THE FORMATION OF SHEAR LAYERS IN A FLUID WITH TEMPERATURE-DEPENDENT VISCOSITY

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The presence of viscosity normally has a stabilizing effect on the flow of a fluid. However, experiments show that the flow of a fluid in which viscosity decreases as temperature increases tends to form shear layers, narrow regions in which the velocity of the fluid changes sharply. In this paper, we present evidence that the formation of shear layers is due to an inherent instability in the fundamental conservation laws governing fluid flow when the viscosity decreases sufficiently quickly with temperature.

1. Introduction

To understand the effects of the dependence of the viscosity on temperature and, in particular, to determine if the dependence of viscosity upon temperature can destabilize the flow of a fluid, Dafermos and Hsiao[2] proposed a test problem that models an adiabatic rectilinear shearing flow in an incompressible Newtonian viscous fluid between parallel plates. To describe the model, we choose Cartesian coordinates so that the x-axis is perpendicular to the plates located at $x = 0$ and $x = 1$. We assume that the plate at $x = 0$ is at rest and the plate at $x = 1$ moves with constant velocity V in a direction orthogonal to x and that between the plates, the flow is parallel to the plates and uniform in the directions orthogonal to x. The Lagrangian description of the shear flow in terms of the balance laws of mass, momentum and energy, with density and the specific heat normalized to be one, yields

$$\begin{cases} v_t(x,t) = \sigma_x(x,t), & 0 < x < 1, 0 < t, \\ \theta_t(x,t) = \sigma(x,t)v_x(x,t), & 0 < x < 1, 0 < t, \\ v(0,t) = 0,\ v(1,t) = V, & 0 < t, \\ v(x,0) = v_0(x),\ \theta(x,0) = \theta_0(x), & 0 < x < 1, \end{cases} \tag{1}$$

where v denotes the velocity, σ the shear stress, and θ the temperature (or internal energy). For a constitutive relation, we assume that the fluid is linearly viscous, that is, the shear stress is related to the temperature and velocity gradient as follows

$$\sigma = \mu(\theta)v_x. \tag{2}$$

An important feature of (1) is that the uniform shear flow, $v(x,t) = Vx$, $\theta(x,t) = h(t)$, where $h(t)$ is determined by

$$\int_a^{h(t)} \frac{ds}{\mu(s)} = t,$$

is a solution. In other words, the flow keeps the uniform shear profile if it begins with a uniform profile. Much of the analysis of (1) in the literature concentrates on understanding the stability of the uniform shear flow, i.e., what is the long-time behavior of the flow when the initial velocity is close to $v_0(x) = Vx$ and the initial temperature θ_0 is almost constant.

As the material is being sheared, energy is pumped into the system. Since the flow is adiabatic, temperature will keep rising and tend to infinity with time. The distribution of temperature can either go to infinity uniformly in space or it could localize. Note that in the uniform shear flow, the temperature grows with t uniformly in x. If the temperature tends to infinity in a localized region in x, then a shear layer can develop in the same region. Shear layers, or bands, are narrow regions in which the velocity of the fluid changes sharply or narrow layers of concentrated shearing deformation observed during the plastic shearing of materials.

Various mechanisms and associated continuum thermomechanics models have been proposed for the explanation of the existence of shear layers and there is an extensive literature on the formation of shear layers[3]. Much of the existing literature study (1) or closely related models. The motivation for studying this simple model is to obtain a better understanding of the phenomenon of localization of the temperature and the formation of shear layers. Moreover, the conservation laws that define (1) lie at the heart of any more sophisticated model describing shear layer phenomena.

Detailed mathematical analyses of (1) have been carried out by Dafermos and Hsiao[2], Tzavaras[4], and Bertsch, Peletier, and Verduyn Lunel[1]. The analysis shows that if the viscosity $\mu = \mu(\theta)$ tends monotonically to a positive constant as $\theta \to \infty$ and either μ^2 is concave or μ is convex, then for all smooth initial data, there is a unique solution of (1) that converges to the uniform shear flow as $t \to \infty$. In other words, the uniform shear flow is a stable solution attracting all smooth solutions. The analysis also shows that the situation is more delicate if the viscosity tends to zero as temperature increases, i.e. $\mu \to 0$ as $\theta \to \infty$. In this case, the existence of solutions and the stability of the uniform shearing flow depend on the rate of decrease of μ with θ. In particular, when

$$\mu(\theta) = \theta^{-\alpha}, \quad \alpha > 0, \tag{3}$$

the mathematical analysis shows that shear bands do not form in (1) when $\alpha \leq 1$, i.e., the rate of decrease of the viscosity as the temperature increases is not too big. If $\alpha > 1$, then the uniform shear flow becomes unstable. However, in this case, the analysis reveals little about the long-time behavior of the solutions.

