THE COMPLEX GINZBURG-LANDAU EQUATION*

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1 The formal derivation of the Ginzburg-Landau equation

1.1 Introduction

The bifurcation that is described by the Ginzburg-Landau equation can appear in a very large class of models. In this Chapter we will present the formal derivation process by which the Ginzburg-Landau equation can be derived. We will do this by focusing on various explicit model problems. These model problems are all within the class described by the following general system:

$$\mathcal{S}_\mu \frac{\partial U}{\partial t} = \mathcal{L}_\mu U + \mathcal{N}_\mu(U). \tag{1.1}$$

Essential to the derivation of the Ginzburg-Landau equation is assumption that the spatial variables of the vector field $U(x, y, t)$ are defined on a *cylindrical domain*. This means that $(x, y) \in \mathbf{R}^m \times \Omega$, where $\Omega \subset \mathbf{R}^n$ is a open and bounded domain (and $m \geq 1$, $n \geq 0$), so that $U : \mathbf{R}^m \times \Omega \times \mathbf{R}^+ \to \mathbf{R}^N$. The $N \times N$ constant coefficient matrix $\mathcal{S}_\mu$ is assumed to be non-negative, in the sense that all its eigenvalues are real and ≥ 0. Note that this implies that $\mathcal{S}_\mu$ is not necessarily invertible. In most situations, $\mathcal{S}_\mu$ will be trivial, i.e. the identity matrix. However, for instance when (1.1) describes a model in fluid dynamics, $\mathcal{S}_\mu$ can have a zero eigenvalue. This eigenvalue is then associated to the equation that governs the conservation of mass, which does not have an explicit time derivative in it. The differential operators $\mathcal{L}_\mu$ and $\mathcal{N}_\mu$ are of course essential to the dynamics of (1.1). The linear operator $\mathcal{L}_\mu$ is assumed to be elliptic; $\mathcal{N}_\mu$ is a nonlinear operator of order less than $\mathcal{L}_R$. Neither $\mathcal{L}_\mu$ nor $\mathcal{N}_\mu$ is allowed to depend explicitly on x or t. However, both $\mathcal{L}_\mu = \mathcal{L}_\mu(y)$ and $\mathcal{N}_\mu = \mathcal{N}_\mu(y)$, i.e both operators may depend directly on the bounded variable $y \in \Omega$. At this point we do not need to impose any additional constraints on $\mathcal{L}_\mu$ or $\mathcal{N}_\mu$.

In general there will be a family of (bifurcation) parameters in a problem like (1.1). However, since the Ginzburg-Landau bifurcation is a co-dimension 1 bifurcation, we assume the bifurcation parameter μ is one-dimensional, i.e. $\mu \in \mathbf{R}$. Note that this implies that one should identify μ with one of the (sometimes many) parameters of a given 'practical' problem (see sections ??). Of course one needs to add explicit boundary (and initial) conditions to (1.1). For the moment, we do not pay attention to these.

We will first develop the derivation process by a number of explicit model problems, and only come back

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to the general situation described by (1.1) after that. Our first example is the so-called Swift-Hohenberg equation:

$$\frac{\partial U}{\partial t} + (1 + \frac{\partial^2}{\partial x^2})^2 U = \mu U - U^3 \tag{1.2}$$

Here, $\Omega = \{0\}$ and $m = 1$, so that the 'cylindrical domain' is just $\mathbf{R}$. Note that also $N = 1$ in this example. The linear operator $\mathcal{L}_\mu = -(1 + \frac{\partial^2}{\partial x^2})^2 + \mu$ has $-\frac{\partial^4}{\partial x^4}$ as leading, fourth order, component and is thus clearly elliptic. The nonlinear operator has no quadratic components, we will see that this simplifies the analysis considerably.

Among the other model problems we consider in this section are explicit systems of two-component reaction-diffusion equations, i.e. $N = 2$ in (1.1) and $\mathcal{S}_\mu = \mathrm{Id}$. In this context, the Ginzburg-Landau equation can be both associated to a Turing bifurcation and to a Hopf bifurcation, as we shall show in section ?.?. The final example will be that of a model for convection between to (unbounded) plates. Thus, x is associated to the coordinates along the plates, while the bounded variable y corresponds to the direction perpendicular to the plates.

A next essential ingredient of the derivation process is the assumption that (1.1) has a 'trivial' *laminar* solution,

$$U(x, y, t) = U_0(y). \tag{1.3}$$

Thus, the laminar solution U_0 is supposed to be only a function of the bounded variable $y \in \Omega$. Note that $U_0 \equiv 0$ in the case of the Swift-Hohenberg model problem (1.2). The Ginzburg-Landau equation describes the nonlinear evolution of perturbations of such a laminar solution 'at near-critical conditions', i.e. when the laminar solution is 'a little' unstable. We will measure the 'near criticality' and the 'smallness' of the perturbations by an asymptotically small parameter ε.

Hence, before one can derive a Ginzburg-Landau equation it is necessary to determine the linear, or spectral, stability of the laminar solution. By the structure of the cylindrical domain $\mathbf{R}^m \times \Omega$ we can study the linear stability of $U_0(y)$ by setting

$$U(x, y, t) = U_0(y) + f(y)e^{i(k,x)+\omega t}, \tag{1.4}$$

where, $f(y) : \Omega \to \mathbf{C}^N$ is a yet unknown vector valued, a priori complex, function. Since neither (1.1), nor U_0 depends explicitly on the unbounded variable x, the x-dependence of the perturbation of U_0 can be represented by the *wave number* $k \in \mathbf{R}^m$ in the linear stability analysis, as can be seen by taking the Fourier transform with respect to x. The evolution in time of $U - U_0$ is governed by $\omega \in \mathbf{C}$. The behavior of the *dispersion relation* $\omega = \omega(k, \mu)$ as function of the bifurcation parameter μ in (1.1) and the wave number k of $U - U_0$ (1.4) is another crucial ingredient of the Ginzburg-Landau bifurcation. Both the vector field $f(y)$ and the dispersion relation ω are determined by the eigenvalue problem that is obtained by substitution of the expansion (1.4) into (1.1) and neglecting all terms that are nonlinear in $f(y)$.

The eigenvalue problem will in general be an elliptic partial differential equation on the domain Ω for the complex function $f(y)$. Note that this equation will reduce to an ordinary differential equation when $n = 1$, i.e. when Ω is an interval, as is the case in the convection problem discussed in section ??. Furthermore, in many applications, and in the first model problems considered in this Chapter, $\Omega = \{0\}$, so that U does not depend on a bounded variable y. As a consequence, the eigenvalue problem is just that of a constant coefficients $N \times N$-matrix and f will just be a vector in $\mathbf{C}^N$. Such a matrix will have N eigenvalues: $\omega_j(k, \mu) \in \mathbf{C}$, $j = 1, ..., N$. In the general case $\Omega \neq \{0\}$ there will (in general) be countably many eigenvalues $\omega_j(k, \mu) \in \mathbf{C}$ with associated eigenfunctions $f_j(y) : \Omega \to \mathbf{C}^N$, $j = 1, 2, ...$ Equation (1.1) is real valued, thus we note by (1.4) that the eigenvalue problem will be symmetric with respect to

$$k \to -k, \ \ \omega \to \bar{\omega}, \tag{1.5}$$

where $\bar{\omega}$ is the complex conjungate of ω.

