Effects of time delays on the dynamics of feedback systems*

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Abstract

Feedback systems are important in applications, for example, optical feedback lasers, phase-locked frequency synthesizers and wave equations with feedback stabilization at the boundary, and the problem regarding sensitivity and robustness of the feedback system with respect to time delays in the loop has attracted a lot of attention. In this paper we shall discuss some recent developments.

1 Introduction

In the implementation of any feedback control system, it is very likely that time delays will occur. Therefore, it is of importance to understand the sensitivity of the control system with respect to the introduction of small delays in the feedback loop. For some systems, small delays lead to destabilization while other systems are robust with respect to small time delays. One of the earliest papers to address this topic using frequency domain techniques was [1]. A more recent and more complete discussion using this approach is [12, 13, 14]. At approximately the same time, studies were made directly on the characteristic equation associated with a neutral delay differential equations to determine robustness of stability with respect to variations in the delays [10, 16, 17].

In [8] a first attempt was made for a unifying theory which explains the underlying mechanisms and the role difference equations are playing in these results. In this paper we shall discuss a result from [8] concerning the problem of stabilization and control of partial differential equations through the application of forces on the boundary and present some new developments [9].

The control of partial differential equations through the boundary has received a considerable amount of attention and the mathematical theory is very complete in the situation when the boundary forces are applied with no delays in time (see [11] for a review). However, it is known that small delays in the boundary forces can lead to a destabilization of the system (see [2, 4, 5, 12, 13] and the references in [8]).

Before we present some examples in detail, it is instructive to make some remarks about why the control system might be sensitive with respect to the introduction of small

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delays in the feedback loop. Suppose that τ is a parameter varying in a metric space S and $T_\tau(t) : X \to X$ is a $\mathcal{C}_0$-semigroup of linear transformations defined on a Banach space X which is continuous in τ; that is, $T_\tau(t)x$ is continuous in $(\tau, t, x) \in S \times [0, \infty) \times X$. Let $r_\sigma(T_\tau(1))$ denote the spectral radius of $T_\tau(1)$. The asymptotic behavior of the semigroup $T_\tau(t)$ is determined by the spectral radius of $T_\tau(1)$.

If $r_\sigma(T_\tau(1)) < 1$, then each orbit $\{T_\tau(t)\varphi\}_{t\geq 0}$ will approach zero exponentially and uniformly. If $r_\sigma(T_\tau(1)) > 1$, then there is an unbounded orbit and the semigroup is unstable. Although, the eigenvalues of $T_\tau(1)$ can be chosen to be continuous functions of τ, the spectral radius $r_\sigma(T_\tau(1))$ might not be continuous with respect to τ. In general, the spectral radius $r_\sigma(T_\tau(1))$ is only lower semicontinuous (see Thm II.4.1 of [6]) and, in order to be continuous, further conditions are needed. For example, the spectral radius $r_\sigma(T_\tau(1))$ will be continuous at a point τ_0 if there exists a $\epsilon > 0$ such that the differences

$$T_\tau(1) - T_{\tau_0}(1), \quad |\tau - \tau_0| < \epsilon \tag{1.1}$$

are compact (see Thm II.4.2 of [6]).

If the operators in (1.1) are only bounded, then there is the possibility of a large shift in the spectral radius if we vary τ. So, if $r_\sigma(T_\tau(1)) < 1$, and the perturbation is only bounded, it is possible to have $r_\sigma(T_{\tau_j}(1)) > 1$ for a sequence of $\tau_j \to \tau$ and the semigroup $T_{\tau_j}(t)$ will be unstable for each j. In Section 3 we shall present some examples. In many of the results that have been given in the literature for which stability is destroyed by the introduction of a small delay, this instability is due to the fact that the operators in (1.1) are only bounded.

A fundamental role in the study of stability and the preservation of stability under perturbations in the parameter τ is played by the essential spectrum $\sigma_e(T_\tau(1))$ of $T_\tau(1)$. This is, by definition, the part of the spectrum of $T_\tau(1)$ that cannot be removed by a compact perturbation. Let $r_e(T_\tau(1)))$ denote the radius of the essential spectrum of $T_\tau(1)$.

It is known that, if $r_e(T_\tau(1))) < 1$ then the stability or instability of 0 is determined by eigenvalues of $T_\tau(1)$. Furthermore, if 0 is unstable, then it is due to only a finite number of eigenvalues and the instability occurs in a finite dimensional subspace of X. If the latter situation occurs, then it is natural to expect that the stabilization of the system could be accomplished using a bounded finite rank control and the stabilization is insensitive to small changes in the parameter.

