Assignment-set 3 Introduction to Perturbation Methods

Deadline to hand in: 4 May 2017, 11.15u

1.) Find a composite expansion (first-term) which is uniformly valid on $x \in (0, 1)$ to

$$x^3 \frac{dy}{dx} = \varepsilon\{(1+\varepsilon)x + 2\varepsilon^2\}y^2$$

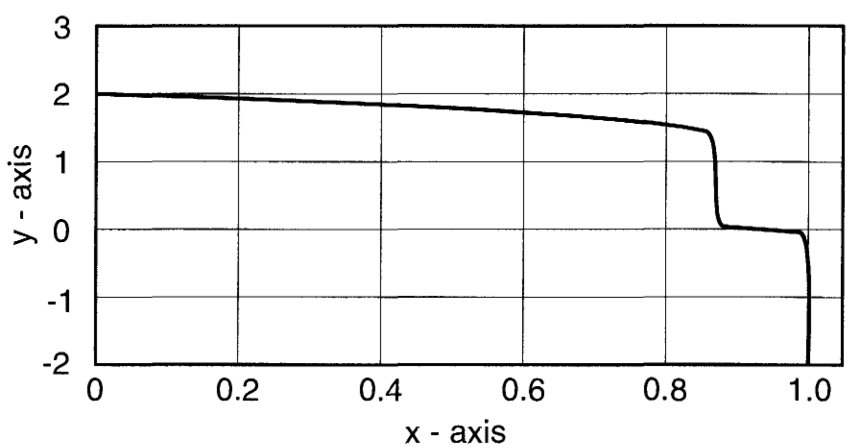
with $y(1) = 1 - \varepsilon$.

Hint: There are *two* layers.

2.) Consider the problem

$$\varepsilon^2 y'' + y(1-y)y' - xy = 0, \text{ for } 0 < x < 1,$$

where $y(0) = 2$ and $y(1) = -2$. The numerical solution is shown in the figure below, in the case where $\varepsilon = 10^{-3}$. You can use this figure when deriving a first-term approximation of the solution in this exercise.



(a) Find the first term in the expansion of the outer solution.

(b) Assume there is a boundary layer at $x = 1$. Determine a first-term approximation in the boundary layer and show that it does not match with the outer solution you found in part (a).

(c) Assume that there is an interior layer across which the solution jumps from the outer solution to the solution in the boundary layer. Find a first-term approximation in the layer. From the matching show that the layer is located at $x_0 = \frac{\sqrt{3}}{2}$. Note that your layer solution will contain an undetermined constant.

(d) Correct the boundary layer analysis in part (b) based on the results from part (c).

3.) In the study of Josephson junctions, the following problem appears (Sanders, 1983):

$$\phi'' + \varepsilon(1 + \gamma \cos\phi)\phi' + \sin\phi = \varepsilon\alpha, \text{ for } t > 0,$$

where $\phi(0) = \phi'(0) = 0$ and γ is a positive constant. Find a first-term approximation of $\phi(t)$ that is valid for large t.

4.) This problem considers how to apply the multiple-scale approach to the first-order system

$$y' = Ay + f(y, \varepsilon),$$

where $y(0) = a$. It is assumed f is smooth with $f(y, 0) = 0$. Also, letting $y^T = (u, v)$, it is assumed that

$$A = \begin{bmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{bmatrix},$$

where $\alpha^2 < \beta\gamma$. The matrix A and vector a do not depend on ε.

(a) Introduce the time scales $t_1 = t$ and $t_2 = \varepsilon t$, and then show that the first term in the expansion of the solution has the form

$$y_0(t_1, t_2) = q(t_2)e^{i\omega t} + \bar{q}(t_2)e^{-i\omega t}$$

where $\omega = \sqrt{\beta\gamma - \alpha^2}$ and the overbar indicates a complex conjugate. Here q is an eigenvector associated with the first-order problem. Also, determine $q(0)$.

(b) Suppose the solution is known to be $\frac{2\pi}{\omega}$ periodic in t_1. This information can be used to remove secular terms. To show how, let $p(t_1)$ be the solution of the (first-order) adjoint equation $p' = -A^T p$. From the equation for the $\mathcal{O}(\varepsilon)$ term, show that

$$\int_0^{\frac{2\pi}{\omega}} (\partial_{t_2} y_0 - \partial_\varepsilon f(y_0, 0))\dot{p}\, dt_1 = 0.$$

Explain why this is a first-order differential equation that completes the determination of the vector q.

(c) Consider the system of equations

$$\begin{aligned} u' &= u + v - \varepsilon u(2 + \varepsilon uv) \\ v' &= -4u - v + \varepsilon u(2 + \varepsilon uv). \end{aligned}$$

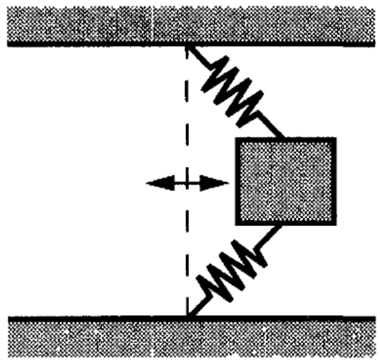
After identifying the matrix A and the function f, use the results of parts (b) and (c) to find a first-term approximation of $y(t)$ that is valid for large t (assume the solution satisfies the periodicity condition mentioned in (c)).

(d) In the study of chemical oscillators, the following system of equations is found (Schnakenberg, 1979):

$$\begin{aligned} u' &= u + v + \varepsilon u(2u + 2v + \varepsilon uv) \\ v' &= -4u - v - \varepsilon u(2u + 2v + \varepsilon uv). \end{aligned}$$

In this case, show that the result in part (c) indicates that the wrong time scale has been introduced. Extend the results of parts (b) and (c) to include this situation, and in the process find a first-term approximation of $y(t)$ that is valid for large t (assume the solution satisfies the periodicity condition mentioned in part (c)).

5.) Suppose a mass is situated between two parallel walls and is connected to the walls by springs, as shown in the following figure.



For a small periodic forcing the equation for the transverse oscillations of the mass is

$$y'' + y\left[1 - \frac{\lambda}{\sqrt{1+y^2}}\right] = \varepsilon\cos(\kappa t + \varepsilon^{\frac{2}{3}}\omega t) \text{ for } t > 0,$$

where λ and κ are positive constants with $0 < \lambda < 1$. Also, $y(0) = y'(0) = 0$. Set $\kappa^2 = 1 - \lambda$, and for small ε find a first-term approximation of the solution that is valid for large t. If you are not able to solve the problem that determines the t_2 dependence, then find, and sketch, the possible steady states (if any) for the amplitude.