

# The Riesz part of a positive bilinear form

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## Abstract

It is shown that below every positive bilinear form on a Riesz space there exists a greatest Riesz (orthosymmetric) bilinear form, called its Riesz part. An explicit formula is given. If a positive inner product on a Riesz space induces a norm such that the positive cone is closed and the space complete, then its Riesz part is an inner product that induces an equivalent norm.

This paper goes back to research on inner products on Riesz spaces that I started in 1995, during my first year as a PhD student with Arnoud. Soon, our attention shifted towards seminorms, leaving results we obtained unfinished and our suspicions unconfirmed. Since then I had the vague plan to revisit these first steps and try my new experience and insights. The occasion of this special book seemed the right moment to do so, and I did with dearest memories.

## 1 Introduction

The presence of inner products in the theory of Riesz spaces (vector lattices) is not as prominent as those of norms and vector space topologies. There is literature available, though, especially focussing on bilinear forms, often in connection with algebra structures or tensor products, and Hilbert lattices. In this paper we study inner products that are positive as bilinear forms (called positive inner products) in relation with inner products that induce Riesz norms (called Riesz inner products).

We will show that for every positive bilinear form  $\langle \cdot, \cdot \rangle$  on a Riesz space  $E$  there exists a greatest positive bilinear form  $\langle\langle \cdot, \cdot \rangle\rangle$  below  $\langle \cdot, \cdot \rangle$  with the property that  $|x| \wedge |y| = 0$  implies that  $\langle\langle x, y \rangle\rangle = 0$ . We will call  $\langle\langle \cdot, \cdot \rangle\rangle$  the *Riesz part* of  $\langle \cdot, \cdot \rangle$  and construct it by an explicit formula.

In case  $\langle \cdot, \cdot \rangle$  is an inner product, the question arises whether its Riesz part is an inner product as well. It turns out to be a semi-inner product, but it fails in general to be positive definite. The situation improves if we assume norm completeness. We will show that if  $E$  is norm complete and  $E^+$  is closed with respect to the norm  $\|\cdot\|$  induced by  $\langle \cdot, \cdot \rangle$ , then the Riesz part of  $\langle \cdot, \cdot \rangle$  is an inner product that induces a norm that is equivalent to  $\|\cdot\|$ . Further, if  $\langle \cdot, \cdot \rangle$  is such that  $\langle x, y \rangle \geq 0$  for all  $x \geq 0$  implies that  $y \geq 0$ , then norm completeness of the space yields that  $\langle \cdot, \cdot \rangle$  is itself a Riesz inner product.

Throughout this paper,  $E$  will denote a Riesz space, and all our vector spaces will be real. We assume the Axiom of Choice.

## 2 Generalities on bilinear forms

A bilinear form  $B: E \times E \rightarrow \mathbb{R}$  is called *positive* if  $B(x, y) \geq 0$  for all  $x, y \in E^+$ , and *Riesz* if moreover  $B(x, y) = 0$  for all  $x, y \in E$  with  $|x| \wedge |y| = 0$  (i.e., *orthosymmetric* in the terminology of [2]). Note that  $B$  is positive if and only if it is increasing on  $E^+ \times E^+$  in both of its arguments. The cone of positive bilinear forms on  $E$  generates a partially ordered vector space. Statements involving ordering relations of bilinear forms refer to this ordering.

A bilinear form  $B$  on a vector space  $V$  is called *symmetric* if  $B(x, y) = B(y, x)$  for all  $x, y \in V$  and (semi-) positive definite if  $B(x, x) > 0$  ( $\geq 0$ ) for all  $x \in V \setminus \{0\}$ . A (semi-) *inner product* is a (semi-) positive definite symmetric bilinear form.

Every Riesz bilinear form  $B$  is semi-positive definite. Indeed,  $B(x, x) = B(x^+, x^+) + B(x^-, x^-) \geq 0$  for all  $x$ . If  $E$  is Archimedean, [2, Cor. 2] yields that  $B$  is also symmetric and hence a semi-inner product.

**2.1 Lemma** *Let  $\langle \cdot, \cdot \rangle$  be a semi-inner product on  $E$  and let  $p$  be its induced seminorm. Then*

- (i)  $\langle \cdot, \cdot \rangle$  is positive if and only if  $p$  is Fremlin, i.e.,  $p(x - y) \leq p(x + y)$  for all  $x, y \in E^+$  (see [4], [5]).
- (ii)  $\langle \cdot, \cdot \rangle$  is Riesz if and only if  $p$  is Riesz.

PROOF. (i):  $\langle x, y \rangle = (p(x + y) - p(x - y))/4$  for all  $x, y \in E$ . (ii): Observe that  $\langle |x|, |x| \rangle - \langle x, x \rangle = 4\langle x^+, x^- \rangle$  equals zero for all  $x \in E$  if and only if  $\langle x, y \rangle = 0$  for all  $x, y \in E$  with  $|x| \wedge |y| = 0$  and combine with (i).  $\square$

A function  $f: E^+ \rightarrow \mathbb{R}$  is said to be *additive* if  $f(x + y) = f(x) + f(y)$  for all  $x, y \in E^+$  and *positively homogeneous* if  $f(\lambda x) = \lambda f(x)$  for all  $x \in E^+$  and  $\lambda \in [0, \infty)$ .