If $r_e(T_\tau(1))) > 1$, then the instability of the system is of such a nature that, in general, it cannot be controlled by a finite rank control which is robust with respect to small changes in the parameter (see [8, 9]).

2 Delayed boundary control: an example

In this section, we consider the one dimensional linear wave equation on a compact interval with zero boundary conditions at one end fixed and strain at the other end. The strain is being controlled by the velocity with a small delay. Various methods have been used to show that a stable system can be destabilized even with small time delays in the boundary forces. We shall show how this particular problem can be reduced to a discussion of a difference equation with three delays.

Let $w : [0,1] \times [0,\infty) \to \mathbb{R}$ denote the solution of the linear wave equation

$$w_{tt} + 2aw_t - w_{xx} + bw = 0, \quad 0 < x < 1, \quad t > 0, \tag{2.1}$$

with initial data $w(x,0) = w_0(x)$ and $w_t(x,0) = w_1(x)$ and with the boundary conditions

$$w(0,t) = 0, \quad w_x(1,t) = -kw_t(1,t-r), \tag{2.2}$$

where $a \geq 0, k > 0, r \geq 0$ and b are constants. System (2.1)-(2.2) corresponds to a boundary stabilization problem where the control function is $u(1,t) = kw_t(1,t-r)$.

If $r = 0$, system (2.1)-(2.2) generates a $\mathcal{C}_0$-semigroup $S_{0,k}(t)$ on $X = H^1(0,1) \times L^2(0,1)$. In this space X, the control function $kw_t(1,t)$ is a bounded perturbation but not compact. Therefore, it might be possible to shift the radius of the essential spectrum of $S_{0,k}(1)$ by varying k. In fact, even if $a = 0$, it is easy to see that $r_e(S_{0,k}(1)) < 1$ for any $k > 0$ (note that $r_e(S_{0,0}(1)) = 1$). Therefore for $k > 0$, any remaining instability in the system is due to finitely many eigenvalues of $S_{0,k}(1)$ with positive real part. These unstable modes are easily controlled by a finite dimensional control (see [9]).

If we introduce a time delay r in the problem then the situation becomes more complicated. For the space of initial data, let us choose an $h > 0$ and define

$$X_h = C([-h,0]; H^1(0,1)) \times C([-h,0]; L^2(0,1)).$$

For any $0 \leq r \leq h$, system (2.1)-(2.2) with a time delay generates a $\mathcal{C}_0$-semigroup $S_{r,k}(t)$ on the state space X_h. In the space X_h, the control function $kw_t(1,t-r)$ is a bounded perturbation but not compact. Therefore, the radius of the essential spectrum of $S_{r,k}(1)$ may change with r for fixed k.

Therefore we shall study the dependence of the radius of the essential spectrum of $S_{r,k}(1)$ upon the parameters k and r. By definition, the radius of the essential spectrum $r_e(S_{r,k}(1))$ does not change when we apply a compact perturbation to $S_{r,k}(1)$. So, in order to compute $r_e(S_{r,k}(1)$ we can reduce the equation accordingly.

If we replace $w(x,t)$ by $e^{-at}w(x,t)$ and use the fact that, for any constant c, a term $cw(x,t)$ corresponds to a compact perturbation, then we see that

$$r_e(S_{r,k}(1)) = e^{-a} r_e(T_{r,k}(1)),$$

where $T_{r,k}(t)$ is the semigroup generated by the wave equation

$$w_{tt} - w_{xx} = 0, \quad 0 < x < 1, \quad t > 0, \tag{2.3}$$

with the boundary conditions

$$w(0,t) = 0, \quad w_x(1,t) = -ke^{ar} w_t(1,t-r). \tag{2.4}$$

We can now easily compute the eigenvalues of (2.3)–(2.4). If we let $w(x,t) = e^{\lambda t} v(x)$, then it is easy to see that $\lambda \neq 0$ and that λ must satisfy the equation

$$e^{\lambda} + e^{-\lambda} = -ke^{ar}(e^{\lambda} - e^{-\lambda})e^{-\lambda r}.$$

If we let $z = \lambda - a$, we obtain the equation

$$1 + ke^{-zr} - ke^{-2a}e^{-z(2+r)} + e^{-2a}e^{-2z} = 0 \tag{2.5}$$

and we obtain $r_e(S_{r,k}(1)) = e^{\rho}$, where ρ is determined by the supremum of the real parts of the solutions of (2.5). If this supremum is negative, then system (2.1)–(2.2) has finitely many independent solutions which are exponentially unbounded as $t \to \infty$.

