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ON ONE-COMPLEMENTED SUBSPACES OF MINKOWSKI SPACES WITH SMOOTH RIESZ NORMS

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ABSTRACT. In this paper we study one-complemented subspaces of Minkowski spaces. The main objective is to examine norms on $\mathbf{R}^n$ for which every one-complemented subspace has a block basis, i.e., a basis of vectors with mutually disjoint supports. We introduce a collection of norms on $\mathbf{R}^n$ and show that, for these norms, each one-complemented subspace has a block basis. This collection contains, among others, finite sums of ℓ_p-norms, where $1 < p < \infty$ and $p \neq 2$, and their duals. In the proofs an important role is played by the derivative of the (scaled) duality map and, in particular, its behavior near the coordinate planes.

1. Introduction. This paper concerns one-complemented subspaces of Minkowski spaces, that is, subspaces of $\mathbf{R}^n$ that are the range of a projection of norm one. A classic result of Kakutani [**9**] says that if X is a real Banach space of dimension at least three, then X is Euclidean if and only if every subspace of X is one-complemented. This implies that the one-complemented subspaces of a Banach space that is not Euclidean are somehow special. The special nature of these subspaces is manifested in the following result, compare Bohnenblust [**4**] and Lindenstrauss and Tzafriri [**10**, Theorem 2.a.4]. If $\mathbf{R}^n$ is equipped with an l_p-norm, where $1 < p < \infty$ and $p \neq 2$, then a subspace is one-complemented if and only if it is the linear span of a family of vectors with mutually disjoint supports. With this result in mind it is natural to ask for which norms on $\mathbf{R}^n$ the one-complemented subspaces are spanned by vectors with mutually disjoint supports.

This question has been examined in general Banach spaces. It is known, for instance, to have a positive answer for L_p-spaces, where $1 \leq p < \infty$ and $p \neq 2$, see Ando [**1**], Bernau and Lacey [**3**], Douglas [**7**] and Tzafriri [**17**], and for some natural generalizations of L_p-spaces such as Lorenz sequence spaces and Orlicz sequence spaces, see Randrianantoanina [**12–14, 16**] and Jamison, Kamińska and Lewicki [**8**]. On the other hand, there exist one-complemented subspaces of $\mathbf{R}^3$

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that do not admit such a basis if the norm is the ℓ_∞-norm, see [**6**]. An extensive overview of many of these results and related problems is given in [**15**].

The purpose of this paper is to extend some of the ideas used by Lindenstrauss and Tzafriri in the proof of Theorem 2.a.4 in [**10**]. This theorem asserts that a subspace of an l_p-sequence space, where $1 < p < \infty$ and $p \neq 2$, is one-complemented if and only if it is spanned by a set of vectors with mutually disjoint supports. The extension allows us to introduce a class of norms on $\mathbf{R}^n$ and show that for these norms each one-complemented subspace has a basis of vectors with mutually disjoint supports. The class of norms include, among others, positive linear combinations of l_p-norms, where $1 < p < \infty$ and $p \neq 2$, and their duals.

Besides the introduction the paper contains six sections. In Section 2 several definitions and basic facts are collected. Subsequently we introduce in Section 3 a class of norms on $\mathbf{R}^n$, denoted by $\mathcal{N}^n$, and show that for these norms each one-complemented subspace is spanned by vectors with mutually disjoint supports. To decide whether a norm belongs to $\mathcal{N}^n$, one has to verify several properties of its dual norm. As the dual norm is often not at hand this can be difficult. Therefore we examine in Sections 4 and 5 simpler conditions for a norm to be in $\mathcal{N}^n$. In Section 6 the results are applied to sums of ℓ_p-norms. The final section contains a proof of a technical lemma, which is used in Section 5.

2. Basic definitions and facts. Vectors in $\mathbf{R}^n$ will sometimes be viewed as functions from $\{1, \dots, n\}$ to $\mathbf{R}$. Accordingly we write xy for the coordinate-wise product of x and y in $\mathbf{R}^n$. The *support* of $x \in \mathbf{R}^n$ is denoted by $S(x) = \{i : x_i \neq 0\}$. Further we let $\chi(x)$ denote the indicator of the support of x, so $\chi(x)_i = 1$ if $i \in S(x)$, and $\chi(x)_i = 0$ otherwise.

For simplicity, we say that a subspace R of $\mathbf{R}^n$ has a *block basis* if it is the linear span of a set of vectors $\{v^1, \dots, v^k\}$ in $\mathbf{R}^n$, with mutually disjoint supports, that is, the intersection of $S(v^i)$ with $S(v^j)$ is empty for all i and j distinct. To verify that a subspace has a block basis one can use the following simple observation.