**2.2 Lemma** *Let  $B: E^+ \times E^+ \rightarrow \mathbb{R}$  be such that  $B$  is additive and positively homogeneous in both arguments. Then there is one and only one bilinear form  $\bar{B}$  on  $E$  extending  $B$ , and*

- (i) if  $B(x, y) \geq 0$  for all  $x, y \in E^+$ , then  $\bar{B}$  is positive,
- (ii) if  $B(x, y) = 0$  for all  $x, y \in E^+$  with  $x \wedge y = 0$ , then  $\bar{B}(x, y) = 0$  for all  $x, y \in E$  with  $|x| \wedge |y| = 0$ .

PROOF. Define for  $x, y \in E$ :

$$\bar{B}(x, y) := B(x^+, y^+) - B(x^-, y^+) - B(x^+, y^-) + B(x^-, y^-). \quad (1)$$

Then  $\bar{B} = B$  on  $E^+ \times E^+$ . If  $a \in E^+$ ,  $x \in E$ , and  $x_1, x_2 \in E^+$  are such that  $x = x_1 - x_2$ , then  $x_1 + x^- = x^+ + x_2$  and, by additivity of  $B$ ,  $B(x_1, a) + B(x^-, a) = B(x^+, a) + B(x_2, a)$ , so that

$$B(x^+, a) - B(x^-, a) = B(x_1, a) - B(x_2, a).$$

Similarly,

$$B(a, x^+) - B(a, x^-) = B(a, x_1) - B(a, x_2).$$

Then for  $x, y \in E$  and  $x_1, x_2, y_1, y_2 \in E^+$  with  $x = x_1 - x_2$  and  $y = y_1 - y_2$  one has

$$\begin{aligned} \bar{B}(x, y) &= B(x^+, y^+) - B(x^-, y^+) - B(x^+, y^-) + B(x^-, y^-) \\ &= B(x_1, y^+) - B(x_2, y^+) - B(x_1, y^-) + B(x_2, y^-) \\ &= B(x_1, y_1) - B(x_1, y_2) - B(x_2, y_1) + B(x_2, y_2). \end{aligned}$$

For  $x_1, x_2, a \in E$  it follows that

$$\begin{aligned} \bar{B}(x_1 + x_2, a) &= B(x_1^+ + x_2^+, a^+) - B(x_1^+ + x_2^+, a^-) \\ &\quad - B(x_1^- + x_2^-, a^+) + B(x_1^- + x_2^-, a^-) \\ &= \bar{B}(x_1, a) + \bar{B}(x_2, a), \end{aligned}$$

and, similarly,

$$\bar{B}(a, x_1 + x_2) = \bar{B}(a, x_1) + \bar{B}(a, x_2).$$

It is clear from the definition of  $\bar{B}$  that for  $x, y \in E$  and  $\lambda \in [0, \infty)$  one has that  $\bar{B}(-x, y) = -\bar{B}(x, y)$ ,  $\bar{B}(x, -y) = -\bar{B}(x, y)$ ,  $\bar{B}(\lambda x, y) = \lambda \bar{B}(x, y)$ , and  $\bar{B}(x, \lambda y) = \lambda \bar{B}(x, y)$ . Thus,  $\bar{B}$  is a bilinear form on  $E$  that extends  $B$ . That it is the only one can be seen from (1).

Assertion (i) is obvious. To see (ii), let  $x, y \in E$  be such that  $|x| \wedge |y| = 0$ . Then  $x^+, x^-, y^+, y^-$  are pairwise disjoint, so that (1) yields that  $\bar{B}(x, y) = 0$ .  $\square$

*Example.* Let  $X$  be a compact Hausdorff space, let  $E = C(X)$ , and let  $B$  be a positive bilinear form on  $E$ . According to a theorem by Fremlin[3, 3.6], there is a finite regular Borel measure  $\mu$  on  $X \times X$  such that

$$B(f, g) = \int_{X \times X} f(x)g(y)d\mu(x, y) \quad \text{for all } f, g \in E.$$

If  $\mu$  is concentrated on the diagonal  $\Delta := \{(x, x): x \in X\}$ , then  $B$  is Riesz. Also the converse is true. Indeed, assume that  $B$  is Riesz. First, let  $R = A \times B$  be a closed rectangle in  $X \times X$  with  $R \cap \Delta = \emptyset$ . Then  $A \cap B = \emptyset$ , so with aid of Urysohn's lemma we can find  $f, g \in E$  with disjoint supports and  $f = 1$  on  $A$ ,  $g = 1$  on  $B$ , and  $0 \leq f, g \leq 1$ . Then  $\mu(R) \leq \int f(x)g(y)d\mu(x, y) = B(f, g) = 0$ . Now let  $K \subset X \times X \setminus \Delta$  be compact. We claim that  $K$  is covered by finitely many closed rectangles that do not intersect  $\Delta$ . Indeed, there exists an open set  $V \supset K$  with  $\bar{V} \cap \Delta = \emptyset$ . As  $V$  is open,  $V$  is the union of open rectangles and

finitely many of them cover  $K$ , by compactness. The closures of the rectangles are contained in  $\bar{V}$  and the claim follows. Thus we have that  $\mu(K) = 0$  and, by regularity of  $\mu$ , that  $\mu(X \times X \setminus \Delta) = 0$ .