The roots of equation (2.5) will be studied in the next section and we return to the delayed boundary control problem (2.1)-(2.2) in Section 4.

It is not surprising that the radius of the essential spectrum of the semigroup generated by (2.1)–(2.2) is determined by the characteristic equation of a difference equation. In fact, any linear wave equation in one space dimension (even with nonlinear boundary conditions) is equivalent to a neutral delay differential equation (see [7] for references), and for neutral equations, the radius of the essential spectrum of the solution operator is determined by the difference equation obtained from the neutral part in the equation.

3 Difference equations

In this section we shall explain how variations in parameters affect the property of difference equations being stable. For simplicity, we shall only be concerned with scalar difference operators $D = D(r, a)$ of the form

$$D(r, a)\varphi = \varphi(0) - \sum_{j=1}^{M} a_j\varphi(-r_j), \quad \varphi \in C[-h, 0], \tag{3.1}$$

where $r_k > 0$ are constants, $a = (a_1, \ldots, a_M)$, $r = (r_1, \ldots, r_M)$ and $h = \max_{j=1,\ldots,M} r_j$. For details and the corresponding results for matrix-valued equations we refer to [8].

In a problem, both the a_k and the r_k are generally parameters that are not known exactly. It is important, therefore, to know the effect that small changes in the parameters will have on the stability of the zero solution of the difference equation

$$D(r, a)x_t = 0,$$

where $x_t(\theta) = x(t + \theta)$, $-h \le \theta \le 0$. Equivalently, we can write

$$x(t) - \sum_{j=1}^{M} a_j x(t - r_j) = 0. \tag{3.2}$$

With the difference equation (3.2), we can associate a semigroup of operators (see [7]). Let $C = C[-h, 0]$ be the space of continuous functions from $[-h, 0]$ to $\mathbb{R}$ with the usual supremum topology. If we define the closed subspace $C_D \subset C$ as follows

$$C_D = \{\varphi \in C : \varphi(0) - \sum_{j=1}^{M} a_j\varphi(-r_j) = 0\},$$

then for any $\varphi \in C_D$, there is a unique solution $x(t;\varphi)$ of (3.2) which is continuous for $t \geq -h$ and coincides with φ on $[-h,0]$. This implies that translation along the solutions of (3.2)

$$T_{D(r,a)}(t)\varphi = x_t(\,\cdot\,;\varphi) \tag{3.3}$$

defines a $\mathcal{C}_0$-semigroup on C_D.

Let $r_\sigma(T_{D(r,a)}(1))$ denote the spectral radius of $T_{D(r,a)}(1)$ and let $r_e(T_{D(r,a)}(1))$ denote the radius of the essential spectrum of $T_{D(r,a)}(1)$. We would like to study the dependence of $r_\sigma(T_{D(r,a)}(1))$ upon the vector-valued parameters r and a. Variations in a lead to compact perturbations of the semigroup $T_{D(r,a)}(1)$, whereas variations in r lead to bounded perturbations which are not compact. So, according to the discussion in the introduction, it is this fact which is the source of the strange behaviour that can occur from variations in the delays.

The next result is due to Henry [10].

Theorem 3.1 *Let $T_{D(r,a)}(t)$ be the $\mathcal{C}_0$-semigroup associated with* (3.2). *The spectral radius of $T_{D(r,a)}(1)$ equals the radius of the essential spectrum of $T_{D(r,a)}(1)$. Furthermore, if the components of r are rationally independent, then*

$$r_\sigma(T_{D(r,a)}(1)) = e^{\rho_0}, \tag{3.4}$$

where ρ_0 is the solution to the equation

$$\sum_{j=1}^{M} |a_j| e^{-\rho_0 r_j} = 1. \tag{3.5}$$

and the spectral radius of $T_{D(r,a)}(1)$ is continuous at r.

The proof of (3.4) follows from basic properties of almost periodic functions and Kronecker's theorem (see [8] for further details and references).

Note that for difference equations, the spectral radius equals the radius of the essential spectrum of $T_{D(r,a)}(1)$. Thus, if a difference equation is unstable, there are infinitely many unstable modes. The next corollary shows that this leads to severe conditions in order to have robustness of stability with respect to the delays.

