MIXED-TYPE FUNCTIONAL DIFFERENTIAL EQUATIONS, HOLOMORPHIC FACTORIZATION, AND APPLICATIONS

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In this note we outline recent developments in the study of mixed-type, or forward-backward, functional differential equations. Such equations arise naturally in various contexts, for example, in the study of traveling waves in discrete spatial media such as lattices. Since initial value problems for mixed-type equations are not in general well-posed, it is our goal to decompose solutions of such equations as sums of "forward" solutions and "backward" solutions. We show that for autonomous equations, the set of all forward solutions defines a semigroup which can be realized by a retarded functional differential equation except for possibly finitely many modes, and similarly for the set of backward solutions as an advanced functional differential equation. Holomorphic factorizations play a crucial role in our results. We also study the boundary value problem associated to an interval $[-\tau, \tau]$ of finite length, with emphasis on the asymptotic case of large τ.

1. Introduction

Consider the simplest scalar mixed-type functional differential equation

$$\dot{x}(t) = ax(t) + bx(t-1) + cx(t+1) \tag{1.1}$$

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defined on the real line or on a subinterval of the line. In this paper we discuss equation (1.1) and its inhomogeneous counterpart

$$\dot{x}(t) = ax(t) + bx(t-1) + cx(t+1) + h(t), \tag{1.2}$$

to illustrate and motivate our recent results in [14] for a more general class of linear functional differential equations of mixed type. In [14] are found results for the homogeneous autonomous system

$$\dot{x}(t) = \int_{-q}^{p} d\eta(\theta)\, x(t+\theta),$$

where the solution $x(t)$ is an M-vector and $d\eta(\theta)$ is an $M \times M$ matrix of finite Lebesgue-Stieltjes measures on $[-q, p]$, following [7]. Some of our results in [14] also cover nonautonomous systems, and in particular the scalar nonautonomous equation

$$\dot{x}(t) = a(t)x(t) + b(t)x(t-1) + c(t)x(t+1), \tag{1.3}$$

and vector-valued generalizations. We refer to [14] for a complete description and proofs of our results.

Mixed-type equations as above, both linear and nonlinear, occur naturally in problems of traveling waves in discrete spatial media such as lattices. Earlier papers by Chi, Bell, and Hassard[5] and by Keener[10] were followed by many others which developed the basic theory; see, for example, [4, 11, 14] and the references given there. Mixed-type equations also arise in spatially nonlocal equations of convolution type[2], in spatial discretizations of shock-wave problems[3], and in singularly perturbed time-delay problems[6] again in convolution form. Applications from optimal control theory are discussed and analyzed by Rustichini[15,16]. Some very general classes of problems related to ours, namely elliptic partial differential equations with spatially nonlocal terms, along with applications, are to be found in the book of Skubachevskiĭ[17].

As an illustration of how mixed-type equations arise, consider the system

$$\dot{u}_{i,j} = u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} - f(u_{i,j}), \qquad (i,j) \in \mathbf{Z}^2, \tag{1.4}$$

which is a lattice differential equation. In particular, (1.4) is an infinite system of ordinary differential equations modeled on the lattice $\mathbf{Z}^2$. Here $f : \mathbf{R} \to \mathbf{R}$ is a given nonlinearity. To obtain traveling wave solutions of

(1.4) moving in a given direction $(\theta_1, \theta_2) \in S^1$ with speed c, where S^1 is the unit circle in $\mathbf{R}^2$, one makes the Ansatz

$$u_{i,j}(t) = \psi(i\theta_1 + j\theta_2 - ct)$$

for an unknown function $\psi : \mathbf{R} \to \mathbf{R}$, and one sees that ψ must satisfy the differential-difference equation

$$\begin{aligned} -c\psi'(\xi) &= \psi(\xi + \theta_1) + \psi(\xi - \theta_1) + \psi(\xi + \theta_2) \\ &\qquad + \psi(\xi - \theta_2) - 4\psi(\xi) - f(\psi(\xi)), \qquad \xi \in \mathbf{R}. \end{aligned} \tag{1.5}$$

Typically, boundary conditions are also imposed, for example $\psi(\pm\infty) = \pm 1$ where $f(\pm 1) = 0$ is assumed, which here means that ψ is a heteroclinic orbit for (1.5). In [13] an extensive study of this problem is given for f of bistable type, and the analysis there rests upon key results for the associated (nonautonomous) linear problem. In this direction, theorems on linear operators $\Lambda : W^{1,p}(\mathbf{R}, \mathbf{C}^M) \to L^p(\mathbf{R}, \mathbf{C}^M)$ of the form

$$(\Lambda x)(t) = \dot{x}(t) - \sum_{j=1}^{N} A_j(t)x(t + r_j)$$

are given in [12]. A basic result of [12] is that if the coefficient matrices possess limits $A_j(\pm\infty)$ and if the limiting autonomous problems at $\pm\infty$ are hyperbolic, then the operator Λ is Fredholm. Further information about the exponential decay of solutions is given there. In addition, there is given an effective method for computing the Fredholm index of Λ in terms of a crossing number.

In [14] we approach such linear problems from a more dynamic perspective. There we decompose solutions of mixed-type equations as sums of "forward" solutions on an interval $[t_1, \infty)$, and "backward" solutions on an interval $(-\infty, t_2]$, obtaining therewith the structures and estimates of exponential dichotomies. Such a decomposition is needed to represent solutions with a variation of constants formula, in order to obtain invariant (stable, unstable, center) manifolds for nonlinear equations such as (1.5). We are following here the pioneering work of Rustichini[15,16], who used spectral methods to obtain such decompositions for autonomous systems, although the techniques we employ are quite different and rely on Fredholm properties of the operator Λ. In particular, our techniques apply also in the nonautonomous case, for example to the equation (1.3), and to more general nonautonomous systems. In addition, we obtain extensive and detailed information about the spaces Q and P on which the forward and backward solutions reside, namely information about the kernels, ranges, and Fredholm indices of certain associated operators.