### 3 The Riesz part of a positive bilinear form

We will now present a construction of the Riesz part of a positive bilinear form. It will be convenient to use the following terminology.

Let  $x \in E^+$ . A finite family  $(x_1, \dots, x_n)$  of elements of  $E^+$  is called a *partition* of  $x$  if  $x = x_1 + \dots + x_n$ . If  $X$  and  $Y$  are two partitions of  $x$ , then  $Y$  is called *finer* than  $X$  if every element  $u$  of  $X$  corresponds to a subfamily of  $Y$  that is a partition of  $u$ . From the Riesz decomposition property (cf. [1, Thm 1.15, p.14]) it is clear that for any two partitions  $X$  and  $Y$  there is a *common refinement*, i.e., a partition that is finer than both  $X$  and  $Y$ . It follows that the set of all partitions of  $x$  endowed with the ‘finer than’ relation is directed.

Given a positive bilinear form  $B$  on  $E$  and  $x, y \in E^+$ , we will associate with every partition  $X = (x_1, \dots, x_n)$  of  $x$  and  $Y = (y_1, \dots, y_m)$  of  $y$  the sum of  $B(x_k, y_l)$  over all  $k \in \{1, \dots, n\}$  and  $l \in \{1, \dots, m\}$  with  $x_k \wedge y_l \neq 0$ , denoted shortly by

$$\sum_{k,l: x_k \wedge y_l \neq 0} B(x_k, y_l).$$

We will see that the infimum of these sums over all partitions yields the greatest Riesz bilinear form below  $B$ . First we show that the sums form a decreasing net.

**3.1 Lemma** *Let  $B$  be a positive bilinear form on  $E$  and let  $x, y \in E^+$ . Let  $X = (x_1, \dots, x_n)$  and  $X' = (x'_1, \dots, x'_{n'})$  be partitions of  $x$  and  $Y = (y_1, \dots, y_m)$  and  $Y' = (y'_1, \dots, y'_{m'})$  of  $y$ . If  $X'$  is finer than  $X$  and  $Y'$  is finer than  $Y$ , then*

$$\sum_{i,j: x'_i \wedge y'_j \neq 0} B(x'_i, y'_j) \leq \sum_{k,l: x_k \wedge y_l \neq 0} B(x_k, y_l).$$

PROOF. Since  $X'$  is finer than  $X$  and  $Y'$  finer than  $Y$ , one has that  $\{1, \dots, n'\} = I_1 \cup \dots \cup I_n$  with  $I_k \cap I_l = \emptyset$  for all  $k \neq l$  and  $x_k = \sum_{i \in I_k} x'_i$  for all  $k$ , and similarly  $\{1, \dots, m'\} = J_1 \cup \dots \cup J_m$  with  $J_k \cap J_l = \emptyset$  for  $k \neq l$  and  $y_l = \sum_{j \in J_l} y'_j$  for all  $l$ .

If  $k$  and  $l$  are such that  $x_k \wedge y_l = 0$ , then  $x'_i \wedge y'_j = 0$  for all  $i \in I_k$  and  $j \in I_l$ . Therefore

$$\begin{aligned} \sum_{i,j: x'_i \wedge y'_j \neq 0} B(x'_i, y'_j) &= \sum_{k,l} \sum_{i \in I_k, j \in I_l: x'_i \wedge y'_j \neq 0} B(x'_i, y'_j) \\ &\leq \sum_{k,l: x_k \wedge y_l \neq 0} \sum_{i \in I_k, j \in I_l} B(x'_i, y'_j) \\ &= \sum_{k,l: x_k \wedge y_l \neq 0} B(x_k, y_l). \end{aligned}$$

□

**3.2 Theorem** Let  $B$  be a positive bilinear form on  $E$ . Define for  $x, y \in E^+$ :

$$D(x, y) := \inf_{\substack{x=x_1+\dots+x_n \\ y=y_1+\dots+y_m \\ x_k, y_l \geq 0, \text{ all } k, l}} \sum_{k, l: x_k \wedge y_l \neq 0} B(x_k, y_l). \quad (2)$$

Then  $D$  extends uniquely to a Riesz bilinear form on  $E$  and that one is the greatest Riesz bilinear form on  $E$  below  $B$ .