The laminar solution $U_0(y)$ is (linearly) asymptotically stable against a perturbation with wave number k when $\|U(x, y, t) - U_0(y)\|$ decays in time. By (1.4), this is clearly the case when $\mathrm{Re}(\omega_j(k, \mu)) < 0$ for all possible j. Thus, the laminar solution is linearly asymptotically stable when $\mathrm{Re}(\omega_j(k, \mu)) < 0$ for all $k \in \mathbf{R}^m$

and all $j = 1, 2, ...$

The critical, or bifurcation, value μ_c of μ is defined by the condition that $U_0(y)$ is asymptotically stable for $\mu < \mu_c$, but not any longer for $\mu > \mu_c$. This implies that there must be a special value of k, denoted by k_c, and a special value of j, that can be relabeled to be $j = 1$, such that

$$\omega_1(\pm k_c, \mu_c) = \pm i\omega_c, \text{ i.e. } \mathrm{Re}(\omega_1(\pm k_c, \mu_c)) = 0 \tag{1.6}$$

(see (1.5)), while $\mathrm{Re}(\omega_j(k, \mu_c)) < 0$ for $j \geq 2$ and $k \in \mathbf{R}^m$ and $\mathrm{Re}(\omega_1(k, \mu_c)) < 0$ for $k \neq \pm k_c$. Note that this implies that we assume that the manifold $\mathrm{Re}(\omega_j(k, \mu_c))$ has a non-degenerate maximum at $k = k_c$ (and at $k = -k_c$). Hence, at $\mu = \mu_c$ there is a *linear 'wave'* that does not decay exponentially. By the symmetry (1.5), the same is true for the complex conjungate of the wave. This linear wave is called *'the linearly most unstable wave'*, and it is given by

$$f_c(y)E(x,t) \stackrel{\text{def}}{=} f_c(y)e^{i[(k_c,x)+\omega_c t]}, \tag{1.7}$$

where $f_c(y)$ is the (in general complex) eigenfunction associated to the eigenvalue $i\omega_c = \omega_1(k_c, \mu_c)$.

The assumption that the nonlinear evolution of perturbations of the laminar solution $U_0(y)$ is dominated by this 'most unstable wave' for μ close to μ_c can be seen as the fundamental *'Ansatz'* of the weakly nonlinear stability theory . The 'closeness' of μ to μ_c is captured by the introduction of the asymptotically small parameter ε, $0 < \varepsilon \ll 1$, and r in

$$\mu = \mu_c + r\varepsilon^2, \tag{1.8}$$

so that $\mu - \mu_c = r\varepsilon^2 \ll 1$. A more precise formulation of the above 'Ansatz' reads as follows. For μ as in (1.8) we assume that

$$U(x,y,t) = U_0(y) + \varepsilon \left[A(\xi,\tau)f_c(y)e^{i[(k_c,x)+\omega_c t]} + \bar{A}(\xi,\tau)\bar{f}_c(y)e^{-i[(k_c,x)+\omega_c t]} \right] + \text{ h.o.t.}, \tag{1.9}$$

where h.o.t. means 'higher order terms'. These higher order terms will be both in ε as well as in E (1.7), i.e. (1.9) gives the leading order terms of double expansion of $U - U_0$ in a asymptotic series in ε and in a Fourier series in E. The *amplitude* $A(\xi, \tau) : \mathbf{R}^m \times \mathbf{R}^+ \to \mathbf{C}$ is a function of a *long (in general traveling) spatial variable* ξ associated to x and t and a *slow temporal variable* τ associated to t. Here, 'long' and 'slow' will be determined in terms of the small parameter ε. Note that this implies that A does not depend on the bounded variable y.

The complex amplitude A describes a *modulation* of the linearly most unstable wave $f_c(y)E(x,t)$ (1.7). If this Ansatz is correct and if we can derive an equation that governs the behavior of A, we have obtained a leading order description of the behavior of $U(x,y,t) - U_0(y)$, i.e. the behavior of perturbations of $U_0(y)$. The Ginzburg-Landau equation is this *modulation*, or *amplitude equation*.

Of course there are many question to be answered. We just list here a few of them. Why should $U - U_0$ have this structure? Why should $|U - U_0| = \mathcal{O}(\varepsilon) = \mathcal{O}(\sqrt{\mu - \mu_c})$? What is the exact definition of ξ and τ? Why does A not depend on (a scaled version of) y?

In this Chapter we will present *formal* answers to these questions by developing the derivation process for the Ginzburg-Landau equation. We will do this first in the context of a number of explicit (and relevant) model problems. We return to the general setting of system (1.1) in section ?.?.

1.2 The Swift-Hohenberg model

The Swift-Hohenberg model (1.2) has originally been derived in the context of a convection problem [7]. It has served as an important, and probably 'the most simple', model in the development of the Ginzburg-Landau theory of modulation equations.

The stability of the 'laminar solution' $U_0 \equiv 0$ is determined by the symbol the linear operator, the dispersion relation

$$\omega(k,\mu) = \mu - (1-k^2)^2 \in \mathbf{R}, \tag{1.10}$$

that can be obtained by inserting (1.4) into (1.2) and neglecting the nonlinear term. Note that the eigenvalue problem associated to this stability analysis reduces to a scalar algebraic equation, since $N = 1$ (and $\Omega = \{0\}$). In Figure ?.? we show the graph of ω as function of k. It is clear that U_0 is asymptotically stable for $\mu < 0$ and that $k_c = 1$.

Following (1.7) we introduce $E = E(x) = e^{ix}$. Note that E does not vary with t, since $\omega(k,\mu) \in \mathbf{R}$ (1.10). This wave is not the only one that is linearly unstable when $\mu = r\varepsilon^2$ (1.8), see Figure ?.?. There are two intervals centered around $k = \pm 1$ on which $\omega(k, r\varepsilon^2) > 0$: $I_+ \cup I_- = (-k_+(\varepsilon), -k_-(\varepsilon)) \cup (k_-(\varepsilon), k_+(\varepsilon))$ with $k_\pm(\varepsilon) = \pm\frac{1}{2}\sqrt{r}\varepsilon + \mathcal{O}(\varepsilon^2)$. By the linear theory, a wave $e^{ikx+\omega(k,r\varepsilon^2)t}$ associated to a $k \in I_+$ (or $\in I_-$) will grow exponentially. We introduce K by $k = 1 + \varepsilon K$, so that

$$k \in I_+ \leftrightarrow K \in (-\frac{1}{2}\sqrt{r} + \mathcal{O}(\varepsilon), \frac{1}{2}\sqrt{r} + \mathcal{O}(\varepsilon)) \tag{1.11}$$

We can now write the family of linearly unstable, spatially periodic, perturbations of $U_0 = 0$ (1.4) as

$$e^{i(1+\varepsilon Kx)+\varepsilon^2(r-4K^2+\mathcal{O}(\varepsilon))t} = e^{iK(\varepsilon x)+(r-4K^2+\mathcal{O}(\varepsilon))(\varepsilon^2 t)} e^{ix} = A_{\mathrm{lin}}(\varepsilon x, \varepsilon^2 t)E(x). \tag{1.12}$$

Here, we have chosen $f_c(y) = 1$ in (1.9), which is, of course, possible in a case as this with $\Omega = \{0\}$ and $N = 1$. Equation (1.12) shows that the linear theory 'predicts' the existence of complex modulations A_{lin} of the linearly most unstable wave $E(x)$ that vary on a long spatial scale ξ and a slow temporal scale τ with

$$\xi = \varepsilon x, \quad \tau = \varepsilon^2 t. \tag{1.13}$$

Hence, it is natural to assume that the nonlinear modulation $A(\xi,\tau)$ in (1.9) varies on the same spatial and temporal scale. It is not possible to use 'linear arguments' to motivate the choice of the $\mathcal{O}(\varepsilon)$ magnitude of the perturbation of U in (1.9) for $\mu = r\varepsilon^2$, since the factor ε in front of the $A(\xi,\tau)$ in (1.9) is determined by a nonlinear mechanism. Let us for the moment assume that the perturbation in the 'Ansatz' (1.9) is of $\mathcal{O}(\varepsilon^\nu)$ for some, a priori unknown, exponent ν, and consider the action of the $\frac{\partial}{\partial t}$ operator in (1.2) on the expression in (1.9):

$$\frac{\partial U}{\partial t} = \varepsilon^{2+\nu}\frac{\partial A}{\partial \tau}E + \text{ c.c } + \text{ h.o.t.}, \tag{1.14}$$

where 'c.c.' means 'complex conjungate'. Substitution of (1.9) into the nonlinear term U^3 in (1.2) yields

$$U^3 = \varepsilon^{3\nu}[A^3E^3 + 3A|A|^2E] + \text{ c.c. } + \text{ h.o.t.} \tag{1.15}$$

We have already seen that the linear theory predicts an exponential, and thus unbounded, growth on the τ time-scale (1.12). If we want to derive an equation for $A(\xi,\tau)$ that has bounded solutions, we need another effect that is strong enough to 'balance' this linear growth. The nonlinear term U^3 in (1.2) might be able to produce that desired effect, as soon as the magnitude of U^3 is comparable to that of $\frac{\partial U}{\partial t}$, i.e., by (1.14) and (1.15), bounded behavior of $A(\xi,\tau)$ is possible when $\nu = 1$. This argument motivates the $\mathcal{O}(\varepsilon)$ magnitude of $U - U_0$ in (1.9).