Corollary 3.1 *Exponential stability of the difference equation* (3.2) *is preserved under small perturbations in the delays if and only if*

$$\sum_{j=1}^{M} |a_j| < 1. \tag{3.6}$$

The proof of this corollary is an easy consequence of Theorem 3.1. If the components of r are rationally independent, then exponential stability of (3.2) is equivalent to having $\rho_0 < 0$ in (3.4). Also, from the definition of ρ_0 in (3.5), we see that $\rho_0 < 0$ if and only if (3.6) is satisfied. Since the rationals are dense in $\mathbb{R}$, we obtain the result.

Example 3.1. (*Destruction of stability with small changes in delays.*) Suppose that $0 < r_1 \leq r_2 \leq h$ are constants and consider the difference equation

$$x(t) + \frac{1}{2}x(t - r_1) + \frac{1}{2}x(t - r_2) = 0. \tag{3.7}$$

Let $\Delta_{(r,a)}(z) = 1 + \frac{1}{2}e^{-r_1 z} + \frac{1}{2}e^{-r_2 z}$ denote the characteristic equation associated with (3.7). If $\Delta_{(r,a)}(\rho + i\tau) = 0$, then

$$\frac{1}{2}e^{-\rho r_1}e^{-i\tau r_1} + \frac{1}{2}e^{-\rho r_2}e^{-i\tau r_2} = -1.$$

So

$$2 \leq e^{-\rho r_1} + e^{-\rho r_2}$$

and it follows that $\rho \leq 0$. We shall compute the spectral radius of $T_{(r_1,r_2)}(t)$, the $\mathcal{C}_0$-semigroup associated with (3.7) for various values of (r_1, r_2).

For $(r_1, r_2) = (1, 2)$, the function $\Delta_{(1,2)}(z)$ is quadratic in e^{-z} and it is easy to show that the zeros are on the line

$$\operatorname{Re} z = -\log \sqrt{2}.$$

Thus,

$$r_\sigma(T_{(1,2)}(1)) = \frac{1}{\sqrt{2}} < 1,$$

which implies that $T_{(1,2)}(t)$ is uniformly exponentially stable.

For $(r_1, r_2) = (1 - \frac{1}{2k+1}, 2)$, where $k \geq 0$ is an odd integer. It is easy to verify that the equation $\Delta_{(1-1/(2k+1),2)}(z) = 0$ has solutions

$$\lambda_k = i(k + \frac{1}{2})\pi.$$

The corresponding eigenfunctions are solutions of (3.7) which are periodic of period $2\pi(\lambda_k)^{-1}$. This implies, for all odd integers k, that

$$r_\sigma(T_{(1-\frac{1}{2k+1},2)}(1)) = 1.$$

Finally, for (r_1, r_2) rationally independent, we can apply Theorem 3.1 to conclude that

$$r_\sigma(T_{(r_1,r_2)}(1)) = 1.$$

This example illustrates that $r_\sigma(T_{(r_1,r_2)}(1))$ is not continuous with the parameters r_1 and r_2.

Example 3.2. In this example we shall analyse the zeros of equation (2.5)

$$\Delta_r(z) = 1 + ke^{-zr} - ke^{-2a}e^{-z(2+r)} + e^{-2a}e^{-2z}, \tag{3.8}$$

where $a \geq 0, k > 0$ are constants. This equation arises as the characteristic equation of a difference equation with three delays

$$x(t) + kx(t - r) - ke^{-2a}x(t - (2 + r)) + e^{-2a}x(t - 2) = 0. \tag{3.9}$$

If $r = 0$, then $\Delta_0(z) = 0$ if and only if

$$e^{-2(z+a)} = \frac{k+1}{k-1}.$$

Therefore, for every $k > 0$,

$$\operatorname{Re} z + a = -\frac{1}{2} \log\left|\frac{k+1}{k-1}\right| < 0. \tag{3.10}$$

So, if $r = 0$, then the zeros of $\Delta_0(z)$ are in the left half plane $\{z \in \mathbb{C} : \operatorname{Re} z < 0\}$ for every $k > 0$.