We remark that Härterich, Sandstede, and Scheel[8] have independently and simultaneously extended the spectral decomposition approach to nonautonomous equations, to obtain results on exponential dichotomies.

An important issue is to understand solutions of equations such as (1.3) on finite intervals with appropriately chosen boundary conditions. This is particularly relevant in doing numerical studies, as one may wish to consider such equations on intervals $[-\tau, \tau]$ where τ is finite but large. The interval $[-\tau, \tau]$ arises as a numerical approximation to the line $\mathbf{R}$, and one goal of our analysis is to obtain norm estimates on the inverse $(\Lambda^\tau)^{-1}$ of the finite interval operator Λ^τ which are uniform in τ. In the case of equation (1.3) the operator $\Lambda^\tau : W_0^{1,p}(-\tau, \tau) \to L^p(-\tau, \tau)$ is given by

$$(\Lambda^\tau x)(t) = \dot{x}(t) - a(t)x(t) - b(t)x(t-1) - c(t)x(t+1), \ |t| \leq \tau, \tag{1.6}$$

where $W_0^{1,p}(-\tau, \tau) \subseteq W^{1,p}(-\tau, \tau)$ is the subspace of $x \in W^{1,p}(-\tau, \tau)$ for which $x(-\tau) = x(\tau) = 0$. In evaluating (1.6) we extend $x(t) = 0$ outside $[-\tau, \tau]$, namely

$$x(t) = 0, \qquad t \in [-\tau - 1, -\tau] \cup [\tau, \tau + 1]. \tag{1.7}$$

Typically, with $p = \infty$, one seeks an estimate of the form

$$\|\Lambda^\tau x\|_{L^\infty(-\tau,\tau)} \geq K \|x\|_{W_0^{1,\infty}(-\tau,\tau)} \tag{1.8}$$

where K is independent of (large) τ, that is, a uniform estimate $\|(\Lambda^\tau)^{-1}\| \leq K^{-1}$ on the inverse operator. In general the estimate (1.8) does not hold, even in the constant coefficient case. However, it can be obtained under appropriate assumptions which include conditions on the limiting coefficients $a(\pm\infty)$, $b(\pm\infty)$, and $c(\pm\infty)$. Even if (1.8) fails, such an estimate does hold under quite general conditions for x in an appropriately chosen finite-codimension subspace of $W_0^{1,\infty}(-\tau, \tau)$.

The outline of this paper is as follows. In Section 2 we recall the dynamical systems approach to functional differential equations. In Section 3 we obtain the exponential splitting for equation (1.1). In Section 4 we present a holomorphic factorization of the characteristic function of equation (1.1) which realizes the forward and backward semigroups corresponding to the exponential splitting by, respectively, a retarded and advanced functional differential equation. Finally, in Section 5 we discuss the finite-interval problem.

Generally, if $L : X \to Y$ is a linear operator between two Banach spaces, we let $\mathcal{K}(L) \subseteq X$ and $\mathcal{R}(L) \subseteq Y$ respectively denote the kernel and range of L. If L is a Fredholm operator we let $\mathrm{ind}(L)$ denote its Fredholm index.

2. A Functional Analytic Approach

A dynamical systems approach to the autonomous linear retarded functional differential equation

$$\dot{x}(t) = \int_{-1}^{0} d\eta(\theta)\, x(t+\theta), \tag{2.1}$$

where η is a matrix of Lebesgue-Stieltjes measures, is to associate to (2.1) a semigroup $T(t)$ acting on the space of initial data. The semigroup is defined by the time evolution of segments of solutions. In the scalar case $T(t)$ acts on the Banach space $C[-1,0]$ of continuous functions $\varphi : [-1,0] \to \mathbf{C}$ with the sup norm $\|\varphi\|_{C[-1,0]} = \sup_{[-1,0]} |\varphi(\theta)|$ as follows. Letting $x(t;\varphi)$ denote the solution of (2.1) with initial condition $x(\theta) = \varphi(\theta)$ for $\theta \in [-1,0]$, one defines the state $x_t(\varphi) \in C[-1,0]$ of the solution at any $t \geq 0$ by $x_t(\varphi) = x_t(\theta;\varphi) = x(t+\theta;\varphi)$ for $\theta \in [-1,0]$. Setting

$$T(t)\varphi = x_t(\varphi), \qquad t \geq 0,$$

then defines a strongly continuous semigroup on $C[-1,0]$.

In the case of the mixed-type equation (1.1), however, the initial value problem is not in general well-posed, and translation along solutions does not generate a strongly continuous semigroup. Nevertheless it is still useful to consider $C[-1,1]$ as a state space for this equation and to define $x_t \in C[-1,1]$ by

$$x_t(\theta) = x(t+\theta), \qquad \theta \in [-1,1]. \tag{2.2}$$

The evolution of the state x_t is still given by an abstract ordinary differential equation in $C[-1,1]$, which, for equation (1.1), takes the form

$$\frac{du}{dt} = Au, \qquad u(0) = \varphi \in C[-1,1], \tag{2.3}$$

where $A : \mathcal{D}(A) \to C[-1,1]$ is an unbounded operator defined by

$$\begin{aligned} &A\varphi = \varphi', \\ &\mathcal{D}(A) = \{\varphi \in C^1[-1,1] \mid \varphi'(0) = a\varphi(0) + b\varphi(-1) + c\varphi(1)\}. \end{aligned} \tag{2.4}$$

The solutions of (1.1) are in one-to-one correspondence with the solutions of the infinite-dimensional ordinary differential equation (2.3), the correspondence being given by $u(t)(\theta) = x(t+\theta)$. However, if $c \neq 0$ then the operator A defined in (2.4) is not the infinitesimal generator of a semigroup $T(t)$ because it does not satisfy the Hille-Yosida conditions. This can be seen from the fact that the spectrum of A, which is given by $\sigma(A) = \{\lambda \in \mathbf{C} \mid \Delta(\lambda) = 0\}$ where

$$\Delta(\lambda) = \lambda - a - be^{-\lambda} - ce^{\lambda} \tag{2.5}$$

denotes the characteristic function associated with (1.1), is not contained in any left half-plane of the complex plane.