PROOF. Let  $x, y \in E^+$ . Let  $I$  be the set of all pairs  $(X, Y)$  where  $X$  and  $Y$  are partitions of  $x$  and  $y$ , respectively, directed by the relation ensuing from ‘finer than’. By the previous lemma, the net  $((x_1, \dots, x_n), (y_1, \dots, y_m)) \mapsto \sum_{k, l: x_k \wedge y_l \neq 0} B(x_k, y_l)$ ,  $((x_1, \dots, x_n), (y_1, \dots, y_m)) \in I$ , is a decreasing net of positive reals, hence convergent, and its limit is the infimum  $D(x, y)$ . From this it is straightforward that  $D$  is positively homogeneous in both arguments. To see that  $D$  is additive, let  $x, y, z \in E^+$  and let  $I$  be the directed set of pairs of partitions of  $x + y$  and  $z$ . Let  $I'$  be the directed subset of  $I$  where of the partitions of  $x + y$  only those are included that are refinements of  $(x, y)$ . Then for every  $((u_1, \dots, u_p), (z_1, \dots, z_q)) \in I'$  the partition  $(u_1, \dots, u_p)$  splits into a partition  $(x_1, \dots, x_n)$  of  $x$  and  $(y_1, \dots, y_m)$  of  $y$ , and

$$\sum_{r, s: u_r \wedge z_s \neq 0} B(u_r, z_s) = \sum_{k, s: x_k \wedge z_s \neq 0} B(x_k, z_s) + \sum_{l, s: y_l \wedge z_s \neq 0} B(y_l, z_s).$$

By taking limits it follows that  $D(x + y, z) = D(x, z) + D(y, z)$ . Similarly,  $D(x, y + z) = D(x, y) + D(x, z)$ .

It is clear from the definition that  $D$  is positive and that  $x \wedge y = 0$  implies that  $D(x, y) = 0$ . Thus, Lemma 2.2 yields that  $D$  extends uniquely to a Riesz bilinear form on  $E$ , which we denote by  $D$  as well.

For  $x, y \in E^+$  one has that  $D(x, y) \leq B(x, y)$ , so that  $D$  is below  $B$  in the ordering of bilinear forms on  $E$ . Suppose that also  $F$  is a Riesz bilinear form on  $E$  below  $B$ . Then for any  $x, y \in E^+$  and partitions  $(x_1, \dots, x_n)$  of  $x$  and  $(y_1, \dots, y_m)$  of  $y$ , one has that  $\sum_{k, l: x_k \wedge y_l \neq 0} B(x_k, y_l) \geq \sum_{k, l: x_k \wedge y_l \neq 0} F(x_k, y_l) = \sum_{k, l} F(x_k, y_l) = F(x, y)$  and then the definition of  $D$  yields that  $D \geq F$ . Thus,  $D$  is the greatest Riesz bilinear form on  $E$  below  $B$ . □

We will call the greatest Riesz bilinear form below a positive bilinear form  $B$  the *Riesz part* of  $B$ .

*Remarks.* 1. In case  $x = y$  in formula (2), it follows with aid of common refinements and Lemma 3.1 that

$$D(x, x) = \inf_{\substack{x=x_1+\dots+x_n \\ x_k \geq 0, \text{ all } k}} \sum_{k, l: x_k \wedge x_l \neq 0} B(x_k, x_l). \quad (3)$$

2. As pointed out in Section 2, every Riesz bilinear form on  $E$  is semi-positive definite, and a semi-inner product if  $E$  is Archimedean. It is immediate from our construction that the Riesz part of a symmetric positive bilinear form is symmetric. Hence the Riesz part of a positive bilinear form  $B$  is a semi-inner product if  $B$  is symmetric or  $E$  Archimedean.

3. If the Riesz part  $D$  of a positive bilinear form  $B$  on  $E$  is a semi-inner product, we can follow a slightly different approach. Indeed,  $D$  can be recovered by polarisation from its induced Riesz seminorm, which may be constructed by means of formula (3), or, in case  $E$  is Dedekind complete, by (4) below.

In case  $E$  is Dedekind complete, there is a useful reformulation of formula (3). We derive it in two steps.

**3.3 Lemma** *Assume that  $E$  is Dedekind complete. Let  $x, x_1, \dots, x_n \in E^+$  be such that  $x = x_1 + \dots + x_n$ . Then there exist finite subsets  $U_1, \dots, U_m$  of  $E^+$  such that their elements form a partition of  $x$  that is finer than  $(x_1, \dots, x_n)$  and such that*

$$\begin{aligned} U_i \perp U_j & \quad \text{for } i \neq j \text{ and} \\ u, v \in U_i & \Rightarrow u \wedge v \neq 0 \quad \text{for all } i. \end{aligned}$$

( $A \perp B$  means that  $|a| \wedge |b| = 0$  for all  $a \in A$  and  $b \in B$ .)

