Coco seminar, 9 april

The Brown representability theorem | Alexander Tonkelaar

<u>Def</u> $(X, x_0), (Y, y_0) \in \mathrm{Top}_\bullet$, $f, g : X \to Y$. pointed.

A <u>homotopy</u> from f to g is a map $H : X \times I \to Y$ such that

$$\begin{tikzcd}
X \arrow[d, hook, "i_0"'] \arrow[dr, "f"] & \\
X \times I \arrow[r, "H"] & Y \\
X \arrow[u, hook', "i_1"] \arrow[ur, "g"'] &
\end{tikzcd}$$

commutes. and $H(x_0, t) = y_0$ for all $t \in I$.

If it exists, f and g are <u>homotopic</u>. Notation: $f \simeq g$.

$$\begin{aligned} \mathrm{Ho}(\mathrm{Top}_\bullet) &= \mathrm{Top}_\bullet/\simeq \\ &= \mathrm{Top}_\bullet[\text{htpy-eq}^{-1}]. \end{aligned}$$

<u>Examples</u>

$\pi_1 = [S^1, -]_\bullet : \mathrm{Ho}(\mathrm{Top}_\bullet) \longrightarrow \mathrm{Grp}$

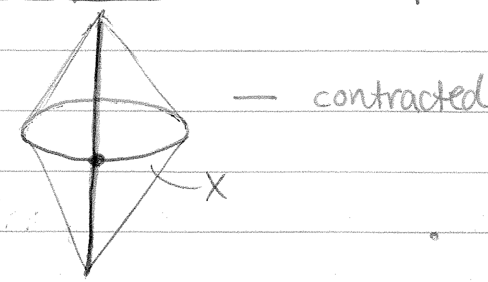
$\pi_0 = [S^0, -]_\bullet : \mathrm{Ho}(\mathrm{Top}_\bullet) \longrightarrow \mathrm{Set}_\bullet$

$\pi_n = [S^n, -]_\bullet : \mathrm{Ho}(\mathrm{Top}_\bullet) \longrightarrow \mathrm{Ab}$ for $n \geq 2$

This follows from the Eckmann-Hilton argument.

So, for $n \geq 1$, S^n is an H-cogroup (i.e. group in $\mathrm{Ho}(\mathrm{Top}_\bullet)^{op}$)
with comultiplication $S^n \longrightarrow S^n \vee S^n$
and counit $S^n \longrightarrow *$.

Example Reduced suspension $\Sigma : \mathrm{Top}_* \to \mathrm{Top}_*$

$\Sigma X =$ 

Loop space $\Omega : \mathrm{Top}_* \to \mathrm{Top}_*$

$$\Omega Y = \{ \gamma \in Y^I : \gamma(0) = \gamma(1) = y_0 \}.$$

This are adjoints $\begin{tikzcd} \mathrm{Top}_* \arrow[r, bend right, "\Omega"'] \arrow[r, phantom, "\perp"] & \mathrm{Top}_* \arrow[l, bend right, "\Sigma"'] \end{tikzcd}$. Also modulo homotopy, we get

$$[\Sigma X, Y]_* \cong [X, \Omega Y]_*$$

Def A CW-complex is a space X with

$$\emptyset = X^{(-1)} \subseteq X^{(0)} \subseteq X^{(1)} \subseteq \dots \subseteq X^{(n)} \subseteq \dots \subseteq X$$

and $\{\chi_\alpha : \partial e^n \to X^{(n-1)}\}_{\alpha \in \mathcal{J}_n}$ such that

$$\begin{tikzcd} \coprod_{\alpha \in \mathcal{J}_n} \partial e^n \arrow[r, "\chi"] \arrow[d, hook] & X^{(n-1)} \arrow[d, hook] \\ \coprod_{\alpha \in \mathcal{J}_n} e^n \arrow[r] & X^{(n)} \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

is a pushout diagram, and $X = \operatorname{colim}_n X^{(n)}$.

A map $f: X \to Y$ of CW-complexes is <u>cellular</u> if $f(X^{(n)}) \subseteq Y^{(n)}$ for all n.

<u>Def</u> A <u>pointed CW-complex</u> is a CW-complex equipped with a 0-cell.

<u>Example</u>
- $\emptyset, *, S^0, S^1, \ldots, I$.
- if X is a CW-complex, then so is ΣX, $X \times I$, $\bigvee_\alpha X_\alpha$.

<u>Thm (cellular approximation)</u> $X, Y \in CW$, $A \subseteq X$ subcomplex. Let $f: X \to Y$ a map such that $f|_A$ is cellular. Then there is $f': X \to Y$ cellular such that $f \simeq f'$ relative to A.

<u>Example</u> $[S^k, X^{(n)}]_\circ \longrightarrow [S^k, X]_\circ$ is an iso for $k<n$, and epi for $k=n$. We say $i: X^{(n)} \to X$ is an <u>n-equivalence</u>.