3. The Exponential Splitting

Since equation (1.1) does not generate a dynamical system, it is important to define precisely the notion of a solution of this equation.

Definition. A solution of equation (1.1) on an interval $[t_1, t_2] \subseteq \mathbf{R}$ is a continuous function $x : [t_1 - 1, t_2 + 1] \to \mathbf{C}$ which is absolutely continuous on $[t_1, t_2]$ and which satisfies equation (1.1) for almost every $t \in [t_1, t_2]$.

The analogous definition of a solution on an unbounded interval $[t_1, \infty)$, $(-\infty, t_2]$, or on $\mathbf{R}$ is taken, as well as solutions of other equations such as (1.2), (1.3), and their generalizations. We shall say that a solution on an interval I is bounded if $\sup_I |x(t)| < \infty$.

Throughout this section we assume that equation (1.1) is **hyperbolic**, that is,

$$\Delta(i\sigma) \neq 0 \ \text{ for every } \ \sigma \in \mathbf{R},$$

where Δ is given by (2.5). It turns out that this is not a significant restriction for much of our analysis, as one can shift the roots of Δ by a change of variables $x(t) = e^{\mu t} y(t)$ to obtain a hyperbolic equation, and then read off the relevant results for the shifted equation.

It was shown in [12, Theorem A] that for equation (1.1), under the assumption of hyperbolicity, there are no nontrivial bounded solutions on the real line. On the other hand, there are many bounded solutions on a half-line. In this spirit we decompose solutions of (1.1) into forward and backward components by means of the spaces

$$\mathcal{P} = \{x : (-\infty, 1] \to \mathbf{C} \mid x \text{ is a bounded solution of (1.1) on } (-\infty, 0]\},$$

$$\mathcal{Q} = \{x : [-1, \infty) \to \mathbf{C} \mid x \text{ is a bounded solution of (1.1) on } [0, \infty)\}.$$

Recalling the notation (2.2), in particular for $t = 0$, let us set

$$P = \{\varphi \in C[-1,1] \mid \varphi = x_0 \text{ for some } x \in \mathcal{P}\},$$

$$Q = \{\varphi \in C[-1,1] \mid \varphi = x_0 \text{ for some } x \in \mathcal{Q}\},$$

which are the initial states for solutions in $\mathcal{P}$ and $\mathcal{Q}$. A solution $x \in \mathcal{P}$ associated to an initial state $x_0 = \varphi \in P$ is called a **left prolongation** of φ, and we take the analogous definition of **right prolongation** of an element of Q. We note that if $x \in \mathcal{P}$ and $t \leq 0$ is fixed, then the translated function $s \to x(s+t)$ for $s \leq 1$ also belongs to $\mathcal{P}$, and so $x_t \in P$. The corresponding property holds for $\mathcal{Q}$, for $t \geq 0$. The first result provides exponential estimates for such solutions.

Theorem 1. *There exist $K, \alpha > 0$ such that*

$$\begin{aligned} |x(t)| + |\dot{x}(t)| &\leq K e^{\alpha t} \|x_0\|_{C[-1,1]}, \qquad t \leq 0, \\ |y(t)| + |\dot{y}(t)| &\leq K e^{-\alpha t} \|y_0\|_{C[-1,1]}, \qquad t \geq 0, \end{aligned} \tag{3.1}$$

for every $x \in \mathcal{P}$ and $y \in \mathcal{Q}$.

Note that (3.1) implies that the solution x or y is uniquely determined by its initial state x_0 or y_0, namely by its restriction to $[-1,1]$. In fact this uniqueness follows directly from the hyperbolicity assumption. Indeed, suppose that $x \in \mathcal{P}$ is a nonzero function which is a left prolongation of zero. That is, suppose $x(t)$ is bounded and satisfies (1.1) on $(-\infty, 0]$, that $x(t) = 0$ for every $t \in [-1,1]$, but that $x(t_0) \neq 0$ for some $t_0 < -1$. Extend x to be identically zero on the interval $[1, \infty)$. One checks that the extended function x is also a solution on $[0, \infty)$, and thus is a nontrivial bounded solution of (1.1) on $\mathbf{R}$. However, as noted above, such solutions do not exist.

Sketch of the Proof of Theorem 1. We sketch the proof of the estimate for x in (3.1) and refer to [14, Section 3] for details. It is enough to work with the space $\mathcal{P}$ and to show the estimate

$$|x(t)| \leq K e^{\alpha t} \|x_0\|_{C[-1,1]}, \qquad t \leq 0, \tag{3.2}$$

for every $x \in \mathcal{P}$, as the estimate in (3.1) involving the derivative will then follow from the differential equation (1.1). The constants K in (3.1) and (3.2) will in general be different, but they should not depend on the choice of $x \in \mathcal{P}$.