PROOF. Denote for each  $k \in \{1, \dots, n\}$  the band generated by  $x_k$  by  $A_k$  and let  $A_x$  be the band generated by  $x$ . Denote  $A'_k := A_x \cap A_k^d$ , where  $A_k^d$  stands for the disjoint complement of  $A_k$ . Consider the following collection of bands:

$$\mathcal{B} := \{D_1 \cap \dots \cap D_n : \text{the band } D_k = A_k \text{ or } A'_k \text{ for } k = 1, \dots, n\} \setminus \{\{0\}\}.$$

Clearly  $\mathcal{B}$  is a finite set; number its elements as  $B_1, \dots, B_m$ . We need three observations on  $B_1, \dots, B_m$ .

Firstly,  $B_i \perp B_j$  for  $i \neq j$ . Indeed, let  $i, j \in \{1, \dots, m\}$  and write  $B_i = D_1 \cap \dots \cap D_n$  and  $B_j = E_1 \cap \dots \cap E_n$ , where for each  $k$  one has  $D_k = A_k$  or  $A'_k$  and  $E_k = A_k$  or  $A'_k$ . If  $i \neq j$ , then  $B_i \neq B_j$  and then there is a  $k$  with  $D_k \neq E_k$ . Then it follows that  $D_k \perp E_k$  and therefore  $B_i \perp B_j$ .

Secondly,  $B_1 \oplus \dots \oplus B_m = A_x$ . Indeed,  $E$  is Dedekind complete and by induction to  $l$  one can straightforwardly show that

$$\oplus \{D_1 \cap \dots \cap D_l : \text{the band } D_k = A_k \text{ or } A'_k \text{ for } k = 1, \dots, l\} = A_x$$

for every  $l \in \{1, \dots, n\}$ .

Thirdly, for every  $i \in \{1, \dots, m\}$  and  $k \in \{1, \dots, n\}$  one has either  $B_i \perp A_k$  or  $B_i \subset A_k$ . Indeed, write  $B_i = D_1 \cap \dots \cap D_n$  with  $D_l = A_l$  or  $A'_l$  for each  $l$ . Then  $D_k$  is either  $A_k$  or  $A'_k$ , hence either  $B_i \subset A_k$  or  $B_i \perp A_k$ .

Now let  $P_i$  be the projection on the band  $B_i$  and define

$$U_i := \{P_i x_1, \dots, P_i x_n\} \setminus \{0\},$$

$i = 1, \dots, m$ . Then  $U_i \perp U_j$  for  $i \neq j$ , because  $B_i \perp B_j$  for  $i \neq j$ . Further, if  $u, v \in U_i$ , then  $u = P_i x_l$  and  $v = P_i x_k$  for certain  $k$  and  $l$  and  $u$  and  $v$  are both

nonzero, so that the third observation above yields that  $B_i \subset A_l$  and  $B_i \subset A_k$ . Then  $u$  and  $v$  both generate the band  $B_i$  and therefore  $u \wedge v \neq 0$ .

Because of the second observation,  $\sum_i P_i x_k = x_k$  for each  $k \in \{1, \dots, n\}$ , which means that the elements of  $U_1, \dots, U_m$  form a partition of  $x$  that is finer than  $(x_1, \dots, x_n)$ .  $\square$

**3.4 Theorem** *Let  $B$  be a positive bilinear form on  $E$  and let  $D$  be its Riesz part. If  $E$  is Dedekind complete, then for every  $x \in E^+$  one has*

$$D(x, x) = \inf_{\substack{x=x_1+\dots+x_n \\ x_k \wedge x_l=0, k \neq l \\ x_k \geq 0, \text{ all } k}} \sum_{k=1}^n B(x_k, x_k). \quad (4)$$

PROOF. From the formula in Theorem 3.2 it follows that  $D$  is not greater than the infimum at the right hand side of (4). To prove that it is not less either, let  $x_1, \dots, x_n \in E^+$  be such that  $x = x_1 + \dots + x_n$ . According to the previous lemma, there is a partition  $(u_1, \dots, u_r)$  of  $x$  that is finer than  $(x_1, \dots, x_n)$  and such that  $\{1, \dots, r\} = I_1 \cup \dots \cup I_m$  with  $I_1, \dots, I_m$  pairwise disjoint and  $U_i := \{u_p: p \in I_i\}$ ,  $i = 1, \dots, m$ , are such that

$$\begin{aligned} U_i \perp U_j & \text{ for } i \neq j \text{ and} \\ u, v \in U_i & \Rightarrow u \wedge v \neq 0 \text{ for all } i. \end{aligned}$$

Let  $z_i$  be the sum of the elements of  $U_i$ ,  $i = 1, \dots, m$ . Then  $x = z_1 + \dots + z_m$ ,  $z_i \geq 0$  for all  $i$ , and  $z_i \wedge z_j = 0$  for  $i \neq j$ . For each  $i$ ,  $u \wedge v \neq 0$  for all  $u, v \in U_i$  and hence

$$B(z_i, z_i) = \sum_{p, q \in I_i} B(u_p, u_q) = \sum_{p, q \in I_i, u_p \wedge u_q \neq 0} B(u_p, u_q).$$

It follows that

$$\begin{aligned} \sum_{p, q: u_p \wedge u_q \neq 0} B(u_p, u_q) &= \sum_{i=1}^m \sum_{p, q \in I_i, u_p \wedge u_q \neq 0} B(u_p, u_q) \\ &= \sum_{i=1}^m B(z_i, z_i). \end{aligned}$$

As  $(u_1, \dots, u_r)$  is finer than  $(x_1, \dots, x_n)$ , Lemma 3.1 yields that

$$\sum_{k=1}^n B(x_k, x_k) \geq \sum_{i=1}^m B(z_i, z_i),$$

and  $\sum_{i=1}^m B(z_i, z_i)$  is greater than or equal to the right hand side of (4).

