Topics in Analysis 1 – Real Functions

Assignment 3

Let $f\colon [a,b] \to \mathbb{R}$ be a function and let $x \in [a,b]$. We say that f *has a right limit at* x if $x < b$ and

$$f(x+) := \lim_{y \downarrow x} f(y) \text{ exists}$$

and f *has a left limit at* x if $x > a$ and

$$f(x-) := \lim_{y \uparrow x} f(y) \text{ exists.}$$

The function f is called *right continuous* if $f(x) = f(x+)$ for all $x \in [a,b)$ and *left continuous* if $f(x-) = f(x)$ for all $x \in (a,b]$. f is called a *càdlàg function* (from the french phrase "continue à droite, limite à gauche") if f is right continuous and has a left limit at every $x \in (a,b]$. Similarly, f is *càglàd* if f is left continuous and has a right limit at every $x \in [a,b)$. (Authors avoiding french sometimes use 'RCLL' and 'LCRL'.)

1. **Discontinuities of functions with left and right limits.** Let $f\colon [a,b] \to \mathbb{R}$ be a functions that has a right limit at every $x \in [a,b)$ and a left limit at every $x \in (a,b]$. Let

$$D = \{x \in [a,b]\colon\ f \text{ is not continuous at } x\}.$$

   (a) Show that $D = \{x \in (a,b]\colon\ f(x-) \neq f(x)\} \cup \{x \in [a,b)\colon\ f(x) \neq f(x+)\}$.

   (b) Let $\varepsilon > 0$. Show that for each $x \in (a,b)$ there exists a $\delta > 0$ such that

$$|f(y) - f(x-)| < \varepsilon \text{ for all } y \in (x-\delta, x) \cap [a,b]$$

   and

$$|f(x+) - f(y)| < \varepsilon \text{ for all } y \in (x, x+\delta) \cap [a,b].$$

   (c) Show that f is bounded.

   (d) Show that for each $\alpha > 0$ the set

$$\{x \in (a,b]\colon\ |f(x-) - f(x)| > \alpha\}$$

   is finite.
   (Hint: use (b) and compactness of $[a,b]$.)

   (e) Show that D is at most countable.

2. **Càdlàg versions.** Let $f\colon [a,b] \to \mathbb{R}$ be a functions that has a right limit at every $x \in [a,b)$ and a left limit at every $x \in (a,b]$. Define

$$h(x) := \begin{cases} f(x+), & x \in [a,b), \\ f(b), & x = b. \end{cases}$$

   (The function h is called a *càdlàg version* of f.)

(a) Show that h is a càdlàg function.

(b) Show that $\{x \in [a,b] \colon h(x) \neq f(x)\}$ is at most countable.

(c) Let $f \colon [a,b] \to \mathbb{R}$ be of bounded variation. Show that f has a right limit at every $x \in [a,b)$ and a left limit at every $x \in (a,b]$. (Hence f has a càdlàg version.)

3. **An integral equation.** Let a continuous function $f \colon [0,\infty) \to \mathbb{R}$ with $f(0) = 0$ and a constant $c \in \mathbb{R}$ be given. Consider the integral equation

$$y(t) = c + \int_0^t y(s)\,\mathrm{d}s + f(t), \quad t \geq 0. \tag{1}$$

Show that there exists a unique continuous function $y \colon [0,\infty) \to \mathbb{R}$ satisfying (1).

4. **Convex functions.** If $f \colon [a,b] \to \mathbb{R}$ is a convex function and $x \in (a,b)$, then we denote the *left derivative of f at x* by

$$D_l f(x) = \lim_{y \uparrow x} \frac{f(y) - f(x)}{y - x}$$

and the *right derivative of f at x* by

$$D_r f(x) = \lim_{y \downarrow x} \frac{f(y) - f(x)}{y - x}.$$

(a) Let $f \colon [a,b] \to \mathbb{R}$ be convex and let $x_0 \in (a,b)$ be such that

$$f(x_0) \leq f(x) \text{ for all } x \in (a,b).$$

Show that $D_l f(x_0) \leq 0$ and $D_r f(x_0) \geq 0$.

(b) Let $f \colon [a,b] \to \mathbb{R}$ be convex and let $x_0 \in (a,b)$. Show that for $\alpha \in \mathbb{R}$ we have

$$D_l f(x_0) \leq \alpha \leq D_r f(x_0) \quad \Longleftrightarrow \quad \alpha(x - x_0) + f(x_0) \leq f(x)\, \forall x \in [a,b].$$

(c) Let $f \colon \mathbb{R} \to \mathbb{R}$ be a convex function which is *coercive*, that is,

$$\lim_{x \to -\infty} f(x) = \infty \text{ and } \lim_{x \to \infty} f(x) = \infty.$$

Show that there exists $x_0 \in \mathbb{R}$ such that $f(x_0) \leq f(x)$ for all $x \in \mathbb{R}$.

(d) Let $f \colon \mathbb{R} \to \mathbb{R}$ be *strictly convex*, that is,

$$\left.\begin{array}{l} x, y \in \mathbb{R},\ x \neq y \\ \lambda \in (0,1) \end{array}\right\} \Longrightarrow f(\lambda x + (1-\lambda) y) < \lambda f(x) + (1-\lambda) f(y).$$

Show that there exists at most one $x_0 \in \mathbb{R}$ such that $f(x_0) \leq f(x)$ for all $x \in \mathbb{R}$. (Hence (c) and (d) together yield: a strictly convex coercive function from $\mathbb{R}$ to $\mathbb{R}$ has a unique minimizer.)

(e) Let $f \colon \mathbb{R} \to \mathbb{R}$ be a convex function. Show that there exists an affine function $g(x) = \alpha x + \beta$, $x \in \mathbb{R}$, such that $f(x) \geq g(x)$ for all $x \in \mathbb{R}$.

— *Please hand in before May 27, 2008* —

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