We first show there exists $\tau > -1$ such that

$$|x(t)| \leq \frac{1}{2} \sup_{(-\infty,1]} |x(s)|, \qquad t \leq -\tau, \tag{3.3}$$

for every $x \in \mathcal{P}$. Observe that such an estimate immediately implies that

$$|x(t)| \leq \frac{1}{2^m} \sup_{(-\infty,1]} |x(s)|, \qquad t \leq -m(\tau+1)+1,$$

for every $m \geq 1$, by repeatedly applying (3.3) to the translates of x. Assuming that τ satisfying (3.3) does not exist, we see there exists a sequence $\tau^n \to \infty$ and a sequence $x^n \in \mathcal{P}$ such that

$$|x^n(-\tau^n)| \geq \frac{1}{2}, \qquad \text{with the normalization } \sup_{(-\infty,1]} |x^n(s)| = 1.$$

Define $z^n(t) = x^n(t - \tau^n)$ and observe that z^n is a solution of (1.1) on the interval $(-\infty, \tau^n]$. On every compact subinterval of $\mathbf{R}$ the sequence of functions z^n is uniformly bounded and equicontinuous for large n, and so for some subsequence one has the limit $z^n(t) \to z(t)$ uniformly on compact intervals, by Ascoli's theorem. One easily checks that z is a solution of (1.1) on the line. Moreover $z(t)$ is uniformly bounded there, and is a nontrivial solution as $|z(0)| \geq \frac{1}{2}$. However, this contradicts the fact that (1.1) has no nontrivial bounded solutions on the line.

Having shown the existence of τ as above, we see with (3.3) that in order to conclude (3.2), and hence conclude the theorem, it is enough to prove that there exists $K > 0$ (possibly different from the K in (3.2)) such that

$$|x(t)| \leq K \|x_0\|_{C[-1,1]}, \qquad t \leq 1, \tag{3.4}$$

for every $x \in \mathcal{P}$. We prove (3.4) by contradiction. If (3.4) were false for every K, then there would exist sequences $K^n \to \infty$ and $x^n \in \mathcal{P}$ such that

$$\sup_{(-\infty,1]} |x^n(t)| = K^n \|x_0^n\|_{C[-1,1]} = 1, \tag{3.5}$$

where the second equality in (3.5) is a normalization. Then there exists some $t = -\tau^n$ at which the maximum $|x^n(-\tau^n)| = 1$ of $|x^n(t)|$ is attained, where clearly $-\tau^n \in [-\tau, -1]$ with τ as above. By passing to a subsequence, we take the limit $x^n(t) \to x(t)$ uniformly on compact subsets of $(-\infty, 1]$, where we note here that $\|x_0^n\|_{C[-1,1]} = (K^n)^{-1} \to 0$ and so $x_0 = 0$ for the limit. We may also assume the limit $\tau^n \to \tau^0$ exists as τ^n is a bounded sequence. We have that the limit x is a solution of equation (1.1) on $(-\infty, 0]$, and it is nontrivial in light of the fact that $|x(-\tau^0)| = 1$. But then x is a nontrivial left prolongation of zero, which we observed earlier is impossible. ■

For any $\psi \in Q$ let $y \in \mathcal{Q}$ denote the unique right prolongation of this element and define the operator $T_Q(t) : Q \to Q$ for any $t \geq 0$ by $T_Q(t)\psi = y_t$. In a similar fashion define $T_P(t)\varphi = x_t$ for $t \leq 0$ where $x \in \mathcal{P}$ is the left prolongation of $\varphi \in P$. We wish to think of the subspaces P and Q as surrogates for the phase spaces $C[0,1]$ and $C[-1,0]$ for advanced and retarded functional differential equations, respectively. In order to make this notion precise, we define the maps

$$\pi^+ : C[-1,1] \to C[0,1], \qquad \pi^+\varphi = \varphi|_{[0,1]},$$

$$\pi^- : C[-1,1] \to C[-1,0], \qquad \pi^-\varphi = \varphi|_{[-1,0]},$$

given by restricting any $\varphi \in C[-1,1]$ to the interval $[0,1]$ or $[-1,0]$ as indicated. Let π_P^+ and π_Q^+ denote the restrictions $\pi_P^+ = \pi^+|_P$ and $\pi_Q^+ = \pi^+|_Q$ of the operator π^+ to the subspaces $P \subseteq C[-1,1]$ and $Q \subseteq C[-1,1]$, respectively. Define π_P^- and π_Q^- similarly.

The following result provides precise information on which $x_0 \in P$ and $y_0 \in Q$ can occur as initial states. We refer to [14, Section 3] for a proof, although we mention that Theorem 1 above plays a key role in the proof.

Theorem 2. *Let the sets P and Q be defined as above. Then the following hold.*

(i) *P and Q are closed subspaces of $C[-1,1]$.*
(ii) *$P \oplus Q = C[-1,1]$.*
(iii) *The operators $\pi_P^- : P \to C[-1,0]$ and $\pi_Q^+ : Q \to C[0,1]$ are compact.*
(iv) *The operators $\pi_P^+ : P \to C[0,1]$ and $\pi_Q^- : Q \to C[-1,0]$ are Fredholm, and*

$$\operatorname{ind}(\pi_P^+) + \operatorname{ind}(\pi_Q^-) = -1$$

for the Fredholm indices of these operators.
(v) *The operators $T_Q(t) : Q \to Q$ and $T_P(-t) : P \to P$ defined as above by translation along the unique prolongation for $t \geq 0$ are exponentially stable strongly continuous semigroups.*

We remark that in fact Theorems 1 and 2 can be extended to a general class of nonautonomous equations using the framework of linear skew product semiflows. See [14] for the details.

