J. Fluid Mech. (1971), *vol.* 48, *part* 3, *pp.* 529–545
Printed in Great Britain

A non-linear instability theory for a wave system in plane Poiseuille flow

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(Received 22 October 1970)

The initial-value problem for linearized perturbations is discussed, and the asymptotic solution for large time is given. For values of the Reynolds number slightly greater than the critical value, above which perturbations may grow, the asymptotic solution is used as a guide in the choice of appropriate length and time scales for slow variations in the amplitude A of a non-linear two-dimensional perturbation wave. It is found that suitable time and space variables are ϵt and $\epsilon^{\frac{1}{2}}(x+a_{1r}t)$, where t is the time, x the distance in the direction of flow, ϵ the growth rate of linearized theory and $(-a_{1r})$ the group velocity. By the method of multiple scales, A is found to satisfy a non-linear parabolic differential equation, a generalization of the time-dependent equation of earlier work. Initial conditions are given by the asymptotic solution of linearized theory.

1. Introduction

We consider an incompressible viscous fluid in steady motion at a Reynolds number R close to the critical value R_c, above which small velocity perturbations, or vorticity waves, may be amplified. If their amplitude becomes too large, a non-linear theory is required in order to follow the evolution of such perturbations and, ten years ago, Stuart (1960) suggested that the development could be treated by means of an expansion in powers of $(R-R_c)$ or of some parameter close to that Reynolds number difference. After some analysis it was deduced, amongst other results, that the time-dependent amplitude (A_1) of the leading-Fourier mode of the expansion satisfied the non-linear ordinary differential equation.

$$\frac{d}{dt}|A_1|^2 = k_1|A_1|^2+k_2|A_1|^4, \tag{1.1}$$

an equation whose validity for such systems was originally conjectured by Landau (1944). From this equation a number of inferences were drawn about the behaviour of nearly monochromatic waves in parallel shear flows.

Specifically Stuart's paper was concerned with plane Poiseuille flows in an infinite channel, and subsequently the ideas were extended and assessed mathematically by Watson (1960, 1962), Eckhaus (1965), Pekeris & Shkoller