Thus, the proof is complete.  $\square$

## 4 Positive inner products

In this section we will apply our construction of Riesz parts to inner products. If  $\langle \cdot, \cdot \rangle$  is a positive inner product on  $E$ , then its Riesz part  $\langle\langle \cdot, \cdot \rangle\rangle$  is a semi-inner product (see Remark 2 after Theorem 3.2). The question arises whether it is an inner product or not.

*Example.* A positive inner product with Riesz part equal to zero. Let  $E = L^1(\mathbb{R})$ , and  $a = \mathbb{1}_{[-1,1]}$ , and let

$$\langle x, y \rangle := \int_{\mathbb{R}} (x * a)(y * a), \quad x, y \in E,$$

where  $*$  denotes the convolution product. It is clear that  $\langle \cdot, \cdot \rangle$  is a positive semi-inner product on  $E$  and if  $\langle x, x \rangle = 0$ , then  $x * a = 0$ , so that  $\hat{x}\hat{a} = 0$ , hence  $\hat{x} = 0$  and therefore  $x = 0$ .

Let  $\langle\langle \cdot, \cdot \rangle\rangle$  be the Riesz part of  $\langle \cdot, \cdot \rangle$ . We will show that  $\langle\langle \cdot, \cdot \rangle\rangle = 0$ . Let  $S_{n,k} := [k/n, (k+1)/n]$ ,  $k = -n^2 + 1, \dots, n^2 - 1$ ,  $S_{n,-n^2} := (-\infty, -n + 1/n)$ ,  $S_{n,n^2} := [n, \infty)$ , and  $A_n := \cup_{k=-n^2}^{n^2} S_{n,k} \times S_{n,k}$ ,  $n \in \mathbb{N}$ . For  $x \in E^+$  and  $n \in \mathbb{N}$  we have  $x = \sum_{k=-n^2}^{n^2} x_k$ , with  $x_k = x \mathbb{1}_{S_{n,k}}$ , so that

$$\begin{aligned} \langle\langle x, x \rangle\rangle &\leq \sum_k \langle x_k, x_k \rangle = \sum_k \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} x_k(r) a(t-r) x_k(s) a(t-s) dr ds dt \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} x(r) x(s) \mathbb{1}_{A_n}(r, s) a(t-r) a(t-s) dr ds dt. \end{aligned}$$

Since  $\mathbb{1}_{A_n} \rightarrow 0$  a.e., Lebesgue's dominated convergence theorem yields that  $\langle\langle x, x \rangle\rangle = 0$ .

In case of norm completeness and a closed positive cone, the Riesz part of a positive inner product turns out to be an inner product. To show this, we start with the following lemma. Its proof follows standard arguments.

**4.1 Lemma** *Let  $\langle \cdot, \cdot \rangle$  be a positive inner product on  $E$  such that  $E^+$  is closed with respect to the norm induced by  $\langle \cdot, \cdot \rangle$ . If  $E$  is norm complete, then  $E$  is Dedekind complete.*

PROOF. Denote by  $\|\cdot\|$  the norm induced by  $\langle \cdot, \cdot \rangle$ . Let  $A$  be a subset of  $E$  that is bounded from above. It has to be shown that  $A$  has a supremum. Without restriction one may assume that  $A \subset E^+$  and that  $A$  is closed under finite suprema. Let  $\alpha := \sup_{x \in A} \|x\| < \infty$ . Then there is a sequence  $(x_n)_n$  in  $A$  with  $\|x_n\| \rightarrow \alpha$  and it can be chosen to be increasing. Since

$$\|y\|^2 = \|y - x\|^2 + \|x\|^2 + 2\langle y - x, x \rangle \geq \|y - x\|^2 + \|x\|^2$$

for all  $0 \leq x \leq y$ , one has for  $m > n$  that  $\|x_m\|^2 \geq \|x_m - x_n\|^2 + \|x_n\|^2$ , so  $\|x_m - x_n\|^2 \leq \|x_m\|^2 - \|x_n\|^2 \rightarrow 0$ . As  $E$  is norm complete,  $x := \lim_{n \rightarrow \infty} x_n$  exists and because  $E^+$  is closed it follows that  $x \geq x_n$  for all  $n$ .