Items (i) and (ii) of Theorem 2 can be viewed as a spectral splitting for the unbounded operator A defined in (2.4) which is associated to equation (1.1). To explain this connection, consider, generally, a closed unbounded linear operator $A : \mathcal{D}(A) \subseteq X \to X$ on a Banach space X. Suppose there is a direct sum decomposition $X = P \oplus Q$ of X which reduces A, that is, for which A maps $\mathcal{D}(A) \cap P$ into P and $\mathcal{D}(A) \cap Q$ into Q. Letting $A_P = A|_{\mathcal{D}(A)\cap P}$ and $A_Q = A|_{\mathcal{D}(A)\cap Q}$ denote the restrictions of A to these spaces, we say that A is **exponentially dichotomous** if the operators $-A_P$ and A_Q are infinitesimal generators of exponentially stable semigroups $T_P(-t)$ and $T_Q(t)$, respectively. In the case of an exponentially dichotomous operator, the splitting $X = P \oplus Q$ turns out to be unique and the associated projections onto P and Q are called the separating projections for A. We refer to the paper of Bart, Gohberg, and Kaashoek[1] and as well to [9] for details and further information. For equation (1.1) this spectral decomposition was already obtained by Rustichini[15,16]. We observe, directly from Theorem 1, the exponential estimates $\|T_Q(t)\| \leq K'e^{-\alpha t}$ and $\|T_P(-t)\| \leq K'e^{-\alpha t}$ on the operator norms for $t \geq 0$, where $K' = Ke^{\alpha}$, to give item (v) of Theorem 2.

The interpretation of item (iv) of Theorem 2 is that P and Q are surrogates for $C[0,1]$ and $C[-1,0]$, as remarked above. The Fredholm operators π_P^+ and π_Q^- need not be isomorphisms, however, and can have nonzero Fredholm indices. In particular, it is possible for there to exist nontrivial elements in the kernels $\mathcal{K}(\pi_P^+) \subseteq P$ or $\mathcal{K}(\pi_Q^-) \subseteq Q$ of these operators. A nontrivial element $\varphi \in \mathcal{K}(\pi_P^+)$ is characterized as $x_0 = \varphi \neq 0$ where $x \in \mathcal{P}$ satisfies $\pi^+ x_0 = 0$. That is, $x : (-\infty, 1] \to \mathbf{C}$ is a bounded solution on $(-\infty, 0]$ which vanishes identically on $[0,1]$ but does not vanish identically on $[-1,1]$. Note that if we extend such a function x to vanish identically on $[0,\infty)$, then x is **not** a solution on $\mathbf{R}$ if $b \neq 0$, as it does not satisfy equation (1.1) on $[0,1]$. Thus while left prolongations are uniquely determined by the values of φ in $[-1,1]$, they need not be determined by the values of φ in $[0,1]$, that is, they are not in general determined by $\pi^+\varphi$. On the other hand, $\pi^+\varphi$ does determine a finite-dimensional set of left prolongations, which is a reflection of the finite-dimensionality of the kernel $\mathcal{K}(\pi_P^+)$.

The elements of the kernels $\mathcal{K}(\pi_P^+)$ and $\mathcal{K}(\pi_Q^-)$ play a crucial role in understanding equation (1.1) on finite intervals $[t_1, t_2]$, particularly when the length $t_2 - t_1$ of the interval is large, as we shall see in Section 5. In order to understand the structure of the kernels $\mathcal{K}(\pi_P^+)$ and $\mathcal{K}(\pi_Q^-)$, we first turn to the question of how to realize the semigroups $T_P(-t)$ and $T_Q(t)$ by explicit equations.

4. Concrete Representations for $T_P(t)$ and $T_Q(t)$

The goal of this section is to provide concrete representations for the semigroups $T_Q(t)$ on Q and $T_P(-t)$ on P obtained in Section 3 from equation (1.1). Specifically, in Theorem 4 we show these semigroups can be realized by equations

$$\dot{x}(t) = \int_{-1}^{0} x(t+\theta)\, d\eta_-(\theta), \qquad \dot{x}(t) = \int_{0}^{1} x(t+\theta)\, d\eta_+(\theta), \tag{4.1}$$

that is, by a retarded functional differential equation and an advanced functional differential equation respectively, at least up to a finite-dimensional subspace, in a sense that is made precise. As usual, $d\eta_\pm(\theta)$ are Lebesgue-Stieltjes measures on the indicated intervals. The results in this section are a special case of much more general results developed in [14, Section 5]. For the time being, we do not need equation (1.1) to be hyperbolic, although this will be assumed later.

We mention that some of the general results as stated in [14] require an atomic condition, which for equation (1.1) means that $b \neq 0$ and $c \neq 0$. In fact, for at least this equation, these conditions are not needed for the results stated here.

We first obtain a factorization of the characteristic function Δ of equation (1.1) in terms of the characteristic functions $\Delta_\pm$ of the equations (4.1). The following result is a special case of [14, Theorem 5.2]. Its proof involves transform techniques, and theorems of Phragmén-Lindelöf type.

Theorem 3. *Let Δ be as in (2.5) and let $\lambda_0 \in \mathbf{C}$ be given. Then there exist finite Lebesgue-Stieltjes measures $d\eta_-(\theta)$ and $d\eta_+(\theta)$ on $[-1,0]$ and on $[0,1]$, respectively, such that the factorization*

$$(\lambda - \lambda_0)\Delta(\lambda) = \Delta_-(\lambda)\Delta_+(\lambda) \tag{4.2}$$

of Δ holds, where

$$\Delta_-(\lambda) = \lambda - \int_{-1}^{0} e^{\lambda\theta}\, d\eta_-(\theta), \qquad \Delta_+(\lambda) = \lambda - \int_{0}^{1} e^{\lambda\theta}\, d\eta_+(\theta). \tag{4.3}$$

If one has a second such factorization of Δ, say with $\widetilde{\lambda}_0 \in \mathbf{C}$ and with measures $d\widetilde{\eta}_\pm(\theta)$ and corresponding functions $\widetilde{\Delta}_\pm$, then

$$\frac{\Delta_-(\lambda)}{\widetilde{\Delta}_-(\lambda)} = \frac{(\lambda - \lambda_0)\widetilde{\Delta}_+(\lambda)}{(\lambda - \widetilde{\lambda}_0)\Delta_+(\lambda)} = r(\lambda) \tag{4.4}$$

where r is a rational function with $r(\infty) = 1$.