Remark that the size of $U - U_0$ is also proportional to $\sqrt{\mu - \mu_c} = \mathcal{O}(\varepsilon)$ in a (non-degenerate) Hopf bifurcation in an ordinary differential equation (see, for instance, [4]). This is not a coincidence, the Ginzburg-Landau equation can be seen as a generalization to partial differential equations on unbounded domains of the (leading order of the) normal form associated to a Hopf bifurcation in an ordinary differential equation. See sections ???.

So far, we did not yet consider the action of the operators $\frac{\partial^2}{\partial x^2}$ and $\frac{\partial^4}{\partial x^4}$ of (1.2) on the expansion (1.9):

$$\frac{\partial^2 U}{\partial x^2} = E[-\varepsilon A + 2i\varepsilon^2\frac{\partial A}{\partial \xi} + \varepsilon^3\frac{\partial^2 A}{\partial \xi^2}] + \text{ c.c. } + \text{ h.o.t.}, \tag{1.16}$$

and

$$\frac{\partial^4 U}{\partial x^4} = E[\varepsilon A - 4i\varepsilon^2 \frac{\partial A}{\partial \xi} - 6\varepsilon^3 \frac{\partial^2 A}{\partial \xi^2}] + \text{ c.c. } + \text{ h.o.t.}, \tag{1.17}$$

where we have used that $\frac{\partial E}{\partial x} = iE$. Thus, we note that the action of the operators $\frac{\partial}{\partial t}$, $\mathcal{L}_\mu$ and $\mathcal{N}_\mu$ of the Swift-Hohenberg model (1.2), see (1.1), generates higher order 'modes' of the structure $\varepsilon^2 E$ (1.16), (1.17), $\varepsilon^3 E$ (1.14), (1.16), (1.17) and $\varepsilon^3 E^3$ (1.15), with their complex conjungates. This motivates the following extension of Ansatz (1.9):

$$\begin{array}{rcccccccc} U(x,t) = E[\varepsilon A & + & \varepsilon^2 X_{12} & + & \varepsilon^3 X_{13} & + & \mathcal{O}(\varepsilon^4)] & + & \text{c.c.} \\ & & & + & E^3[\varepsilon^3 X_{33} & + & \mathcal{O}(\varepsilon^4)] & + & \text{c.c.} \\ & & & & & + & \text{h.o.t.} & & \end{array} \tag{1.18}$$

where X_{ij} are, a priori, unknown (complex) functions of the scaled variables ξ and τ, like the amplitude A. We can now proceed by substituting this expanded combined asymptotic/Fourier decomposition of $U(x,t)$ into the equation (1.2). By construction, the outcome can be written as an equation for a combined asymptotic/Fourier expansion of the form (1.18). We can now try to 'solve' the Swift-Hohenberg equation by considering each term of the expansion separately.

We decompose the Swift-Hohenberg equation (1.2) into 6 parts, $\frac{\partial U}{\partial t}$, U, $2\frac{\partial^2 U}{\partial x^2}$, $\frac{\partial^4 U}{\partial x^4}$, $\mu U = r\varepsilon^2 U$ and $-U^3$. Only the terms U, $2\frac{\partial^2 U}{\partial x^2}$ and $\frac{\partial^4 U}{\partial x^4}$ contribute to the 'level' εE of the asymptotic/Fourier expansion (1.16), (1.17). At this level, the equation reduces to

$$\varepsilon E[A - 2A + A] = 0, \tag{1.19}$$

which, of course, holds for any A. Although this seems to be a trivial observation, it is essential to the derivation process: it once more confirms the validity of the Ansatz (1.9), (1.18). By construction, the equation at the εE level reproduces the linear analysis at $\mu = \mu_c = 0$. Hence, by (1.19) we recover the fact that we can 'modulate' the 'linearly most unstable wave' E with any complex amplitude A that varies on a slow spatial and a long temporal scale. A similar thing happens on the $\varepsilon^2 E$ level. Here, again, only U, $2\frac{\partial^2 U}{\partial x^2}$ and $\frac{\partial^4 U}{\partial x^4}$ contribute:

$$\varepsilon^2 E[X_{12} + (4i\frac{\partial A}{\partial \xi} - 2X_{12}) + (-4i\frac{\partial A}{\partial \xi} + X_{12})] = 0. \tag{1.20}$$

This equation is, again, satisfied for any A, but also for any function $X_{12}(\xi,\tau)$. Hence, we cannot determine a solution for X_{12} and are forced to introduce a second unknown amplitude A_2 by

$$A_2(\xi,\tau) = X_{12}(\xi,\tau). \tag{1.21}$$

Note that we could have defined A in a similar way, by first introducing an unknown function X_{11} at the εE level in (1.18) and concluding from (1.19) that $X_{11} = A_1 = A$. We will see that the second unknown amplitude A_2 has no influence on the derivation of the leading order, cubic, Ginzburg-Landau equation. However, A_2 certainly is important if one is interested in the higher order corrections to the cubic equation, see section ?.?.

All six terms are present at the $\varepsilon^3 E$ level:

$$\frac{\partial A}{\partial \tau} + X_{13} + (2\frac{\partial^2 A}{\partial \xi^2} + 4i\frac{\partial A_2}{\partial \xi} - 2X_{13}) + (-6\frac{\partial^2 A}{\partial \xi^2} - 4i\frac{\partial A_2}{\partial \xi} + X_{13}) = rA - 3A|A|^2. \tag{1.22}$$

The A_2 and X_{13} terms drop out of the equation, so that it reduces to

$$\frac{\partial A}{\partial \tau} = 4\frac{\partial^2 A}{\partial \xi^2} + rA - 3A|A|^2. \tag{1.23}$$

This is equation describes the evolution of $A(\xi,\tau)$, it is the Ginzburg-Landau equation associated to the Swift-Hohenberg model.

Before we discuss the implications of having derived a Ginzburg-Landau equation for the dynamics of (small) solutions of the Swift-Hohenberg equations (at near-critical conditions), we make some additional remarks about the derivation process.

First, we conclude that we have once again not found an equation for the unknown function X_{13}, so that we have to introduce a third unknown amplitude, $A_3(\xi,\tau) = X_{13}(\xi,\tau)$. A similar thing will happen at any $\varepsilon^j E$ level, since the 'operator' acting on X_{1j} is by construction identical to the linear operator acting on A at the εE level (1.19). Moreover, this is not special for the Swift-Hohenberg model, it is an intrinsic property of the derivation process. The operator acting on the equivalent of X_{1j} in the general problem (1.1) is also identical to the operator associated to the linear eigenvalue problem that is recovered at the εE level. This operator has, by construction, a non-trivial one-dimensional kernel, since $\mu = \mu_c$, $k = k_c$ and ω is fixed at the eigenvalue $i\omega_c = \omega(k_c, \mu_c)$. Thus, there will always appear an unknown amplitude $A_j(\xi,\tau) \in \mathbf{C}$ in the solution of the equation at the $\varepsilon^j E$ level. This amplitude represents the (one-dimensional) kernel of the operator. In the context of the Swift-Hohenberg problem, the operator cannot be more than just multiplication by a constant. Since the operator is not invertible, this constant must be 0, as we have seen in (1.19), (1.20), and (1.22). We will return to this issue in section ?.? in which we proceed with the derivation process beyond the $\varepsilon^3 E$ level in order to obtain the higher order corrections to the leading order cubic Ginzburg-Landau equation. Note, however, that if one is interested in the derivation of the cubic Ginzburg-Landau equation, there is no reason to go beyond the $\varepsilon^3 E$ level. It is not even necessary to solve the equation at this level, the Ginzburg-Landau equation appears from the application of a solvability condition at the inhomogeneous terms in the equation. This solvability condition is trivial in the context of a scalar model defined on $\mathbf{R}$, i.e. a model with $N = 1$ and $\Omega = \{0\}$, as is the case for (1.2). We will encounter the first non-trivial solvability condition in section ?.?.