<u>Thm (Whitehead)</u> $X, Y \in CW$, $f: X \to Y$ a weak homotopy eq (i.e. n-equivalence for all n). Then f is a homotopy equivalence. If f is a subcomplex inclusion, then f is the inclusion of a strong deformation retract. (That is, there is $r: Y \to X$ with $rf = \mathrm{id}_X$, $fr \simeq \mathrm{id}_Y$ rel X.)

<u>Thm (Brown)</u> $h: \mathrm{Ho}(CW_\circ^c)^{op} \longrightarrow Set$. (c = connected) Suppose
(1) h preserves products: if $X = \bigvee_\alpha X_\alpha$, $i_\alpha: X_\alpha \hookrightarrow X$ the inclusion, then $h(X) \xrightarrow{(i_\alpha)_\alpha} \prod_\alpha h(X_\alpha)$ is a bijection.
(2; Mayer-Vietoris) $X \in CW_\circ^c$, $A, B \subseteq X$ connected pointed subcomplexes with $X = A \cup B$, $\alpha \in h(A)$, $\beta \in h(B)$ such that $\alpha|_{A \cap B} = \beta|_{A \cap B}$. Then there is a unique $\xi \in h(X)$ such that $\xi|_A = \alpha$, $\xi|_B = \beta$.
("h maps homotopy pushouts to fiber products.")

Then h is representable.

<u>Application</u>: Representability of cohomology functors.

For $X \in CW_\circ$ have short exact sequence

$$0 \to \tilde{C}_\circ(X) \to C_\circ(X) \xrightarrow{\varepsilon} \mathbb{Z}[0] \to 0$$

that is naturally split; so homology is reduced homology $\oplus \mathbb{Z}[0]$.
Apply $- \otimes G$:

$$0 \to \tilde{C}_\circ(X;G) \to C_\circ(X;G) \to G[0] \to 0$$

Apply $\mathrm{Hom}(-,G)$.

$$0 \leftarrow \tilde{C}^\circ(X;G) \leftarrow C^\circ(X;G) \leftarrow G[0] \leftarrow 0.$$

We have $H^n(X) \cong \tilde{H}^n(X_+)$ ← base point added.

By Brown's theorem, $\tilde{H}^n(-;G)$ is representable by some E_n'.
For arbitrary $X \in CW_\circ$, get

$$\begin{aligned} \tilde{H}^n(X;G) &\cong \tilde{H}^{n+1}(\Sigma X;G) && \text{"suspension isomorphism"} \\ &\cong [\Sigma X, E_{n+1}']_\circ \\ &\cong [X, \Omega E_{n+1}']_\circ \end{aligned}$$

and for $X \in CW$,

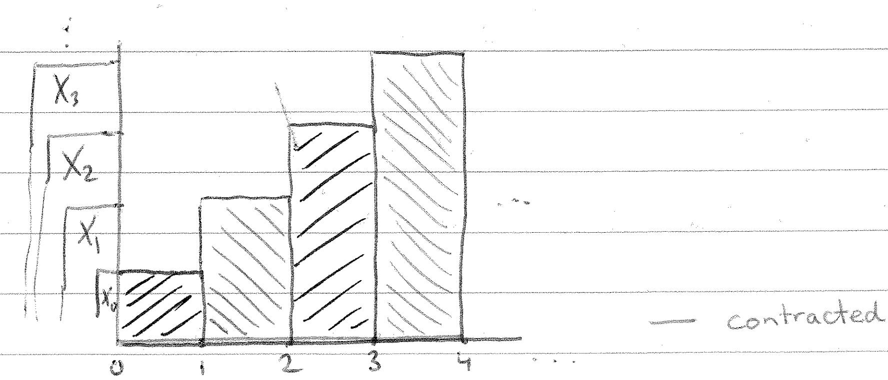
$$\begin{aligned} H^n(X;G) &\cong \tilde{H}^n(X_+;G) \\ &\cong [X_+, \Omega E_{n+1}']_\circ \\ &\cong [X, \underbrace{\Omega E_{n+1}'}_{=:E_n}]. \end{aligned}$$

Fix a functor $h: \mathrm{Ho}(CW_\circ^c)^{op} \longrightarrow \mathrm{Set}$ as in Brown's theorem.

Def A tuple (Y,u) with $Y \in CW_\circ^c$, $u \in h(Y)$ is <u>n-universal</u> if
$$\begin{aligned} [S^k, Y]_\circ &\longrightarrow h(S^k) \\ f &\longmapsto f^* u \end{aligned}$$
is an iso for $1 \leq k < n$, and epi for $k = n$.
A tuple (Y,u) is <u>∞-universal</u> if it is n-universal for all n.