Conversely, suppose one has a particular factorization (4.2), (4.3). Let $\widetilde{\lambda}_0 \in \mathbf{C}$ *and a rational function* r *with* $r(\infty) = 1$ *be given. Suppose also* $\widetilde{\lambda}_0$ *and* r *have the property that both the functions*

$$\frac{\Delta_-(\lambda)}{r(\lambda)}, \qquad \frac{(\lambda - \widetilde{\lambda}_0)\Delta_+(\lambda)r(\lambda)}{(\lambda - \lambda_0)},$$

are entire. Then there exist measures $d\widetilde{\eta}_\pm(\theta)$ *as above such that (4.4) holds, where* $\widetilde{\Delta}_\pm$ *are given by (4.3) but with* $d\widetilde{\eta}_\pm(\theta)$ *in place of* $d\eta_\pm(\theta)$*.*

Finally, suppose that equation (1.1) is hyperbolic. To any factorization (4.2), (4.3) for which $\operatorname{Re}\lambda_0 \neq 0$ *associate the finite quantities*

$$n_- = \operatorname{card}\{\lambda \in \mathbf{C} \mid \Delta_-(\lambda) = 0 \ \textit{and} \ \operatorname{Re}\lambda > 0\},$$

$$n_+ = \operatorname{card}\{\lambda \in \mathbf{C} \mid \Delta_+(\lambda) = 0 \ \textit{and} \ \operatorname{Re}\lambda < 0\},$$

$$n_0 = \operatorname{card}\{\lambda \in \mathbf{C} \mid \lambda - \lambda_0 = 0 \ \textit{and} \ \operatorname{Re}\lambda > 0\},$$

with roots counted according to their multiplicity. Then the quantity

$$n_\sharp = n_+ - n_- + n_0 \tag{4.5}$$

is independent of such factorizations (4.2), (4.3).

As described in Theorem 3, the holomorphic factorization (4.2) is not unique. Indeed, the effect of the rational function r in that result is that arbitrary finite sets of roots of Δ_- and Δ_+ may be swapped, one for one. This is most transparent when $\widetilde{\lambda}_0 = \lambda_0$ is chosen. In that case, if λ_- and λ_+ are roots of Δ_- and Δ_+ respectively, then taking $r(\lambda) = (\lambda-\lambda_-)(\lambda-\lambda_+)^{-1}$ effects such a swap between λ_- and λ_+.

In general all but finitely many of the roots of Δ_- lie in the left half-plane, and all but finitely many of Δ_+ in the right half-plane. The integer $n_\sharp$ defined in (4.5) is an invariant of equation (1.1) and is a function of the number of roots of Δ_- and Δ_+ which lie in the "wrong" half-plane. As we shall see, the dimensions and codimensions, respectively, of the kernels and ranges of the Fredholm operators π_P^+ and π_Q^- are functions of $n_\sharp$.

The next theorem shows, roughly, that solutions of the associated retarded and advanced equations in (4.1), obtained from the holomorphic factorization in Theorem 3, are also solutions of (1.1), and conversely. This result is a combination of [14, Theorem 5.3 and Theorem 5.4]. Its proof, in part, involves results on the completeness of eigenfunctions, as are found for example in [18].

Theorem 4. *Consider any factorization of* Δ *as in the statement of Theorem 3. If* $\Delta_+(\lambda_0) = 0$ *then every solution of the first equation in (4.1) on* $[0, \infty)$ *is also a solution of equation (1.1). If* $\Delta_+(\lambda_0) \neq 0$ *then there exists a closed subspace* $\Gamma \subseteq C[-1, 0]$ *of codimension one which is positively invariant under the semiflow of the first equation in (4.1), such that any solution of this equation on* $[0, \infty)$ *in* Γ *is also a solution of equation (1.1).*

The corresponding results for $\Delta_-(\lambda_0)$ *and the second equation in (4.1) also hold.*

Now let $\mu \in \mathbf{R}$ *be such that all roots* λ *of the quotient function* $\Delta\Delta_-^{-1}$ *satisfy* $\operatorname{Re}\lambda > \mu$*. Then every solution of (1.1) on* $[0, \infty)$ *for which* $e^{-\mu t}x(t)$ *is bounded there is also a solution of the first equation in (4.1).*

The corresponding result for solutions of (1.1) on $(-\infty, 0]$ *and for the second equation in (4.1) also holds.*

We conclude this section with precise descriptions of the kernels and ranges of the Fredholm operators π_P^+ and π_Q^- of Theorem 2. In particular we give the dimensions of their kernels and the codimensions of their ranges. We assume that equation (1.1) is hyperbolic and we fix a factorization of Δ for which $\operatorname{Re}\lambda_0 \neq 0$. Next let $e_\pm$ be the polynomials

$$e_-(\lambda) = \prod_{\substack{\Delta_-(\gamma)=0 \\ \operatorname{Re}\gamma>0}} (\lambda - \gamma), \qquad e_+(\lambda) = \prod_{\substack{\Delta_+(\gamma)=0 \\ \operatorname{Re}\gamma<0}} (\lambda - \gamma),$$

constructed from these roots according to their multiplicities, and let

$$e_0(\lambda) = \begin{cases} \lambda - \lambda_0, & \operatorname{Re}\lambda_0 > 0, \\ \\ 1, & \operatorname{Re}\lambda_0 < 0. \end{cases}$$

Then these polynomials have degrees

$$n_- = \deg e_-, \qquad n_+ = \deg e_+, \qquad n_0 = \deg e_0,$$

with $n_\pm$ and n_0 as in Theorem 3. Now write

$$\Delta(\lambda) = \Delta_{\mathrm{L}}(\lambda)\Delta_{\mathrm{R}}(\lambda),$$

$$\Delta_{\mathrm{L}}(\lambda) = \frac{\Delta_-(\lambda)e_+(\lambda)e_0(\lambda)}{e_-(\lambda)(\lambda - \lambda_0)}, \qquad \Delta_{\mathrm{R}}(\lambda) = \frac{\Delta_+(\lambda)e_-(\lambda)}{e_+(\lambda)e_0(\lambda)},$$

and observe that the functions Δ_{L} and Δ_{R} are entire, and that all roots of Δ_{L} lie in the left half-plane and all roots of Δ_{R} lie in the right half-plane. Moreover, it follows from the root-swapping property described in Theorem 3 that the functions Δ_{L} and Δ_{R} do not depend on the choice of factorization (4.2).