Second, we note that the 'wave' E^3 is, as any other wave E^i with $i \neq 1$, not marginally stable at $\mu = \mu_c = 0$. This implies that the equation for X_{33} at the $\varepsilon^3 E^3$ level can be solved explicitly:

$$X_{33} - 18X_{33} + 81X_{33} = -A^3 : \quad X_{33} = -\frac{1}{64}A^3 \tag{1.24}$$

(note that $\frac{\partial E^3}{\partial x} = 3iE^3$). Thus, we observe that the behavior of $X_{33}(\xi,\tau)$, i.e. the $\varepsilon^3 E^3$ 'mode' of U (1.18) is completely determined by, or 'slaved to', that of $A(\xi,\tau)$. Once again, we do not need this information for the derivation of the Ginzburg-Landau equation (1.23), we have only solved the equation at the $\varepsilon^3 E^3$ level to give some additional insight in the derivation process.

From a formal point of view we have now obtained, by the derivation of the Ginzburg-Landau equation (1.23), a detailed insight in the leading order behavior of small, $\mathcal{O}(\varepsilon)$, solutions of the Swift-Hohenberg equation at near-critical conditions, i.e. with $\mu = r\varepsilon^2$. By (1.18) we know that the $\mathcal{O}(\varepsilon)$ approximation of U is governed by $A(\xi,\tau)$. Thus, we expect to be able to understand the dynamics of the Swift-Hohenberg equation (1.2) at $\mu = r\varepsilon^2$ by studying the dynamics of Ginzburg-Landau equation (1.23).

As an example, we consider the family of stationary spatially periodic solutions of the Ginzburg-Landau equation. We substitute $A(\xi,\tau) = Re^{iK\xi}$ into (1.23) and find

$$3R^2 + 4K^2 = r, \; K \in (-\frac{1}{2}\sqrt{r}, \frac{1}{2}\sqrt{r}). \tag{1.25}$$

By (1.18), (1.13), these solutions correspond to 'nonlinear' spatially periodic solutions of the Swift-Hohenberg model:

$$U(x) = A(\xi)E(x) + \text{ c.c. } + \text{ h.o.t. } = \varepsilon Re^{i\varepsilon Kx}e^{ix} + \text{ c.c. } + \text{ h.o.t. } = \varepsilon Re^{i(1+\varepsilon K)x} + \text{ c.c. } + \text{ h.o.t.}, \tag{1.26}$$

with K, R as in (1.25). By comparing (1.25) to (1.11) and (1.26) to (1.12) we see that there is a direct correspondence between this one parameter family of 'nonlinear' waves and the 'band' of linearly unstable, exponentially growing waves. Thus, the weakly nonlinear Ginzburg-Landau approach 'predicts' the existence of a family of periodic solutions of the Swift-Hohenberg model of which the existence was 'suggested' by the linear theory. Moreover, as we shall see in the forthcoming Chapters, only a subfamily of these the waves described by (1.25) is stable as a solution to Ginzburg-Landau equation (1.23). It is strongly suggested by the formal analysis, that there is, within the family described by (1.26), a corresponding subfamily of stable solutions of the Swift-Hohenberg equation. This (formal) result is in essence a (re)formulation of the so-called Eckhaus (in)stability criterion [1] for the Swift-Hohenberg problem. We will give a rigorous proof of this statement in Chapter ??.

1.3 Ginzburg-Landau dynamics in the Swift-Hohenberg model

This section contains some first remarks about the dynamics of the Ginzburg-Landau equation and the formal transfer of the dynamics to the Swift-Hohenberg model.

The simplest dynamics can be obtained in the subspace of real-valued solutions. There the Ginzburg-Landau equation reduces to

$$\frac{\partial A}{\partial \tau} = 4\frac{\partial^2 A}{\partial \xi^2} + rA - 3A^3.$$

Moreover, we set $r = 1$ throughout this section. We will consider three classes of solutions, namely space-independent solutions, i.e. $A = A(\tau)$, time-independent solutions, i.e. $A = A(\xi)$, and front solutions, i.e. $A = A(\xi - c\tau)$.

The consideration of space-independent solutions leads to the one-dimensional ordinary differential equation

$$\frac{dA}{d\tau} = A - 3A^3.$$

This ordinary differential possesses three equilibria, namely the unstable origin $A = 0$ and the stable fixed points $A = \pm 1/\sqrt{3}$. A solution $A = A(\tau)$ with initial condition $A(0) \in (0, 1/\sqrt{3})$ satisfies $\lim_{\tau\to\infty} A(\tau) = 1/\sqrt{3}$ with some exponential rate. Associated to this solution we find formally in the Swift-Hohenberg model

$$\begin{aligned} U(x,t) &= \epsilon A(\tau)E(x) + \text{c.c.} + \text{h.o.t.} = \epsilon A(\tau)e^{ix} + \epsilon A(\tau)e^{-ix} + \text{h.o.t.} \\ &= 2\epsilon A(\tau)\cos x + \text{h.o.t.} \to 2\epsilon \cos x/\sqrt{3} + \text{h.o.t.} \end{aligned}$$

for $\tau \to \infty$. Thus, we found formally 2π-spatially periodic solutions convergeing towards a spatially periodic equilibrium for $t \to \infty$. This can be made rigorous with the help of the center manifold theorem (cf. [3]) and it can be shown that in the subspace of 2π-spatially periodic solutions the family of all translates of this equilibrium is exponentially attracting.

Next we consider time-independent solutions $A = A(\xi)$ in the subspace of real-valued solutions of the Ginzburg-Landau equation. They can be found as solutions of the scalar second order ordinary differential equation

$$0 = 4\frac{\partial^2 A}{\partial \xi^2} + A - 3A^3.$$

In order to analyse this equation we consider the phase plane of the equivalent first order system

$$\begin{aligned} \frac{\partial A}{\partial \xi} &= B, \\ \frac{\partial B}{\partial \xi} &= -A/4 + 3A^3/4. \end{aligned}$$

We find the three equilibria from above, namely $(A,B) = (0,0)$ and $(A,B) = (\pm 1/\sqrt{3}, 0)$. The last equilibria are connected by two heteroclinic solutions $(A,B) = (A,B)_{\text{heter}}(\xi)$. In the Swift-Hohenberg model to the solution connecting $(A,B) = (-1/\sqrt{3}, 0)$ with $(A,B) = (1/\sqrt{3}, 0)$ we have formally an equilibrium solution

$$U(x) = 2\epsilon A_{\text{heter}}(\xi)\cos x + \text{h.o.t.}$$

with $U(x) \sim 2\epsilon \cos x/\sqrt{3}$ for $x \to \infty$ and $U(x) \sim -2\epsilon \cos x/\sqrt{3} = 2\epsilon \cos(x+\pi)/\sqrt{3}$ for $x \to -\infty$, i.e. these solutions connect the same spatial periodic equilibrium $2\epsilon \cos x/\sqrt{3}$, but with some phase shift of π. The transition from one equilibrium at $x \to -\infty$ to the other at $x \to \infty$, happens on a spatial scale of size $\mathcal{O}(1/\epsilon)$.

