top. spaces $f: X\to Y$ $\mathrm{Ab}\,X$ abelian sheaves on X

$$\begin{tikzcd} \mathrm{Ab}\,X \arrow[rr, "f_*"] \arrow[dr, "\Gamma"'] & & \mathrm{Ab}\,Y \arrow[dl, "\Gamma"] \\ & \mathrm{Ab} & \end{tikzcd}$$

$$E_2^{pq} = H^p(Y; R^qf_*\mathcal{F}) \Rightarrow H^{p+q}(X;\mathcal{F})$$

Application 1 — Applications —

Thm If $f: X\to S$ is proper and S is loc noeth, then $\mathcal{F}\in \mathrm{Coh}\,X \Rightarrow R^if_*\mathcal{F}\in \mathrm{Coh}\,S$ $\forall i\geq 0$.

Pf: assume it is true for f projective. By Chow's lemma $\exists$ $X' \xrightarrow{\pi} X$ st X'/S proj

$$\begin{tikzcd} X' \arrow[r, "\pi"] \arrow[dr, "f'"'] & X \arrow[d, "f"] \\ & S \end{tikzcd}$$

π iso on a dense open of X

know q.coherence, suffices f.g.

$\mathcal{F} \longrightarrow \pi_* \pi^* \mathcal{F}$ $\qquad$ $\mathcal{F}|_U \simeq \tilde{\pi}_* \tilde{\pi}^* \mathcal{F}|_U$ w/ $\tilde{\pi} : \pi^{-1}(U) \simeq U$

$$0 \longrightarrow K \longrightarrow \mathcal{F} \longrightarrow \pi_* \pi^* \mathcal{F} \longrightarrow C \longrightarrow 0$$

K, C are supported on $X \setminus U$ of dim $< X$, using Noetherian induction $\rightsquigarrow$ thm holds on K, C

<u>Enough</u>: to show it for $\pi_* \pi^* \mathcal{F}$ by les of derived functors.

Know $R^i(\overset{f'_*}{f \circ \pi})_* (\pi^* \mathcal{F}) \in \mathrm{Coh}\, S$ but would like $R^i f_* \circ \pi_* (\pi^* \mathcal{F}) \in \mathrm{Coh}\, S$.

a) proper (& separated) $\rightsquigarrow$ $X \hookrightarrow X \times_S X$ closed emb.
by b.c. $\quad X' \simeq X' \times_X X \hookrightarrow X' \times_S X$ closed emb.
$X' \hookrightarrow X' \times_S X \hookrightarrow \mathbb{P}^m_S \times_S X \simeq \mathbb{P}^m_X$ hence π is proj. and $R^q \pi_* (\pi^* \mathcal{F}) \in \mathrm{Coh}\, X$

b) $(R^q \pi_* (\pi^* \mathcal{F}))|_U \simeq R^q \tilde{\pi}_* (\pi^* \mathcal{F}|_{\pi^{-1}(U)})$ hence
$\tilde{\pi}$ iso

$R^q \pi_* (\pi^* \mathcal{F})|_U \simeq 0$ hence $R^q \pi_* (\pi^* \mathcal{F})$ is supported in a proper subscheme for $q > 0$ and by Noetherian induction $R^p f_* R^q \pi_* (\pi^* \mathcal{F})$ is coherent for $q > 0$.

Gr. SS $R^p f_* R^q \pi_* (\pi^* \mathcal{F}) \Rightarrow R^{p+q} (f_* \circ \pi_*)(\pi^* \mathcal{F})$

-) $R^q \pi_* (\pi^* \mathcal{F})$ is coherent as π is projective
-) $R^q \pi_* (\pi^* \mathcal{F})$ is supported in codim ≥ 1 by (b)

hence by NI $R^p f_* R^q \pi_* (\pi^* \mathcal{F})$ is fg $\forall\, p \geq 0$ $\forall\, q > 0$

-) $R^{p+q} (f_* \circ \pi_*)(\pi^* \mathcal{F})$ is fg

$$\underset{f'_*}{\overset{\|}{(f_* \circ \pi_*)}}$$

$q = 0$? Note that $d_r^{p0} : E^{p0} \longrightarrow E^{p+r, -r+1} = 0$ 1st quadr

hence $E_r^{p-r, r-1} \to E_r^{p0} \longrightarrow E_{r+1}^{p0} \longrightarrow 0$

$r = \infty \rightsquigarrow$ get a subquotient of $R^{p = p+q = p+0}(f'_*)(\pi^* \mathcal{F})$ fg and $E_r^{p-r, r-1}$ fg as $q > 0$. Hence by induction

E_r^{p0} fg $\forall\, r \geq 2$ $E_2^{p0} = (R^p f_*) \circ \pi_* (\pi^* \mathcal{F})$ □

Application 2 : Lemma: $X \xrightarrow{f} S$ q. sep q. cpt + S affine then if $\mathcal{F} \in \mathrm{Qcoh}\, S$

$$H^q(X, \mathcal{F}) = H^0(S, R^q f_* \mathcal{F}).$$

~~proof~~: $E_2^{pq} = H^p(S, R^q f_* \mathcal{F}) \Rightarrow H^{p+q}(X, \mathcal{F})$

Now, S affine + $R^q f_* \mathcal{F}$ qcoherent $\Rightarrow H^p(S, R^q f_* \mathcal{F}) = 0$ $\forall\, p > 0$

hence E_2^{pq} degenerates at 2 and there is only one nonzero element on the diagonal (at $p = 0$)

hence $H^0(S, R^q f_* \mathcal{F}) = H^q(X, \mathcal{F})$.

Pf of main theorem. A, d, F chain cpx w/ good filt

$$Z_r^{pq} \doteq F^p A^{p+q} \cap d^{-1}(F^{p+r} A^{p+q+1})$$
$$B_r^{pq} \doteq F^p A^{p+q} \cap d(F^{p-r} A^{p+q-1})$$
$$Z_\infty^{pq} \doteq F^p A^{p+q} \cap \ker d$$
$$B_\infty^{pq} \doteq F^p A^{p+q} \cap \operatorname{Im} d$$

$F^p A \rightsquigarrow F^p H^q(A)$ by $\operatorname{Im} H^q(F^p A)$.

Note: fix any p, q and assume $F^n A^i = A^i$ $n \ll 0$
$F^n A^i = 0$ $n \gg 0$

then for $r \gg 0$ $F^{p+r} A^{p+q+1} = 0$ hence

$$Z_r^{pq} = Z_\infty^{pq}$$

and $F^{p-r} A^{p+q+1} = A^{p+q+1}$ hence

$$B_r^{pq} = B_\infty^{pq}$$

In general $B_0^{p,q} \subseteq B_1^{pq} \subseteq \ldots \subseteq B_\infty^{pq} \subseteq Z_\infty^{pq} \subseteq \ldots \subseteq Z_1^{pq} \subseteq Z_0^{pq}$

define $E_r^{p,q} = \dfrac{Z_r^{pq}}{Z_{r-1}^{p+1\, q+1} + B_{r-1}^{p,q}}$

then e.g. $d(Z_r^{p,q}) \subseteq d(F^p A^{p+q}) \cap F^{p+r} A^{p+q+1} =$

$$= F^{p+r} A^{p+q+1} \cap d(F^p A^{p+q}) =$$

$$B_r^{p+r, q-r+1} \subseteq Z_r^{p+r, q-r+1}$$

and so on with hand calculations. $\square$