If $r > 0$, then $\Delta_r(z) = 0$ if and only if

$$ke^{-zr} = \frac{e^{-2a}e^{-2z} + 1}{e^{-2a}e^{-2z} - 1}. \tag{3.11}$$

Since

$$\left|\frac{e^{-2a}e^{-2z} + 1}{e^{-2a}e^{-2z} - 1}\right| \geq \frac{1 - e^{-2a}e^{-2\operatorname{Re} z}}{1 + e^{-2a}e^{-2\operatorname{Re} z}},$$

we obtain that the zeros of $\Delta_r(z)$ are in the left half plane $\{z \in \mathbb{C} : \operatorname{Re} z < 0\}$ for every

$$k \leq \frac{1 - e^{-2a}}{1 + e^{-2a}}.$$

Next we consider the case that r is irrational. To find the supremum of the real parts of the zeros of $\Delta_r(z)$, $\rho_0 = \sup\{\operatorname{Re} \lambda : \Delta_r(\lambda) = 0\}$, we apply Theorem 3.1. So we have to solve, see (3.5), the following equation for ρ_0

$$1 = ke^{-\rho_0 r} + ke^{-2a}e^{-\rho_0(2+r)} + e^{-2a}e^{-2\rho_0}.$$

Or, equivalently,

$$ke^{-\rho_0 r} = \frac{1 - e^{-2(\rho_0+a)}}{1 + e^{-2(\rho_0+a)}}. \tag{3.12}$$

As a consequence, for any $\rho_0 > 0$, if $k > k_0(\rho_0)$, then we can find an $r = r(\rho_0, k) > 0$ such that (3.12) is satisfied and $\Delta_r(z)$ has zeros in the right half plane $\{z \in \mathbb{C} : \operatorname{Re} z > 0\}$. In particular, if $k > 1$, this implies that we can choose a ρ_0 as large as we want and then choose an r sufficiently small so that (3.12) is satisfied.

Finally, we can analyze the dependence of $\rho_0(r)$ on r as $r \to \infty$. We see that $\rho_0'(r) < 0$ for all r, $\rho_0(r) \to 0$ as $r \to \infty$, $\rho_0(r)r \to c$, $c \geq 0$, where $e^{-c} = k_0/k$ provided that $k \geq k_0$. As a consequence, we can obtain slow exponential growth for the delay r large.

The following example illustrates a general fact for matrix-valued difference equations. Stabilizing an unstable difference equation by feedback is always sensitive to variations in the delays. See [8] for a precise statement and a proof; also see [14] for related results.

Example 3.3. (*Feedback control can be sensitive to delay.*) Consider the system

$$x(t) - a_1x(t - r_1) - a_2x(t - r_2) = 0, \tag{3.13}$$

where r_1 and r_2 are rationally independent. According to Corollary 3.1, this equation is unstable if and only if

$$|a_1| + |a_2| \geq 1.$$

Given $|a_1| + |a_2| \geq 1$, we can apply a feedback control to stabilize (3.13)

$$x(t) - (a_1 + f_1)x(t - r_1) - (a_2 + f_2)x(t - r_2) = 0. \tag{3.14}$$

This system is exponentially stable if and only if

$$|a_1 + f_1| + |a_2 + f_2| < 1.$$

Suppose now that the feedback control cannot be applied instantaneously and that there is a small time delay in the feedback. This leads to the following equation

$$x(t) - a_1 x(t - r_1) - a_2 x(t - r_2) - f_1 x(t - r_1 - \epsilon_1) - f_2 x(t - r_2 - \epsilon_2) = 0. \tag{3.15}$$

We claim that, although (3.15) is exponentially stable for $\epsilon_1 = \epsilon_2 = 0$, there is a sequence $(\epsilon_1^j, \epsilon_2^j)$ tending to zero so that (3.15) is exponentially unstable. To prove the claim, we choose ϵ_1 and ϵ_2 such that r_1, r_2, $r_1 + \epsilon_1$ and $r_2 + \epsilon_2$ are rationally independent. From Corollary 3.1 it follows that (3.15) is always exponentially unstable, since

$$|a_1| + |f_1| + |a_2| + |f_2| > |a_1| + |a_2| \geq 1.$$

4 Delayed boundary control in hyperbolic equations

Equipped with the results from Section 3 we can now continue with our discussion of the one dimensional linear wave equation (2.1)-(2.2). For convenience, we repeat the equations

$$w_{tt} + 2aw_t - w_{xx} + bw = 0, \quad 0 < x < 1, \quad t > 0,$$

with initial data $w(x, 0) = w_0(x)$ and $w_t(x, 0) = w_1(x)$ and with the boundary conditions

$$w(0, t) = 0, \quad w_x(1, t) = -kw_t(1, t - r),$$

where $a \geq 0, k > 0, r \geq 0$ and b are constants.