In the phase plane we also find nonlinear spatially periodic solutions $(A,B) = (A,B)_{\text{per}}(\xi)$ surrounding the origin $(A,B) = (0,0)$. For the Swift-Hohenberg model we find formally the equilibria

$$U(x) = 2\epsilon A_{\text{per}}(\xi)\cos x + \text{h.o.t.}.$$

The envelope $A = A_{\rm per}(\xi)$ is periodic with some period of order $\mathcal{O}(1/\epsilon)$ and modulates the underlying pattern $\cos x$ which has some period of order $\mathcal{O}(1)$. However, such solutions are in general not observed since they are unstable, both for the Ginzburg-Landau equation and the Swift-Hohenberg model.

Finally, we come to the front solutions $A = A(\zeta) = A(\xi - c\tau)$ of the Ginzburg-Landau equation. The real-valued front solutions satisfy

$$0 = 4\frac{\partial^2 A}{\partial \zeta^2} + c\frac{\partial A}{\partial \zeta} + A - 3A^3.$$

Again we consider the phase plane of the equivalent first order system

$$\begin{aligned} \frac{\partial A}{\partial \zeta} &= B, \\ \frac{\partial B}{\partial \zeta} &= -cB/4 - A/4 + 3A^3/4. \end{aligned}$$

For all values of $c \neq 0$ we find heteroclinic connections $(A, B) = (A, B)_{\rm front}(\zeta)$ between the fixed points $(A, B) = (\pm 1/\sqrt{3}, 0)$ and the origin $(A, B) = (0, 0)$. Such solutions describe the spreading of the stable phases $A = \pm 1/\sqrt{3}$ into the unstable phase $A = 0$. The functions $\zeta \mapsto A_{\rm front}(\zeta)$ are monotonic for $|c| \geq 1$. Moreover, it turned out that the non-monotic fronts are unstable [3]. For the Swift-Hohenberg model the associated solutions

$$U(x, t) = 2\epsilon A_{\rm per}(\xi - c\tau)\cos x + \text{h.o.t..}$$

are called modulating front. They are time-periodic in a frame comoving with the envelope A. The velocity is of order $\mathcal{O}(\epsilon)$ due to the scalings $x = \epsilon\xi$ and $t = \epsilon^2\tau$.

There are a number of interesting mathematical questions, as rigorous existence and stability, about the formally constructed solutions in the Swift-Hohenberg model. Note that the situation is already much more complicated in the Ginzburg-Landau equation itself since so far we restricted ourselves to real-valued solutions. We will comment on these questions in subsequent sections.

1.4 A Swift-Hohenberg-Korteweg-deVries equation

As a next example we consider a Swift-Hohenberg-Korteweg-deVries (SHKdV) equation, i.e. a Swift-Hohenberg equation (1.2) with an additional 'Korteweg-deVries like' term $U\frac{\partial U}{\partial x} + \frac{\partial^3 U}{\partial x^3}$:

$$\frac{\partial U}{\partial t} + (1 + \frac{\partial^2}{\partial x^2})^2 U + \alpha(U\frac{\partial U}{\partial x} + \frac{\partial^3 U}{\partial x^3}) = \mu U - U^3 \tag{1.27}$$

for $x \in \mathbf{R}$ and the additional parameter $\alpha \in \mathbf{R}$ that measures the relative magnitude of the new Korteweg-deVries term. We are not aware of any explicit context in which this equation appears as a relevant model. Here, we study the equation at 'near-critical' conditions in order to introduce two new aspects of the derivation process, compared to the Swift-Hohenberg model. The first new aspect is the fact that the eigenvalue, symbol, or dispersion relation, has become complex due to the presence of the $\frac{\partial^3 U}{\partial x^3}$ term:

$$\omega(k, \mu) = \mu - (1 - k^2)^2 + i\alpha k^3 \in \mathbf{C}. \tag{1.28}$$

Note, however, that this term has no influence on the real part of ω, so that we have, as in the Swift-Hohenberg case, that $\mu_c = 0$ and $k_c = 1$. However, $\omega(1, 0) = i\omega_c = i\alpha$, so that we have to define E by $E(x, t) = e^{i(x+\alpha t)}$ (1.7). Thus, by the presence of the $\frac{\partial^3}{\partial x^3}$ operator, E has become a nonstationary traveling wave. The fact that ω has become complex also has its impact on the definition of the long spatial scale ξ. To see this, we follow the arguments of the previous section. There are again two intervals of linearly unstable waves for (1.27) at $\mu = r\varepsilon^2$ centered around $k = \pm 1$, $I_+ \cup I_-$. These intervals are identical to those in the Swift-Hohenberg case. Thus I_+ is given by (1.11) with $k = 1 + \varepsilon K$, and I_- is the reflection of I_+ with respect to $k = 0$. We again decompose the linear unstable waves $e^{ikx+\omega t}$:

$$e^{i(1+\varepsilon Kx)+[\varepsilon^2(r-K^2+\mathcal{O}(\varepsilon))+i(\alpha+3\alpha\varepsilon K+\mathcal{O}(\varepsilon))]t} = e^{i(K+\mathcal{O}(\varepsilon))[\varepsilon(x+3\alpha t)]+(r-4K^2+\mathcal{O}(\varepsilon))[\varepsilon^2 t]}e^{i(x+\alpha t)}, \tag{1.29}$$

see (1.12). Hence, the modulation of the critical wave E is now a function of the traveling coordinate $\varepsilon(x+3\alpha t)$: $A_{\text{lin}} = A_{\text{lin}}(\varepsilon(x+3\alpha t), \varepsilon^2 t)$. As a consequence, we adapt our choice of the spatial coordinate ξ and keep τ as it was in the Swift-Hohenberg case:

$$\xi = \varepsilon(x+3\alpha t), \quad \tau = \varepsilon^2 t. \tag{1.30}$$

Note that the wave speed -3α is identical to the group speed $\frac{1}{i}\frac{d\omega}{dk}(k_c = 1, \mu_c = 0)$ of the package of unstable waves near $k_c = 1$ This gives another, equivalent, motivation of the new definition of ξ. We will find that the fact that ω has become complex (1.28) penetrates the derivation of the Ginzburg-Landau equation at many levels (compare for instance (1.14) to (1.33)).

The second new aspect of equation (1.27) compared to (1.2) is the appearance of the *quadratic* nonlinear term $\alpha U \frac{\partial U}{\partial x}$. A priori, one could think that we have to adapt the $\mathcal{O}(\varepsilon)$ magnitude of $U - U_0$ in Ansatz (1.9), since we motivated the choice of this magnitude by the necessity of having a 'balance' between the exponential growth and the nonlinear mechanisms. In the Swift-Hohenberg case, there was only a cubic nonlinearity, so that it was immediately clear that such a balance was only possible for perturbations of $\mathcal{O}(\varepsilon)$ magnitude, see (1.14) and (1.15). A quadratic nonlinearity yields nonlinear terms of $\mathcal{O}(\varepsilon^2)$, instead of the $\mathcal{O}(\varepsilon^3)$ terms of the cubic case (1.15). Hence, one is inclined to think that these terms are 'too strong' compared to the linear growth generated by $\frac{\partial U}{\partial t}$ that is represented by the $\varepsilon^3 \frac{\partial A}{\partial \tau} E$ term in (1.14). However, this is not the case, as can be seen by writing down the leading order terms of the decomposition of $U\frac{\partial U}{\partial x}$ induced by Ansatz (1.9)

$$U\frac{\partial U}{\partial x} = \varepsilon^2 [iA^2E^2] + \mathcal{O}(\varepsilon^3) + \text{ c.c.} \tag{1.31}$$

Thus, the leading order part of $U\frac{\partial U}{\partial x}$ cannot 'balance' a term involving $\frac{\partial A}{\partial \tau}E$ produced by $\frac{\partial U}{\partial t}$, since this leading order term is at an E^2 'level' of the asymptotic/Fourier expansion of $U\frac{\partial U}{\partial x}$. The $\frac{\partial A}{\partial \tau}$ term always appears at an E level in the decomposition of $\frac{\partial U}{\partial t}$ (see also (1.33)). Note that this is true for any quadratic nonlinearity, since these nonlinearities only produce leading order terms at the E^0 and the E^2 levels (it is, in a sense, a 'coincidence' that there is no $\varepsilon^2 E^0$ term in (1.31)). Hence, there can again only be a 'balance' between the nonlinear terms and the linear growth at $\mathcal{O}(\varepsilon^3)$. We may thus conclude that the $\mathcal{O}(\varepsilon)$ scaling of $U - U_0$ in (1.9) is also justified in the general case with quadratic and cubic nonlinearities.