It turns out that the behavior of solutions for $\alpha > 1$ is much different than in the stable case of $\alpha \leq 1$. To illustrate this, we compute solutions with $\alpha = 2$ beginning with a slight perturbation of the initial data for the uniform shear flow

$$v_0(x) = \int_0^x \frac{C_1^2}{(1+(s-\bar{x})^2)^2}\, ds \quad \text{and} \quad \theta_0(x) = \frac{C_1}{1+(x-\bar{x})^2}, \tag{4}$$

where $\bar{x} = .7$ and $C_1 = 1.108422867$ is fixed to normalize $\theta_0(x)$ to have average value 1. We plot these functions in Fig. 1. In Fig. 2, we show the evolution of the corresponding numerical solution. Very similar solutions are obtained for any initial data consisting of small, smooth perturbations of the uniform shear flow. This particular data has special properties that make analysis easier as we explain below.

A sharp shear layer in the flow is clearly developing accompanied by extreme localization of the temperature. In Fig. 3, we present plots of the maximum and minimum values of the velocity gradient v_x and the temperature θ versus time. The plot of the maximum of v_x versus t suggests

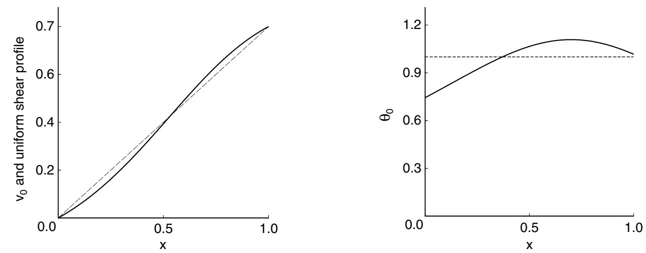

Figure 1. Plots of the initial data for v and θ used for the computation shown in Fig. 2.

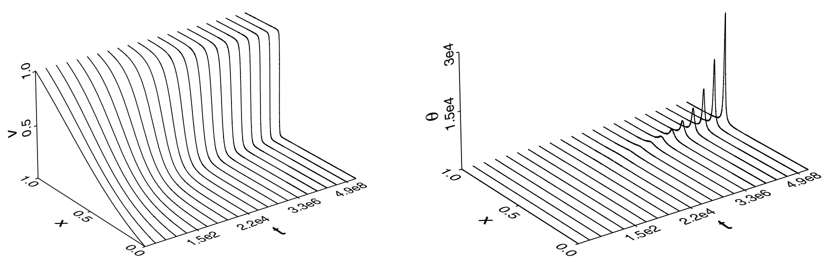

Figure 2. Plots of the evolution of v and *theta* versus time with $\alpha = 2$. The initial data is shown in Fig. 1.

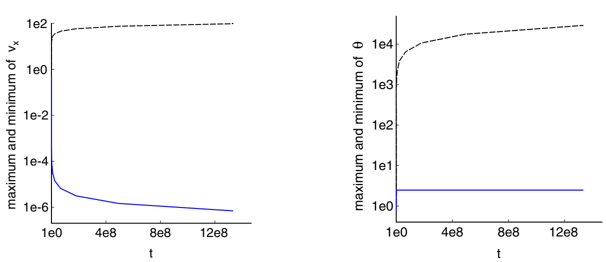

Figure 3. Plots of the maximum and minimum of v_x and θ versus time corresponding to the solution shown in Fig. 2. The vertical axis of these plots is logarithmic.

that the peak height of velocity gradient starts to grow at an exponential rate in time after an initial transient period has passed. This corresponds to a discontinuous velocity profile at $t = \infty$. The temperature θ is growing uniformly in x but grows much faster at the peak. Note that the different growth rates in both plots quantify the extreme degree of localization.