For  $a \in A$  one has that  $x_n \vee a \in A$ , and  $a \vee x_n = x_n + (a - x_n)^+ \geq x_n + (a - x)^+$ , so that  $\|a \vee x_n\|^2 \geq \|x_n + (a - x)^+\|^2 \geq \|x_n\|^2 + \|(a - x)^+\|^2$  for all  $n$ . Then  $\|(a - x)^+\| \leq \|a \vee x_n\|^2 - \|x_n\|^2 \leq \alpha^2 - \|x_n\|^2 \rightarrow 0$ , and therefore  $a \leq x$ . Thus,  $x$  is an upper bound of  $A$ . If  $z$  is also an upper bound of  $A$ , then so is  $x \wedge z$ , which yields that  $\|x \wedge z\| \geq \alpha = \|x\|$  and that is only possible if  $x \wedge z \not\leq x$ . Hence  $x$  is the least upper bound of  $A$ .  $\square$

We will need the following extension of the parallelogram law. It can be proved by a straightforward induction argument.

**4.2 Lemma** *Let  $V$  be a vector space and let  $p$  be a seminorm on  $V$  induced by a semi-inner product. Let  $x_1, \dots, x_n \in V$ . Then*

$$\sum_{\varepsilon_1, \dots, \varepsilon_n \in \{-1, 1\}} p(\varepsilon_1 x_1 + \dots + \varepsilon_n x_n)^2 = 2^n (p(x_1)^2 + \dots + p(x_n)^2).$$

**4.3 Theorem** *Let  $\langle \cdot, \cdot \rangle$  be a positive inner product on  $E$  such that  $E^+$  is closed. If  $E$  is norm complete, then  $\langle \cdot, \cdot \rangle$  is equivalent to a Riesz inner product on  $E$ , i.e., the norm induced by  $\langle \cdot, \cdot \rangle$  is equivalent to a norm induced by a Riesz inner product.*

PROOF. Let  $\|\cdot\|$  be the norm induced by  $\langle \cdot, \cdot \rangle$ . According to Lemma 2.1,  $p$  is a Fremlin norm and hence monotone, i.e., increasing on  $E^+$ . Since  $E$  is norm complete with respect to  $\|\cdot\|$  and  $E^+$  is closed,  $\|\cdot\|$  is equivalent to the Riesz norm  $\rho: x \mapsto \| |x| \|$  on  $E$  (see [5, Cor. 3.48]). (Note that  $\rho$  need not be induced by an inner product.) More precisely, there is a constant  $c > 0$  such that  $c\rho(x) \leq \|x\| \leq \rho(x)$  for all  $x \in E$ . Let  $\langle \langle \cdot, \cdot \rangle \rangle$  be the Riesz part of  $\langle \cdot, \cdot \rangle$  and let  $|||\cdot|||$  be its induced seminorm.

On the one hand, for  $x \in E$  one has  $|||x|||^2 = ||| |x| |||^2 = \langle \langle |x|, |x| \rangle \rangle \leq \langle |x|, |x| \rangle = \rho(x)^2$ . On the other hand,  $E$  is Dedekind complete according to Lemma 4.1 and then Theorem 3.4 says that

$$|||x|||^2 = \langle \langle |x|, |x| \rangle \rangle = \inf_{\substack{|x|=x_1+\dots+x_n \\ x_k \wedge x_l=0, k \neq l \\ x_k \geq 0, \text{ all } k}} \sum_{k=1}^n \langle x_k, x_k \rangle, \quad x \in E.$$

For any  $x_1, \dots, x_n \in E^+$  that are pairwise disjoint, one has with aid of Lemma 4.2 that

$$\begin{aligned} \sum_{i=k}^n \langle x_k, x_k \rangle &= 2^{-n} \sum_{\varepsilon_1, \dots, \varepsilon_n \in \{-1, 1\}} \|\varepsilon_1 x_1 + \dots + \varepsilon_n x_n\|^2 \\ &\geq 2^{-n} \sum_{\varepsilon_1, \dots, \varepsilon_n \in \{-1, 1\}} c^2 \rho(\varepsilon_1 x_1 + \dots + \varepsilon_n x_n)^2 \\ &= 2^{-n} \sum_{\varepsilon_1, \dots, \varepsilon_n \in \{-1, 1\}} c^2 \rho(|\varepsilon_1 x_1 + \dots + \varepsilon_n x_n|)^2 \end{aligned}$$

$$\begin{aligned}
&= 2^{-n} \sum_{\varepsilon_1, \dots, \varepsilon_n \in \{-1, 1\}} c^2 \rho(x_1 + \dots + x_n)^2 \\
&= c^2 \rho(x_1 + \dots + x_n)^2.
\end{aligned}$$

It follows that for every  $x \in E$

$$\| \|x\| \| \geq c\rho(|x|) = c\rho(x).$$

Thus,  $\| \| \cdot \| \|$  is equivalent to  $\rho$ , and therefore to  $\| \cdot \|$ .  $\square$

**4.4 Corollary** *If  $\langle \cdot, \cdot \rangle$  is a positive inner product on  $E$  such that  $E^+$  is closed and  $E$  is norm complete, then the Riesz part of  $\langle \cdot, \cdot \rangle$  is an inner product.*

Finally, we will consider Riesz spaces with positive inner products that are norm complete. As Hilbert spaces such spaces are isomorphic to their norm duals. How are the orderings related?

