Topics in Analysis 1 – Real Functions

Assignment 2

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1. **Absolute continuity.**

   (a) Let $f : [a, b] \to \mathbb{R}$ be continuously differentiable. Show that there exists a constant M such that

   $$\sum_{i=1}^{n} |f(b_i) - f(a_i)| \leq M \sum_{i=1}^{n} |b_i - a_i|$$

   for any mutually disjoint intervals (a_i, b_i), $i = 1, \dots, n$, in $[a, b]$. Is f absolutely continuous?

   (b) Is $f(x) = \sqrt{x}$, $x \in [0, 1]$, absolutely continuous?

   A function $f : [a, b] \to \mathbb{R}$ is called *Hölder continuous with exponent* α if there exists a constant C such that

   $$|f(x) - f(y)| \leq C|x - y|^{\alpha} \quad \text{for all } x, y \in [a, b].$$

   Here $\alpha > 0$.

   (c) Show that every function on $[a, b]$ that is Hölder continuous with exponent $\alpha \geq 1$ is absolutely continuous.

   (d) Can you describe "all" functions that are Hölder continuous with exponent 2?

2. **Sturm-Liouville problem with Neumann boundary conditions.** Consider

   $$\begin{cases} -u''(x) + xu(x) = f(x), & x \in (0, 1), \\ u'(0) = 0, \quad u'(1) = 0. \end{cases} \tag{1}$$

   (A fixed value for u at a boundary is called a *Dirichlet* boundary condition, a fixed value for u' at a boundary is called a *Neumann* boundary condition.)

   (a) Show that for each $f \in L^2[0, 1]$ there exists a unique $u \in W^{1,2}[0, 1]$ such that

   $$\int_0^1 (u'(x)v'(x) + xu(x)v(x))\,\mathrm{d}x = \int_0^1 f(x)v(x)\,\mathrm{d}x \text{ for all } v \in W^{1,2}[0, 1].$$

   (b) Show that if $f \in C[0, 1]$, then the weak solution u of (a) satisfies

   $$\int_0^1 (u'(x) - H(x))v'(x)\,\mathrm{d}x + H(1)v(1) - H(0)v(0) = 0 \text{ for all } v \in W^{1,2}[0, 1],$$

   where

   $$H(x) = \int_0^x (tu(t) - f(t))\,\mathrm{d}t.$$

(c) Show that for each $f \in C[0,1]$ there exists a unique $u \in C^2[0,1]$ such that (1) holds. (Hint: use (b) first for all $v \in W^{1,2}[0,1]$ with $v(0) = v(1) = 0$ and then for $v \equiv 1$.)

3. **Derivative in sense of distribution.**

(a) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and with distributional derivative in $C^1(\mathbb{R})$ (i.e., continuously differentiable). Show that f is twice continuously differentiable.

(b) Let $f : [a,b] \to \mathbb{R}$ be integrable and such that its distributional derivative is identically zero. Show that f is constant.

(c) (i) Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} 1, & x \in [0,1], \\ 0, & x \in \mathbb{R} \setminus [0,1]. \end{cases}$$

Compute its derivative in sense of distribution.

(ii) Let $f : (0,1) \to \mathbb{R}$ be given by

$$f(x) = 1, \quad x \in (0,1).$$

Compute its derivative in sense of distribution.

(d) Compute the second order distributional derivative of

$$f(x) = \begin{cases} x, & x \in [0,1], \\ 2 - x, & x \in [1,2], \\ 0, & \mathbb{R} \setminus [0,2]. \end{cases}$$

— *Please hand in before April 8, 2008* —

Onno van Gaans