Note, however, that one will have to adapt the scaling of $U - U_0$ in the basic Ansatz (1.9) in a (highly degenerate) case in which the leading order nonlinear corrections to the linearized problem are quartic or quintic (or of even higher order). Such systems cannot be described by a cubic Ginzburg-Landau equation. Nevertheless, the derivation procedure can also be applied in such cases and one will in general obtain a quintic Ginzburg-Landau equation, i.e. a Ginzburg-Landau equation in which the term $A|A|^2$ has been replaced by $A|A|^4$.

The structure of the decomposition of $U\frac{\partial U}{\partial x}$ (1.31) also implies that we need to introduce an extension of Ansatz (1.9) that differs from that of the Swift-Hohenberg problem (1.18):

$$\begin{array}{rcccccccc} & + & E^0[\varepsilon^2 X_{02} & + & \varepsilon^3 X_{03} & + & \mathcal{O}(\varepsilon^4)] & & \\ U(x,t) = E[\varepsilon A & + & \varepsilon^2 X_{12} & + & \varepsilon^3 X_{13} & + & \mathcal{O}(\varepsilon^4)] & + & \text{c.c.} \\ & + & E^2[\varepsilon^2 X_{22} & + & \varepsilon^3 X_{23} & + & \mathcal{O}(\varepsilon^4)] & + & \text{c.c.} \\ & & & + & E^3[\varepsilon^3 X_{33} & + & \mathcal{O}(\varepsilon^4)] & + & \text{c.c.} \\ & & & & & + & \text{h.o.t.} & & \end{array} \tag{1.32}$$

where, $X_{ij}(\xi,\tau) \in \mathbf{C}$, $i = 1, 2, ...$, and $X_{0j}(\xi,\tau) \in \mathbf{R}$ are to be determined by substitution of the expansion into (1.27). Note that the decomposition of $\frac{\partial U}{\partial t}$ differs significantly of that in the Swift-Hohenberg case (1.14), due to the fact that ω has become complex (1.28):

$$\begin{array}{rcccccccc} & + & E^0[\varepsilon^2(0) & + & \mathcal{O}(\varepsilon^3)] & & & & \\ \frac{\partial U}{\partial t} = E[i\varepsilon\alpha A & + & \varepsilon^2(3\alpha\frac{\partial A}{\partial \xi} + i\alpha X_{12}) & + & \varepsilon^3(\frac{\partial A}{\partial \tau} + 3\alpha\frac{\partial X_{12}}{\partial \xi} + i\alpha X_{13}) & + & \mathcal{O}(\varepsilon^4)] & + & \text{c.c.} \\ & + & E^2[2i\alpha\varepsilon^2 X_{22} & + & \mathcal{O}(\varepsilon^3)] & + & \text{c.c.} & & \\ & & & + & \text{h.o.t.} & & & & \end{array} \tag{1.33}$$

First, we check whether we recover trivial equations at the εE and $\varepsilon^2 E$ levels:

$$\begin{array}{llr} \varepsilon E: & i\alpha A + A - 2A + A - i\alpha A & = 0, \\ \varepsilon^2 E: & (3\alpha\frac{\partial A}{\partial \xi} + i\alpha X_{12}) + X_{12} + (4i\frac{\partial A}{\partial \xi} - 2X_{12}) + (-4i\frac{\partial A}{\partial \xi} + X_{12}) + (-3\alpha\frac{\partial A}{\partial \xi} - i\alpha X_{12}) & = 0. \end{array} \quad (1.34)$$

Note that the terms generated by $\frac{\partial U}{\partial t}$ (1.33) take care of the 'new' terms produced by $\alpha\frac{\partial^3 U}{\partial x^3}$ (compare (1.34) to (1.19) and (1.20)). This can be seen as an alternative, but equivalent, way to motivate our decision to choose a traveling coordinate $\xi = \varepsilon(x+ct)$, with speed $c = 3\alpha$ (1.30). Any other choice of c (including $c = 0$) results in the equation $\frac{\partial A}{\partial \xi} = 0$ at the $\varepsilon^2 E$ level. This equation implies that the evolution of U cannot be described by an amplitude A that evolves in a traveling frame associated to such as choice of ξ.

Since the equation at the $\varepsilon^2 E$ level does not yield a condition on X_{12} (which is, by construction, of course correct), we again have to introduce a second amplitude A_2 by $A_2(\xi,\tau) = X_{12}(\xi,\tau)$.

At the 'nonlinear' $\varepsilon^2 E^0$ level, we find $X_{02} = 0$. Hence we may conclude that it indeed has not been necessary to introduce X_{02} in this case, as the decomposition (1.31) of $U\frac{\partial U}{\partial \xi}$ already implied. Note, however that this term will be relevant in general (i.e. when there are quadratic terms in the underlying equation of a different type). The $\varepsilon^2 E^2$ level yields the following equation:

$$2i\alpha X_{22} + X_{22} - 8X_{22} + 16X_{22} + i\alpha A^2 - 8i\alpha X_{22} = 0, \quad (1.35)$$

so that

$$X_{22} = \frac{\alpha(2\alpha - 3i)}{3(9+4\alpha^2)}A^2 \quad (1.36)$$

Note that this implies that the behavior of the $\varepsilon^2 E^2$ mode in the decomposition of U (1.32) is completely determined by/'slaved to' the behavior of the amplitude A. This is a general phenomenon (see also (1.24) and is true for any $\varepsilon^i E^j$ mode with $j \neq 0$, as we shall see in the forthcoming sections. In order to write down the equation on the $\varepsilon^2 E^2$ level, we first need to expand the decomposition (1.31) of $U\frac{\partial U}{\partial x}$:

$$U\frac{\partial U}{\partial x} = \begin{array}{cccccccc} + & E^0[\varepsilon^2(0) & + & \mathcal{O}(\varepsilon^3)] & & & & \\ & & + & E[i\varepsilon^3(\bar{A}X_{22}) & + & \mathcal{O}(\varepsilon^4)] & + & \text{c.c.} \\ + & E^2[\varepsilon^2(iA^2) & + & \mathcal{O}(\varepsilon^3)] & + & \text{c.c.} & & \\ & & + & \text{h.o.t.} & & & & \end{array} \quad (1.37)$$

where X_{22} is known explicitly (1.36). By construction, the components containing A_2 and X_{13} cancel at the $\varepsilon^3 E$ level, so that we are left with:

$$\frac{\partial A}{\partial \tau} + 2\frac{\partial^2 A}{\partial \xi^2} - 6\frac{\partial^2 A}{\partial \xi^2} + i\alpha\bar{A}X_{22} + 3i\alpha\frac{\partial^2 A}{\partial \xi^2} = rA - 3A|A|^2. \quad (1.38)$$

By (1.36), we have thus found a Ginzburg-Landau equation with complex coefficients:

$$\frac{\partial A}{\partial \tau} = (4 - 3i\alpha)\frac{\partial^2 A}{\partial \xi^2} + rA - \left(\frac{27+13\alpha^2}{9+4\alpha^2} + \frac{2\alpha^3}{3(9+4\alpha^2)}i\right)A|A|^2 \quad (1.39)$$

We may conclude by this formal procedure that the leading order behavior of small solutions of the Swift-Hohenberg-Korteweg-deVries equation (1.27) is governed, at near-critical conditions, by this complex Ginzburg-Landau equation. We will see in the forthcoming Chapters that the dynamics generated by a complex Ginzburg-Landau equation differ significantly from the dynamics of a real Ginzburg-Landau equation as (1.23).