We have seen that the radius of the essential spectrum of the semigroup $S_{r,k}(t)$ associated with system (2.1)-(2.2) is given by

$$r_e(S_{r,k}(t) = e^{\rho_0 t},$$

where ρ_0 is determined by the supremum of the real parts of the solutions of (2.5). The roots of this equation have been studied in Example 3.2 and as a result we have the following theorem.

Theorem 4.1 *Consider the boundary stabilization problem* $(2.1)-(2.2)$. *For* $a \geq 0$ *fixed, the dependence of the solutions upon* k *and* r *is as follows*

(i) If $r = 0$, $k > 0$, *then the problem* (2.1)-(2.2) *is uniformly exponential stabilizable.*

(ii) If $r > 0$ *and* $0 < k < (1 - e^{-2a})/(1 + e^{-2a})$, *then the problem* (2.1)-(2.2) *is still uniformly exponential stabilizable.*

(iii) If $k > (1 - e^{-2a})/(1 + e^{-2a})$ *then there exists a dense set of* $r > 0$ *such that the problem* (2.1)-(2.2) *is exponentially unstable.*

(iv) If $k > 1$, *then we can choose a* ρ *as large as we want and then choose* r *sufficiently small so that there is a solution of problem* (2.1)-(2.2) *which becomes unbounded at the rate* $e^{\rho t}$ *as* $t \to \infty$.

As a consequence of this theorem, it follows that problem (2.1)-(2.2) could have been stabilized with a control which involves no delay and then there are arbitrarily small delays in the control which lead to destabilization. These remarks are contained in [3] with a somewhat different proof.

The above analysis shows that boundary stabilization in the wave equation with no delay leads to very undesirable behavior when the control is implemented with a small delay. So, the main conclusion of this example (and the abstract result that we present in the next section) is that one should try to avoid to stabilize systems with infinitely many unstable modes. Instead one should put more effort to develop models that put less emphasis on the high frequency modes (the eigenvalues and eigenvectors corresponding with large imaginary part). It has been noted in [18, 19] that such an approach is indeed possible. In [8] two illustrative examples are given that are characteristic for the direction in which we hope to further develop the theory (see also [9]).

5 The role of the essential spectrum

We conclude this paper with a result that will appear in [9] and clarifies the role of the essential spectrum.

Let X be a Banach space and $A(X \to X)$ be the infinitesimal generator of a $\mathcal{C}_0$-semigroup of operators $T_A(t) : X \to X$.

Consider the control problem

$$\frac{d}{dt}x(t) = Ax(t) + Bu(t), \quad x(0) = x_0 \in X$$

and suppose that the system is exponentially stabilizable, i.e., if we close the loop by $u(t) = Fx(t)$ then the resulting system

$$\frac{d}{dt}x(t) = Ax(t) + BFx(t), \quad x(0) = x_0 \in X \tag{5.1}$$

is exponentially stable.

We then have the following theorem.

Theorem 5.1 *Let X be a Banach space and $A(X \to X)$ be the infinitesimal generator of a $\mathcal{C}_0$-semigroup of operators $T_A(t) : X \to X$. If there exists $\lambda \notin \sigma(A)$ such that the resolvent $(\lambda I - A)^{-1}$ of A at λ is compact, then the following statements hold*

(i) If $r_e(T_A(1)) < 1$, then the stability is insensitive to small delays in the feedback loop, i.e., if the control system (5.1) is exponentially stable, then there exists an $\epsilon_0 > 0$ such that for every ϵ, $0 < \epsilon < \epsilon_0$, the system

$$\frac{d}{dt}x(t) = Ax(t) + BFx(t - \epsilon), \quad x(0) = x_0 \in X$$

is exponentially stable.

(ii) If $r_e(T_A(1)) \geq 1$, then the control system (5.1) is sensitive to small delays in the feedback, i.e, there exists a sequence $\epsilon_j \to 0$ such that

$$\frac{d}{dt}x(t) = Ax(t) + BFx(t - \epsilon_j), \quad x(0) = x_0 \in X$$

is not uniformly exponentially stable.

The proof will be given in [9] is based on similar arguments as those used in the proof of the corresponding result for difference equations in [8].

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