Assignment-set 2 Introduction to Perturbation Methods

Deadline to hand in: 6 April 2017, 11.15u

1.) This exercise concerns solutions to the boundary value problem

$$-\varepsilon\partial_{xx}u + u\partial_x u = u, u(0) = u(1) = 0, \tag{1}$$

for small ε.

MATLAB part. The first part of the exercise prepares you for the second analytical part: you will see whether, where and at how many locations sharp gradients (layers) develop as $\varepsilon \to 0$. At these locations you will need to introduce different scalings and match these regions asymptotically.

Software. You need to use the matlab files for assignment-set 2 that are available for download from the course website: http://www.math.leidenuniv.nl/~vivi/IPM2017/

After download unzip the file in a directory of your choice - to use it in matlab you have to select this directory as the "current folder". The main file is "BurgersSiva.m" and you can run the preset simulation by using the file name without suffix as a command.

The simulation is a numerical approximation of the solution to the partial differential equation for $u = u(t, x)$ given by

$$\partial_t u - \varepsilon\partial_{xx}u + u\partial_x u = u, \quad u(t, 0) = u(t, 1) = 0,$$

and a certain initial condition that you see in the window that pops up after the program has finished. The simulation starts when pressing any key on your keyboard while that pop-up window is focussed. Note that the horizonal axis is x and the vertical $u(x)$.

To see a simulation again without recomputing the solution you can use the command "replot".

Now open the file 'BurgersSiva.m' in the editor so that you can modify and save it. Find line 4 : "eps = 1;" which sets the value of ε that is used when running the program. Hence the value of ε in the given file is not small; we are interested in smaller values of ε.

Decrease the ε to smaller and smaller values and rerun the computation. In this way you will notice the formation of layer(s) and that the solution becomes stationary: you can assume that the profile you see at the end of the simulation has zero time derivative and thus solves (1). (Note that if you make 'eps' too small, the numerical solution is no longer accurate. This can be compensated by decreasing the spatial grid length 'xstep'.)

Analysis part. In this part, you have to construct an approximate positive solution ($u(x) > 0$, $0 < x < 1$) of the solutions that you found in the numerical part for small ε by using asymptotic analysis.

(a) Print relevant plots from the numerical simulations and label these. Use these plots to illustrate what you do in the analysis.

(b) Using the numerical results, explain how many layers there exist and what the position of each of them is.

(c) Find the explicit solution to the outer problem.

(d) Now, rescale x and u to obtain the several possibilities for the inner problem. Use the numerics to conclude which of the scalings is correct.

(e) Find an explicit formula for the one-parameter family of fronts (try tanh) for the inner problem found in (c).

(f) Use the above results to give an explicit expression for the approximate solution $u(x)$ to the boundary value problem (1) for $0 < \varepsilon \ll 1$.

2.) The Poisson-Nerst-Planck model for flow of ions through a membrane consists of the following equations for $0 < x < 1$

$$\begin{aligned} \frac{dp}{dx} + p\frac{d\phi}{dx} &= -\alpha \\ \frac{dn}{dx} - n\frac{d\phi}{dx} &= -\beta \\ \varepsilon\frac{d^2\phi}{dx^2} &= -p + n \end{aligned}$$

The boundary conditions are $\phi(0) = 1$, $\phi(1) = 0$, $p(0) = 4$ and $n(0) = 1$. In these equations, p and n are the concentrations of the ions with valency 1 and -1, respectively, and ϕ is the potential. Assume that α and β are positive constants with $\alpha \neq \beta$ and introduce

$$\kappa = \frac{\alpha + \beta}{2\sqrt{p(0)n(0)}}$$

which satisfies $\kappa < 1$.

(a) Assuming that there is a boundary layer at $x = 0$, derive the outer and boundary-layer approximations. Explain why, if the outer approximation for ϕ is required to satisfy $\phi(1) = 0$, the approximations do not match. How can this be solved?

(b) Derive an approximation that does satisfy the condition at $x = 1$ and complete the matching that was started in part (a). From this show that

$$p(1) = p(0)e^{\phi(0)}(1-\kappa)^{\frac{2\alpha}{\alpha+\beta}}$$

$$n(1) = n(0)e^{-\phi(0)}(1-\kappa)^{\frac{2\beta}{\alpha+\beta}}$$

to leading order.

3.) Consider the problem of solving

$$\varepsilon y'' - xy' = 0 \quad \text{for } a < x < b,$$

where $y(a) = y_L$ and $y(b) = y_R$. Also, assume that $a < 0$ and $b > 0$.

(a) Using the usual boundary-layer arguments show that

$$y = \begin{cases} c + (y_L - c)e^{a\bar{x}} & \text{boundary layer at } x = a \\ c & \text{outer region} \\ c + (y_R - c)e^{b\tilde{x}} & \text{boundary layer at } x = b \end{cases}$$

to leading order, where c is an arbitrary constant, $\bar{x} = \frac{x-a}{\varepsilon}$ and $\tilde{x} = \frac{x-b}{\varepsilon}$.

(b) Determine the exact solution of the problem.

(c) Using the results from part (b) and Laplace's approximation (see Appendix C in the book) show that

$$c = \begin{cases} y_L & \text{if } |a| < b \\ \frac{1}{2}(y_L + y_R) & \text{if } |a| = b \\ y_R & \text{if } |a| > b \end{cases}$$

Comment on what happens to the assumed boundary layers in the problem in each case.

(d) Sketch the solution when $a = -2,\ b = 1,\ y_L = 1$ and $y_R = 3$. Comment on whether you used part (a) or part (b), and why.