We note that the coefficients of the linear terms of this Ginzburg-Landau equation are directly related to derivatives of $\omega(k,\mu)$ (1.28) at $k = k_c = 1$ and $\mu = \mu_c = 0$:

$$4 - 3i\alpha = -\frac{1}{2}\frac{\partial^2 \omega}{\partial k^2}(1,0), \quad r = r\frac{\partial \omega}{\partial \mu}(1,0). \quad (1.40)$$

At this stage, this might look like a coincidence. However, we will see in the next sections that this feature is a consequence of the structure of the derivation process.

Of course, Ginzburg-Landau equation (1.39) reduces to (1.23) in the limit $\alpha \to 0$. It is more interesting to consider the limit $\alpha \to \infty$. By rescaling time in the SHKdV-model (1.27), we see that this limit corresponds to the Korteweg-deVries equation $\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + \frac{\partial^3 U}{\partial x^3} = 0$. A similar rescaling in (1.39) shows that the Ginzburg-Landau equation *degenerates* into the *nonlinear Schrödinger equation* in this limit:

$$i\frac{\partial A}{\partial \tilde{\tau}} = 3\frac{\partial^2 A}{\partial \xi^2} + \frac{1}{6}A|A|^2. \tag{1.41}$$

This limit is a singular limit in the sense that $\omega(k,\mu)$ becomes purely imaginary in this limit (by the rescaling in time, the real part of ω becomes $\mathcal{O}(\frac{1}{\alpha})$ in (1.28)). Hence, the foundation on which the derivation process is built, the existence of two small intervals of unstable 'waves', while all other waves are asymptotically stable, disappears in the limit $\alpha \to 0$: *all perturbations (1.4) become marginally stable.* The assumption that $\omega(k,\mu)$ has a non-degenerate maximum at $k = k_c$ for $\mu = \mu_c$ is directly related to the ellipticity of the second order spatial operator in the Ginzburg-Landau equation (see section ?.? for the details). Thus, this ellipticity disappears from (1.39) as $\alpha \to \infty$: the nonlinear Schrödinger equation is a hyperbolic equation.

1.5 Turing and Hopf bifurcation in an autocatalytic reaction-diffusion system

In this section we consider an example in which the underlying model problem (1.1) is no longer a scalar problem, as it was in the previous sections. We consider a system of two coupled reaction-diffusion equations, i.e. $N = 2$ in (1.1). This way, we can show how the Ginzburg-Landau equation appears from the derivation process by the application of a *solvability condition.* Moreover, we will also find that the structure of the derivation process is so 'robust' that it can be applied to different types of bifurcations (Turing and Hopf bifurcation in this section).

We consider the model for an autocatalytic reaction due to Schakenberg [6], [5]. Let $\mathcal{U}$ and $\mathcal{V}$ be two chemical 'species' that react irreversibly according to

$$2\mathcal{U} + \mathcal{V} \to 3\mathcal{U}, \;\; \mathcal{U} \to \mathcal{A}, \;\; \mathcal{B} \to \mathcal{V}, \tag{1.42}$$

where $\mathcal{A}$ and $\mathcal{B}$ are two 'products' of which we assume that the concentration is constant (both in time and in space). This (simplified) autocatalytic reaction-diffusion scheme is the described by the Schnakenberg model:

$$\left\{ \begin{array}{rcrrr} \frac{\partial U}{\partial t} & = & \frac{\partial^2 U}{\partial x^2} & -U & + \; UV^2 \\ \frac{\partial V}{\partial t} & = & D\frac{\partial^2 V}{\partial x^2} & +\mu & - \; UV^2, \end{array} \right. \tag{1.43}$$

where $U = U(x,t)$, respectively $V = V(x,t)$, denotes the concentrations of $\mathcal{U}$, resp. $\mathcal{V}$, and $0 < D = D_{\mathcal{V}}/D_{\mathcal{U}}$, the ratio of the diffusion constants of $\mathcal{U}$ and $\mathcal{V}$. Note that we have again assumed, for simplicity, that $\Omega = \{0\}$. The concentration of $\mathcal{B}$ is denoted by μ, the main bifurcation parameter. Note that (1.43) is a simplification of the original Schnakenberg model, in which the reaction $\mathcal{U} \to \mathcal{A}$ is assumed to be reversible (so that there will be another parameter in the equations, associated to the concentration of $\mathcal{A}$ [6], [5]).

In the notation of (1.1) and (1.3) we thus have:

$$\mathcal{S}_\mu = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; \mathcal{L}_\mu(U,V) = \begin{pmatrix} \frac{\partial^2 U}{\partial x^2} - U \\ D\frac{\partial^2 V}{\partial x^2} + \mu \end{pmatrix}, \; \mathcal{N}_\mu(U,V) = \begin{pmatrix} +U^2 V \\ -U^2 V \end{pmatrix}, \; \begin{pmatrix} U_0 \\ V_0 \end{pmatrix} = \begin{pmatrix} \mu \\ \frac{1}{\mu} \end{pmatrix}. \tag{1.44}$$

The ordinary differential equation associated to the reaction process (1.42) (i.e. the equation for $U(x,t) \equiv u(t)$, $V(x,t) \equiv v(t)$ in (1.43)) has been proposed as one of the 'simplest, but chemically plausible' reaction-mechanisms which admit periodic solutions. These chemical oscillations occur through a Hopf bifurcation [6], [5]). The full reaction-diffusion equation has been studied as a 'prototypical system' for biological pattern formation. These stationary, spatially periodic patterns are created by a Turing bifurcation [5].

Before we can do a bifurcation analysis, we first need to determine the linearized stability of the 'laminar solution' (U_0, V_0) (1.44). Following (1.4), we obtain the following eigenvalue problem for the vector $(f, g) \in \mathbf{R}^2$:

$$\begin{pmatrix} -k^2+1 & \mu^2 \\ -2 & -Dk^2-\mu^2 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} = \omega \begin{pmatrix} f \\ g \end{pmatrix}, \tag{1.45}$$

where $k \in \mathbf{R}$ plays the role of a parameter, so that $\omega = \omega(k, \mu)$. Thus, the *two* 'eigenvalue curves' $\omega_{1,2}(k, \mu)$ are determined by

$$\omega^2 - \omega[-(1+D)k^2 + 1 - \mu^2] + [Dk^4 + (\mu^2 - D)k^2 + \mu^2] = 0. \tag{1.46}$$

Note that the stability of the critical point of the reaction system is determined by $\omega_{1,2}(0, \mu)$. The trivial solution is asymptotically stable if $\mathrm{Re}(\omega_{1,2}(k, \mu)) < 0$ for all $k \in \mathbf{R}$. Instability can set in in two ways (as can be checked by direct computations, see also Figure ??):

$$\begin{array}{rlclclc} \text{Turing}: & \mu_c^{\mathrm{T}} = & (\sqrt{2}-1)\sqrt{D} & , \ k_c^{\mathrm{T}} = & \sqrt{\sqrt{2}-1} & , \ \omega_1((\sqrt{2}-1)\sqrt{D}, \sqrt{\sqrt{2}-1}) & = 0 \\ \text{Hopf}: & \mu_c^{\mathrm{H}} = & 1 & , \ k_c^{\mathrm{H}} = & 0 & , \ \omega_1(1, 0) & = i. \end{array} \tag{1.47}$$