Let  $\| \cdot \|$  be a norm on  $E$ . Denote its norm dual by  $(E', \| \cdot \|')$  and endow it with the partial ordering generated by the cone of positive (i.e. increasing) continuous linear functions. Let  $\langle \cdot, \cdot \rangle$  be an inner product on  $E$  and  $\| \cdot \|$  its induced norm. Then  $E'$  and  $E$  are isomorphic as normed spaces if and only if  $E$  is norm complete. To characterize isomorphism with respect to the ordering structure, we will call the inner product  $\langle \cdot, \cdot \rangle$  *order compatible* if for every  $y \in E$ :

$$\langle x, y \rangle \geq 0 \text{ for all } x \geq 0 \iff y \geq 0.$$

For  $y \in E$  one has that

$$x \mapsto \langle x, y \rangle \in E'^+ \iff \langle x, y \rangle \geq 0 \text{ for all } x \in E^+$$

and the latter is equivalent to  $y \geq 0$  if and only if  $\langle \cdot, \cdot \rangle$  is order compatible. Therefore, if  $E$  is norm complete,  $E'$  and  $E$  are isomorphic as partially ordered vector spaces under the natural isomorphism if and only if  $\langle \cdot, \cdot \rangle$  is order compatible. Clearly, order compatible inner products are positive. Note that if  $\langle \cdot, \cdot \rangle$  is order compatible, then  $f(x) \geq 0$  for all  $f \in E'^+$  implies that  $x \geq 0$ , so that  $E^+$  is closed (see, e.g., [7, Cor 4.2], or [5, Prop. 1.54]).

In  $\mathbb{R}^n$  the only order compatible inner product is the standard inner product, possibly with weights. More generally we have the following.

**4.5 Theorem** *Let  $\langle \cdot, \cdot \rangle$  be an order compatible inner product on  $E$ . If  $E$  is complete with respect to the norm induced by  $\langle \cdot, \cdot \rangle$ , then  $\langle \cdot, \cdot \rangle$  is a Riesz inner product.*

PROOF. Let  $\| \cdot \|$  be the norm induced by  $\langle \cdot, \cdot \rangle$ . Because  $\| \cdot \|$  is Fremlin, it is known that for every  $f \in E'$  there are  $f_1, f_2 \in E'^+$  with  $f = f_1 - f_2$  and  $\|f_1 + f_2\|' \leq \|f\|'$  (see, e.g., [8] or [5, Thm 5.3]). Because  $\langle \cdot, \cdot \rangle$  is order compatible,  $E'$  and  $E$  are isomorphic as partially ordered Hilbert spaces, and it follows that for every  $x \in E$  there are  $x_1, x_2 \in E^+$  with  $x = x_1 - x_2$  and  $\|x_1 + x_2\| \leq \|x\|$ .

As  $\|\cdot\|$  is a Fremlin norm, one has  $\|x\| \leq \| |x| \|$  and from the above also  $\| |x| \| \leq \|x\|$  for all  $x \in E$ . Thus,  $\|\cdot\|$  is a Riesz norm and Lemma 2.1 then yields that  $\langle \cdot, \cdot \rangle$  is a Riesz inner product.  $\square$

**4.6 Corollary** *Let  $\langle \cdot, \cdot \rangle$  be an inner product on  $E$  such that  $E$  is norm complete. Then  $E'$  and  $E$  are isomorphic as partially ordered Hilbert spaces if and only if  $\langle \cdot, \cdot \rangle$  is a Riesz inner product.*

I conclude with a word to the honored reader:

*Thanks!*

## References

- [1] Charalambos D. Aliprantis and Owen Burkinshaw, *Positive Operators*, Academic Press, Inc., Orlando, 1985.
- [2] G. Buskes and A. van Rooij, Almost  $f$ -algebras: Commutativity and the Cauchy-Schwarz inequality, *Positivity* **4** (2000), pp. 227–231.
- [3] D.H. Fremlin, Tensor Products of Archimedean Vector Lattices, *Amer. J. of Math.* **94** (1972), pp. 777–798.
- [4] Onno van Gaans, Extending monotone seminorms on partially ordered vector spaces, *Indag. Mathem., N.S.*, **9** (3) (1998), pp. 341–349.
- [5] Onno van Gaans, *Seminorms on ordered vector spaces*, Ph.D. thesis, University of Nijmegen, 1999.
- [6] E. de Jonge and A.C.M. van Rooij, *Introduction to Riesz spaces* (Mathematical Centre Tracts 78), Mathematisch Centrum, Amsterdam, 1977.
- [7] Isaac Namioka, *Partially Ordered Linear Topological Spaces* (Amer. Math. Soc. Memoir **24**), Providence, 1957.
- [8] Kung-Fu Ng, A duality theorem on partially ordered normed spaces, *J. London Math. Soc. (2)* **3** (1971), pp. 403–404.