In Estep, Verduyn Lunel, and Williams[3], we describe an approach for analyzing (1) under the assumption (3) that is based on a combination of analytic and numerical analysis. To overcome the multiscaled nature of the solutions, the long-time transient to the formation of shear layers, and the apparent blow-up at infinity, we first introduce new variables to change (1) into a system of reaction-diffusion equations where the interesting behavior occurs on a more reasonable time scale. Moreover, we use an adaptive finite element method to maintain accuracy as the shear layers develop. In this paper, we interpet our results in terms of the physical variable and examine the consequences for the evolution of the flow.

Changing the problem into a system of reaction-diffusion equations opens up a number of useful analytic and numerical tools for the study of evolution problems. This change is effected by replacing the equations for the velocity v and temperature θ by an equivalent system of reaction-diffusion equations for the physical variables shear stress σ and the temperature θ:

$$\begin{cases} \sigma_t - \theta^{-\alpha}\sigma_{xx} = -\alpha\theta^{\alpha-1}\sigma^3 & 0 < x < 1, 0 < t, \\ \theta_t = \theta^\alpha \sigma^2, & 0 < x < 1, 0 < t, \end{cases} \tag{5}$$

with boundary conditions and initial data given by

$$\begin{cases} \sigma_x(0,t) = \sigma_x(1,t) = 0, & 0 < t, \\ \sigma(x,0) = \sigma_0(x),\ \theta(x,0) = \theta_0(x), & 0 < x < 1. \end{cases} \tag{6}$$

Completing a standard asymptotic analysis on solutions of (5) turns out to be a highly technical and difficult process. As an alternative, we construct a model for solutions of (5) beginning with the data

$$\sigma_0(x) = 1 \quad \text{and} \quad \theta_0(x) = \frac{C_1}{1 + (x - \bar{x})^2}, \tag{7}$$

corresponding to (4), using a different technique. The technique is based on analytically computing a sequence of approximate solutions $\tilde{\sigma}$ and $\tilde{\theta}$ whose residuals (computed by substitution into (6)) are successively smaller and moreover tend to zero as time increases. After one iteration of the construction process, the approximate solutions for temperature and shear stress are, respectively, given by

$$\tilde{\theta}(x,t) = \frac{C_1(t+1)^{1/2}}{1 + (x - \bar{x})^2(t+1)^{1/2}} \tag{8}$$

and

$$\tilde{\sigma}(x,t) = \frac{1}{\sqrt{2C_1}}(t+1)^{-3/4} - \frac{C_1\sqrt{2C_1}}{16}\frac{(t+1)^{-15/12}}{1 + (x - \bar{x})^2(t+1)^{1/2}}\,. \tag{9}$$

To leading order, the corresponding approximate velocity gradient is given by

$$\tilde{v}_x(x,t) = \frac{C_1^{3/2}(t+1)^{1/4}}{2^{1/2}(1+C_2(x-\bar{x})^2(t+1)^{1/2})^2}. \tag{10}$$

Examining the formula for $\tilde{\theta}$ reveals the mechanism of the localization in temperature that drives the formation of the shear layer. When $x = \bar{x}$, the approximate solution $\tilde{\theta}$ increases like $\mathbf{O}(t^{1/2})$ while away from $\bar{x}$, the approximate solution $\tilde{\theta}$ tends to 1 as t increases. Likewise we see that overall the approximate solution $\tilde{\sigma}$ tends to zero like $\mathbf{O}(t^{-3/4})$. But the approximate solution $\tilde{\sigma}$ is not uniform in x and forms a small "dip" corresponding to the "peak" in the profile of temperature to compensate.

This detailed analysis of the special solution of (1) with initial data (4) actually describes the formation of shear layers in a wide variety of solutions. To analyze the formation of shear layers in solutions that start as arbitrary smooth small perturbations of the uniform shear flow, we take the lowest order terms in (8) and (9). Careful numerical experiments show that these approximations accurately describe the behavior of the solutions in the region of a layer to the point of providing accurate quantitative estimates on the rates of decay and growth. These experiments suggest that solutions that begin as smooth perturbations of the uniform shear flow converge to the model function in the limit of large time. Moreover the formation of shear layers appears to be a robust phenomena with respect to lower order variations of initial data and to alterations in the model (1). Perhaps most significantly, it follows from our results that accounting for diffusive effects in the model equation for temperature does not prevent the formation of a shear layer.

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