The local character of the critical 'eigenvalue curve' $\omega_1(k, \mu_c)$ can be obtained through (1.46):

$$\begin{array}{rlclc} \text{Turing}: & \frac{\partial \omega_1}{\partial \mu}(k_c^{\mathrm{T}}, \mu_c^{\mathrm{T}}) = & -\frac{2\sqrt{D}}{D-1} & , \ \frac{\partial^2 \omega_1}{\partial k^2}(k_c^{\mathrm{T}}, \mu_c^{\mathrm{T}}) = & -\frac{4D\sqrt{2}}{D-1} \\ \text{Hopf}: & \frac{\partial \omega_1}{\partial \mu}(k_c^{\mathrm{H}}, \mu_c^{\mathrm{H}}) = & -1+i & , \ \frac{\partial^2 \omega_1}{\partial k^2}(k_c^{\mathrm{H}}, \mu_c^{\mathrm{H}}) = & -(1+D)-(1-D)i \end{array} \tag{1.48}$$

Thus, in both cases $\omega_1(k_c, \mu)$ decreases as function of μ. This implies that the 'trivial pattern' (U_0, V_0) is asymptotically stable for $\mu > \max(\mu_c^{\mathrm{T}}, \mu_c^{\mathrm{H}}$ and there are unstable linear 'waves' for $\mu < \mu_c^{\mathrm{T,H}}$. We observe that $\mu_c^{\mathrm{T}} = \mu_c^{\mathrm{H}}$ for

$$D \stackrel{\mathrm{def}}{=} D^{\mathrm{T-H}} = 3 + 2\sqrt{2} \tag{1.49}$$

(1.47). Hence, we conclude that for $D > D^{\mathrm{T-H}}$ (U_0, V_0) becomes unstable as μ decreases through μ_c^{T} (see Figure ??). This is an example of *a Turing bifurcation* [8]: there is an instability in the reaction-diffusion (partial differential) equation, while (U_0, V_0) is still asymptotically stable as critical point in the (ordinary) reaction system (i.e. $\mathrm{Re}(\omega_\pm(0, \mu_c^{\mathrm{T}}) < 0$, see Figure ??). As a consequence, the 'most unstable linear waves' have a nontrivial spatial character, as is clear from (1.47). This is not the case for the Hopf bifurcation. As $D < D^{\mathrm{T-H}}$, the instability sets in at $\mu_c^{\mathrm{H}} = 1$. Since $k_c^{\mathrm{H}} = 0$, this bifurcation can be identified with the Hopf bifurcation in the ordinary differential equation associated to the reaction (1.42). Hence, the linear most unstable wave has no spatial structure. Note, however, that this does not imply that the patterns that appear as μ decreases through μ^{H} can also not have a spatial structure.

Since $\omega_1(k, \mu) \in \mathbf{R}$ in the neighborhood of $(k_c^{\mathrm{T}}, \mu_c^{\mathrm{T}})$, we expect that, for $|\mu - \mu_c^{\mathrm{T}}| \ll 1$, the nonlinear evolution of small perturbations will be described by a real Ginzburg-Landau equation in the case of the Turing bifurcation. For the Hopf case, we expect a complex Ginzburg-Landau equation.

1.6 A convection problem

Classical hydrodynamical stability problems, as the Taylor-Couette problem, Bénard's problem, or Poiseuille flow can be described near the onset of instability with the help of the Ginzburg-Landau equation. We will consider these systems in detail in subsequent sections.

As a first example of such a system we consider convection in porous media. This plays a role in the description of hot springs and geysers, and of so called black smokers on the ocean floor. This system is more easy since in this case the velocity field of the fluid is determined by a constitutive law, namely Darcy's law, and has not to be computed as a solution of the Navier-Stokes equtions. Convection in the fluid sets in if the temperature induced density difference between the bottom and the top of the strip is above a certain threshold.

Therefore, we are interested in the velocity field $\mathbf{u} = (u, v)$ and the temperature field T in a strip $\mathbf{R} \times [0, 1]$

of porous media, heated from below. If we denote the coordinates in the strip with $(x_1, x_2) \in \mathbf{R} \times [0,1]$, we have to solve

$$\begin{aligned} \nabla \cdot \mathbf{u} &= 0, && (1.50)\\ \mathbf{u} &= -\nabla p + RTe_2, && (1.51)\\ \frac{\partial}{\partial t}T + u \cdot \nabla T &= \Delta T, && (1.52) \end{aligned}$$

under the boundary conditions

$$\begin{aligned} T = 1,\ u_2 = 0 \quad &\text{for} \quad y = 0,\\ T = 0,\ u_2 = 0 \quad &\text{for} \quad y = 1. \end{aligned}$$

A detailed derivation of this system can be found for instance in [2].

Equation (1.50) comes from the conservation of fluid mass and simplifies here to the usual incompressibility condition.

Equation (1.51) is based on the balance of forces. It is a constitutive law which relates the velocity field $\mathbf{u}$, the pressure p and the temperature T. Equation (1.51) comes from Darcy's law which relates the velocity field $\mathbf{u}$, the pressure p and the density ρ. In the derivation of this system the so called Boussinesq approximation has been used. Therefore except of this place the small temperature induced density differences are neglected and ρ is considered to be constant for all $(x, y) \in \mathbf{R} \times [0,1]$. At this place the relation between the temperature T and the density ρ is given by an affine law. Moreover, we used $e_2 = (0,1)^T$.

Equation (1.52) comes from the balance of energy and is the usual nonlinear heat equation.

System (1.50-1.52) contains the parameter

$$R = \frac{\alpha(\delta T)\rho_0 g k d}{\mu \kappa_T}$$

with α the factor in the affine law between the density ρ and the temperature T, (δT) the temperature difference between the upper and lower boundary, ρ_0 the constant density in the Boussinesq approximation, g the gravitational acceleration, k the permeability, d the distance between upper and lower boundary, μ the liquid vicosity, and κ_T the average thermal conductivity.

For R sufficiently small there is no convection and there is one globally attracting basic, purely conducting, state given by

$$u = 0,\ T = 1 - y,\ p = -\frac{R}{2}(1 - y^2). \qquad (1.53)$$

We eleminate the pressure p by introducing the stream function ψ such that

$$u_1 = \frac{\partial \psi}{\partial y} \text{ and } u_2 = -\frac{\partial \psi}{\partial x}.$$

We introduce the deviation θ from the linear tempurature profile by

$$T = 1 - y + \theta.$$

Hence we obtain

$$\begin{aligned} \Delta \psi &= -R\frac{\partial \theta}{\partial x},\\ \frac{\partial \theta}{\partial t} + \frac{\partial \psi}{\partial x} + \left(\frac{\partial \psi}{\partial y}\frac{\partial \theta}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial \theta}{\partial y}\right) &= \Delta \theta. \end{aligned}$$

As before we are interested in the bifurcation scenario as the trivial solution, here the pure conduction state, becomes unstable. In order to analyse the stability of the trivial state we look at the linearised system first, i.e.

$$\begin{aligned}
\nabla^2\psi &= -R\frac{\partial\theta}{\partial x}, \\
\frac{\partial\theta}{\partial t}+\frac{\partial\psi}{\partial x} &= \nabla^2\theta.
\end{aligned}$$

under the boundary conditions

$$\theta=\psi=0 \text{ at } y=0,1.$$

Due to the boundary conditions we make the ansatz

$$\begin{aligned}
\psi &= a\sin(n\pi y)e^{\mu t+ikx} \\
\theta &= b\sin(n\pi y)e^{\mu t+ikx}
\end{aligned}$$

and obtain the system of linear equations

$$\begin{aligned}
-n^2a-k^2f &= ikRg, \\
-n^2g-k^2g &= ikf+\mu g.
\end{aligned}$$

As already said convection in porous media plays a role in the description of hot springs and geysers, and of so called black smokers on the ocean floor. The rock between the air and the sea, respectively, at the top and of the magma chambers at the bottom is highly fractured. The rock is permeated by water and strongly heated by the magma below. In the strongly nonlinear case, i.e. far above the onset of convection the upwelling water is concentrated strongly. In the ocean the hydrostatic pressure prevents the boiling of the water whereas the boiling of the water in geysers causes the periodic eruption of steam and water.

1.7 The general model

1.8 Higher order corrections to the cubic Ginzburg-Landau equation

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