With the above setting, we can state the following result (see [14, Theorem 6.1]).

Theorem 5. *Assume that equation (1.1) is hyperbolic. An element $x \in \mathcal{P}$ is a left prolongation of $x_0 \in \mathcal{K}(\pi_P^+)$, that is, x is a bounded solution of (1.1) on $(-\infty, 0]$ vanishing identically on $[0,1]$, if and only if*

$$\widehat{x}(\lambda) = \frac{\Pi_{\mathrm{L}}(\lambda)}{\Delta_{\mathrm{R}}(\lambda)}, \qquad \deg \Pi_{\mathrm{L}} < -n_\sharp, \qquad \widehat{x}(\lambda) = \int_{-\infty}^{0} e^{-\lambda t} x(t)\, dt,$$

where Π_{L} is a polynomial of the indicated degree and $n_\sharp$ is as in (4.5). Similarly $y \in \mathcal{Q}$ is a right prolongation of $y_0 \in \mathcal{K}(\pi_Q^-)$ if and only if

$$\widehat{y}(\lambda) = \frac{\Pi_{\mathrm{R}}(\lambda)}{\Delta_{\mathrm{L}}(\lambda)}, \qquad \deg \Pi_{\mathrm{R}} < n_\sharp - 1, \qquad \widehat{y}(\lambda) = \int_{0}^{\infty} e^{-\lambda t} y(t)\, dt,$$

for a polynomial Π_{R}. In particular

$$\dim \mathcal{K}(\pi_P^+) = \max\{-n_\sharp, 0\}, \qquad \dim \mathcal{K}(\pi_Q^-) = \max\{n_\sharp - 1, 0\}, \tag{4.6}$$

both hold. It is moreover the case that

$$\operatorname{codim} \mathcal{R}(\pi_P^+) = \max\{n_\sharp, 0\}, \qquad \operatorname{codim} \mathcal{R}(\pi_Q^-) = \max\{-n_\sharp + 1, 0\},$$

and that

$$\operatorname{ind}(\pi_P^+) = -n_\sharp, \qquad \operatorname{ind}(\pi_Q^-) = n_\sharp - 1,$$

hold, where the codimensions are taken in the spaces $C[0,1]$ and $C[-1,0]$, respectively.

Intuitively, the integer $n_\sharp$ in Theorem 5 measures the excess number of roots of Δ in the left half-plane over the number of those in the right half-plane. The larger $n_\sharp$ is, the "easier" it is for equation (1.1) to have solutions on $[0, \infty)$ decaying at $+\infty$, that is, the "larger" is the space $\mathcal{Q}$ and the greater the dimension of the kernel $\mathcal{K}(\pi_Q^-)$. Conversely, the more negative $n_\sharp$ is, the greater is the dimension of $\mathcal{K}(\pi_P^+)$.

5. Finite Intervals

In this section we study equation (1.1), and more generally equation (1.3), restricted to a finite interval $[-\tau, \tau]$ for some $\tau > 0$. In the case of equation (1.3) we assume that the coefficients $a, b, c \in L^\infty(\mathbf{R})$ are defined on all of $\mathbf{R}$, as we allow τ to vary. Recall the operator $\Lambda^\tau : W_0^{1,\infty}(-\tau, \tau) \to L^\infty(-\tau, \tau)$ defined by (1.6), (1.7). As noted earlier, the case of large τ is of particular interest, and we seek estimates of the form (1.8) for Λ^τ which are uniform as $\tau \to \infty$.

Since Λ^τ is a compact perturbation of the differentiation operator $\frac{d}{dt}$, the operator Λ^τ is Fredholm with the same index $\operatorname{ind}(\Lambda^\tau) = -1$ as $\frac{d}{dt}$. In particular, if $\mathcal{K}(\Lambda^\tau) = \{0\}$, then exactly one linear constraint is needed on the inhomogeneous term h in order that the equation

$$\dot{x}(t) = a(t)x(t) + b(t)x(t-1) + c(t)x(t+1) + h(t) \tag{5.1}$$

with zero boundary conditions (1.7) should have a solution. (To solve such a problem where x has nonzero boundary values in $[-\tau-1, -\tau] \cup [\tau, \tau+1]$, in a standard fashion one would write x as a sum of any given smooth function satisfying the boundary conditions, and an unknown function satisfying (5.1) for some h and with zero boundary conditions.) Note that elements of the kernel $\mathcal{K}(\Lambda^\tau)$ are just functions $x : [-\tau-1, \tau+1] \to \mathbf{C}$ which are solutions of equation (1.3) on $[-\tau, \tau]$ and which vanish on $[-\tau-1, -\tau] \cup [\tau, \tau+1]$.