Lemma $X \in CW_\circ^c$, and $\{(X_n, u_n)\}_{n\in\mathbb{N}}$ a sequence of pointed conn. subcomplexes, with $X_0 \subseteq X_1 \subseteq \dots$, $\bigcup_n X_n = X$, $u_n \in h(X_n)$ such that $u_{n+1}|_{X_n} = u_n$. Then there is $u \in h(X)$ such that $u|_{X_n} = u_n$.
(i.e. $h(\operatorname{colim}_n X_n) \twoheadrightarrow \lim_n h(X_n)$.)

Proof In the smash product $[0,\infty]_+ \wedge X$ take
$$T = \bigcup_n X_n \times [n, n+1] \simeq X.$$



$$A = \bigcup_{n \text{ even}} X_n \times [n, n+1] \simeq \bigvee_{n \text{ even}} X_n$$
$$B = \bigcup_{n \text{ odd}} X_n \times [n, n+1] \simeq \bigvee_{n \text{ odd}} X_n$$

$$A \cap B \simeq \bigvee_n X_n$$

Since h preserves products, we have a class on A, and on B and they coincide on $A \cap B$. By Mayer-Vietoris we get a class on

$T \simeq X$, and it is as desired. $\square$

<u>Prop</u> Let (Y, u) a tuple with $Y \in CW_0^c$, $u \in h(Y)$.
There is a tuple (Y', u') with $Y' \supseteq Y$, $u' \in h(Y')$, and $u'|_Y = u$, such that (Y', u') is ∞-universal.

<u>Proof</u> Suppose (Y, u) is n-universal. First extend

$$Y \hookrightarrow \tilde{Y} := Y \vee \bigvee_{\alpha \in h(S^{n+1})} S_\alpha^{n+1}.$$

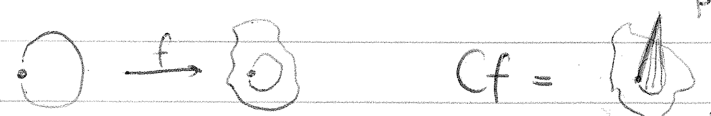
On $\tilde{Y}$ have a natural class $\tilde{u}$, and $[S^{n+1}, \tilde{Y}]_0 \to h(S^{n+1})$, $f \mapsto f^* \tilde{u}$ is epi by construction. Take

$$\{ f_\beta : S^n \longrightarrow \tilde{Y} \}_{\beta \in I}$$

that generate the kernel of the group hom

$$[S^n, \tilde{Y}] \longrightarrow h(S^n), \quad f \mapsto f^* \tilde{u}.$$

Then let Y' be the cofiber of $f : \bigvee_{\beta \in I} S^n \to \tilde{Y}$:



There is a unique class u' on $Cf =: Y'$ that restricts to $\tilde{u}$ on $\tilde{Y}$.
By the cellular approximation theorem, (Y', u') is $(n+1)$-universal.
Now apply the Lemma. $\square$

<u>Lemma</u> (Y, u) ∞-universal, (X, A) a CW_0-pair, $x \in h(X)$, and $f : A \to Y$ such that $f^* u = x|_A$. There there is a map $g : X \to Y$ extending f such that $g^* u = x$.

Proof Wlog $A \xrightarrow{f} Y$ is the inclusion of a subcomplex. Take the pushout

$$\begin{tikzcd} A \arrow[r, hook, "f"] \arrow[d, hook] & Y \arrow[d] \\ X \arrow[r] & X \vee_A Y \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

By Mayer-Vietoris, we get a class $\tilde{u}$ on $X \vee_A Y$, and $(X \vee_A Y, \tilde{u})$ extends to an ∞-universal pair (Y', u'). Let i be the induced map $Y \to Y'$; it is a subcomplex inclusion. We have

$$\begin{tikzcd} {[S^k, Y]_0} \arrow[rr, "i_*"] \arrow[dr, "\cong"'] & & {[S^k, Y']_0} \arrow[dl, "\cong"] \\ & h(S^k) & \end{tikzcd}$$

commutative, so i_* is a bijection. By Whitehead's theorem, i is the inclusion of a strong deformation retract, there is $r: Y' \to Y$ homotopy-inverse to i, and

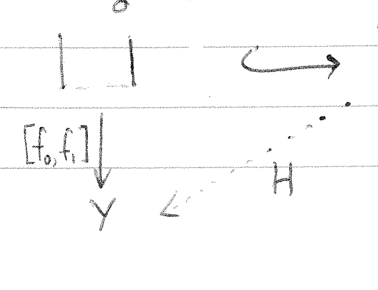
$$g: X \to X \vee_A Y \to Y' \xrightarrow{r} Y$$

is as desired. $\square$

Proof of Brown Let (Y, u) ∞-universal, and consider

$$[X, Y]_0 \longrightarrow h(X), \quad f \longmapsto f^* u.$$

surjective: Apply Lemma to (X, x_0).
injective: Apply lemma to $(X \wedge I_+, X \wedge \partial I_+)$. If $f_0, f_1 : X \to Y$ satisfy $f_0^* u = f_1^* u$, then lemma gives



and H is then a homotopy from f_0 to f_1. $\square$