Even for the constant coefficient equation (1.1) it is surprisingly difficult to determine the dimension of, and to characterize the elements of $\mathcal{K}(\Lambda^\tau)$. If $0 < \tau \leq \frac{1}{2}$ then it is not hard to show for this equation that $\mathcal{K}(\Lambda^\tau) = \{0\}$. On the other hand, suppose that $\frac{1}{2} < \tau \leq 1$. Then $\dim \mathcal{K}(\Lambda^\tau) = 1$ if and only if

$$b = \frac{(-1)^{n+1} e^{a} (n + \frac{1}{2})\pi}{2\tau - 1}, \qquad c = \frac{(-1)^{n} e^{-a} (n + \frac{1}{2})\pi}{2\tau - 1}, \tag{5.2}$$

for some integer $n \geq 0$, and $\mathcal{K}(\Lambda^\tau) = \{0\}$ otherwise. A heuristic argument based on dimension-counting was given in [14, Section 7] which suggests that for any $\tau > 0$, one has $\dim \mathcal{K}(\Lambda^\tau) \leq 1$ for equation (1.1). This argument also suggests that the set of parameters $(a, b, c, \tau) \in \mathbf{C}^3 \times (0, \infty)$ for which $\dim \mathcal{K}(\Lambda^\tau) = 1$ should be a set of real dimension three (which is the case in the range $0 < \tau \leq 1$, in view of the parameterization of b and c in (5.2) by $(a, \tau) \in \mathbf{C} \times (\frac{1}{2}, 1])$. These conjectures, and others given in [14] based on similar heuristic arguments, are as yet unsolved.

However, we do have the following result, which is a special case of [14, Theorem 7.3], and which is proved using lap number arguments.

Theorem 6. *With $a(t)$ real-valued, assume that either $b(t) > 0$ and $c(t) > 0$ for almost every $t \in \mathbf{R}$, or else that $b(t) < 0$ and $c(t) < 0$ for almost every $t \in \mathbf{R}$. Then for equation (1.3) we have that $\mathcal{K}(\Lambda^\tau) = \{0\}$ for every $\tau > 0$.*

Even if $\mathcal{K}(\Lambda^\tau) = \{0\}$, and so $\mathcal{R}(\Lambda^\tau) \subseteq L^\infty(-\tau,\tau)$ is a subspace of codimension one, inverting the operator Λ^τ on $\mathcal{R}(\Lambda^\tau)$ may be problematic as the norm of the inverse can grow rapidly with τ. The following result, which is a special case of [14, Theorem 7.2], shows that a uniform bound on the norm of the inverse can be achieved by restricting Λ^τ to a suitable subspace of finite codimension.

Theorem 7. *Assume that equation (1.1) is hyperbolic, let the integer $n_\sharp$ be as in Theorem 3, and set $n_* = |n_\sharp - \frac{1}{2}| - \frac{1}{2}$. Then there exists $K > 0$, and for every large τ there exists a subspace $V^\tau \subseteq W_0^{1,\infty}(-\tau,\tau)$ satisfying* $\operatorname{codim} V^\tau = n_*$, *such that*

$$\|\Lambda^\tau x\|_{L^\infty(-\tau,\tau)} \geq K\|x\|_{W_0^{1,\infty}(-\tau,\tau)}, \qquad x \in V^\tau.$$

In particular $V^\tau \cap \mathcal{K}(\Lambda^\tau) = \{0\}$, and so $\dim \mathcal{K}(\Lambda^\tau) \leq n_*$ *holds.*

Note that if either $n_\sharp = 0$ or if $n_\sharp = 1$, then $n_* = 0$ and we have $\mathcal{K}(\Lambda^\tau) = \{0\}$ for all large τ. In this case Λ^τ is an isomorphism from $W_0^{1,\infty}(-\tau,\tau)$ onto its range, with a bound $\|(\Lambda^\tau)^{-1}\| \leq K^{-1}$ on the inverse which is independent of (large) τ.

To understand Theorem 7, take any element $y \in \mathcal{K}(\pi_P^+)$. Set $\widetilde{y}(t) = y(t-\tau)$ and observe that $\widetilde{y}$ satisfies the right-hand boundary condition in (1.7), that is, $y(t) = 0$ in the interval $[\tau, \tau+1]$, since $\pi^+\widetilde{y}_\tau = \pi^+ y_0 = 0$. The left-hand boundary condition, in $[-\tau-1,-\tau]$, in general fails, however it almost holds in the sense that $\widetilde{y}(t)$ is exponentially small in that interval. Indeed, by Theorem 1 we have $|y(t)| \leq K' e^{\alpha t}$ for $t \leq 0$ and some $K' > 0$ and $\alpha > 0$, and so $|\widetilde{y}(t)| \leq K' e^{-2\alpha\tau}$ in $[-\tau-1,-\tau]$. One could say that $\widetilde{y}$ is "almost" an element of the kernel $\mathcal{K}(\Lambda^\tau)$. In a similar fashion, functions $\widetilde{z}$ given by $\widetilde{z}(t) = z(t+\tau)$, with $z \in \mathcal{K}(\pi_Q^-)$, also are "almost" elements of $\mathcal{K}(\Lambda^\tau)$. In fact, the subspace V^τ in Theorem 7 can be taken to be the set of all $x \in W_0^{1,\infty}(-\tau,\tau)$ which are orthogonal in $L^2(-\tau,\tau)$ to every such $\widetilde{y}$ and to every such $\widetilde{z}$. One then has, using the formulas in (4.6) of Theorem 5,

$$\begin{aligned} \operatorname{codim} V^\tau &= \dim \mathcal{K}(\pi_P^+) + \dim \mathcal{K}(\pi_Q^-) \\ &= \max\{-n_\sharp, 0\} + \max\{n_\sharp - 1, 0\} = |n_\sharp - \tfrac{1}{2}| - \tfrac{1}{2} = n_*. \end{aligned}$$

We remark finally that a version of Theorem 7 is given in [14] for the nonautonomous equation (1.3). In this result it is assumed that the limits $a(\pm\infty)$, $b(\pm\infty)$, and $c(\pm\infty)$ of the coefficients exist, and that the limiting autonomous equations are hyperbolic. A finite-codimension subspace V^τ is constructed, in part, from elements of $\mathcal{K}(\pi_P^+)$ and $\mathcal{K}(\pi_Q^-)$ for these limiting equations.

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