Lemma 2.1. *For a subspace R of $\mathbf{R}^n$, the following assertions are equivalent*:

(i) *R has a block basis*;

(ii) *for every $x, y \in R$ there exists $z \in R$ with $S(z) = S(x) \cap S(y)$*;

(iii) *for every $x, y \in R$ one has that $\chi(x)y \in R$.*

Let ρ be a norm on $\mathbf{R}^n$. We say that ρ is a C^k-*norm* if ρ is k times continuously differentiable on $\mathbf{R}^n \setminus \{0\}$. We restrict ourselves to *strictly convex* norms, that is, norms for which the unit sphere does not contain any line-segments, or equivalently, $\rho(x+y)/2 < 1$ for every distinct x and y with $\rho(x) = \rho(y) = 1$. The *dual norm* of ρ is denoted by ρ^*, so $\rho^*(y) = \sup\{\langle x, y\rangle : x \in \mathbf{R}^n \text{ and } \rho(x) \leq 1\}$ for all $y \in \mathbf{R}^n$. Here $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbf{R}^n$. There exists a simple relation between the differentiability of the norm and the geometry of its dual. A norm ρ on $\mathbf{R}^n$ is a C^1-norm if and only if its dual ρ^* is strictly convex. Moreover ρ is a C^1-norm if and only if for each $x \in \mathbf{R}^n$ there exists a unique point $x^* \in \mathbf{R}^n$ such that $\rho^*(x^*) = \rho(x)$ and $\langle x, x^* \rangle = \rho^*(x^*)\rho(x)$. The map $J_\rho : \mathbf{R}^n \to \mathbf{R}^n$ given by $J_\rho(x) = x^*$ is called the (*scaled*) *duality map* of ρ. Throughout the text we often write x^* instead of $J_\rho(x)$. The duality map has the following basic properties, see [**2**] or [**11**].

Proposition 2.2. *Let ρ be a C^1-norm on $\mathbf{R}^n$.*

(i) *The duality map $J_\rho : \mathbf{R}^n \to \mathbf{R}^n$ satisfies $J_\rho(0) = 0$ and*

$$J_\rho(x) = (\rho\nabla\rho)(x) = \rho(x)\nabla\rho(x) \quad \textit{for all } x \in \mathbf{R} \setminus \{0\}. \tag{1}$$

(ii) *If ρ is strictly convex, then J_ρ is a continuous bijection from $\mathbf{R}^n$ to $\mathbf{R}^n$ with a continuous inverse. Moreover the inverse is J_{ρ^*} and*

$$(\rho^*\nabla\rho^*)((\rho\nabla\rho)(x)) = x \quad \textit{for all } x \in \mathbf{R}^n \setminus \{0\}. \tag{2}$$

We will often consider norms that have a second derivative as well. If ρ is C^2 on an open subset V of $\mathbf{R}^n$, then we denote its Hesse matrix at $x \in V$ by $H_\rho(x)$, so

$$H_\rho(x)_{ij} = (D_i D_j \rho)(x).$$

A special role will be played by the derivative of the duality map. Observe that if ρ is a C^2-norm on an open subset V of $\mathbf{R}^n$, then $\rho\nabla\rho$ is C^1 on V, and hence by Proposition 2.2 we see that J_ρ is C^1 on V. We denote its derivative at x by $G_\rho(x)$, so

$$G_\rho(x) = D(\rho\nabla\rho)(x) = \nabla\rho(x)\nabla\rho(x)^\top + \rho(x)H_\rho(x) \quad \text{for all } x \in V. \tag{3}$$

Note that $G_\rho(x)$ is a symmetric matrix, as ρ is a C^2-norm, and that $G_\rho(x)$ is positive semi-definite, since ρ is a nonnegative convex function. By applying the inverse function theorem and the chain rule to equation (2), one can establish the following lemma.

Lemma 2.3. *Let ρ be a strictly convex C^1-norm on $\mathbf{R}^n$. If ρ is C^2 on an open set V, then ρ^* is C^2 on $J_\rho(V)$ if and only if* $\det G_\rho(x) \neq 0$ *for all $x \in V$. Moreover, in that case, one has that $G_{\rho^*}(x^*)G_\rho(x) = I$ for all $x \in V$.*

3. One-complemented subspaces. In this section we give an abstract theorem on one-complemented subspaces of $\mathbf{R}^n$. The ideas behind this theorem can be conveniently outlined by considering l_p-norms. Let ρ be an l_p-norm on $\mathbf{R}^n$ with $p \geq 2$. Then both ρ and ρ^* are C^2 on $U = \{x \in \mathbf{R}^n : x_i \neq 0 \text{ for all } i\}$. Moreover, if $p > 2$ and $x \in \mathbf{R}^n$ with $x_i = 0$, then the ith row and column of $G_\rho(x)$ are zero. If $x \in U$, then Lemma 2.3 yields that $\det G_\rho(x) \neq 0$ and $G_{\rho^*}(x^*)G_\rho(x) = I$. By using these observations it is not difficult to show that

$$\lim_{m\to\infty} G_{\rho^*}(u_m^*)G_\rho(x) = \mathrm{Diag}\,(\chi(x))$$

for $x \neq 0$ and $(u_m)_m$ a sequence in U with $u_m \to x$. Further, if R is a one-complemented subspace of $\mathbf{R}^n$ under ρ, then one can prove that $G_{\rho^*}(u_m^*)G_\rho(x)y$ is in R whenever $x, y \in R$ and $u_m \in U \cap R$ for all m. Now if R contains a vector with full support, then for every x in R the set $U \cap R$ contains a sequence that converges to x, and hence we find that $\chi(x)y \in R$ for all $x, y \in R$. If R does not contain a vector with full support the same arguments can be used after a reduction of the dimension. It turns out that the above ideas can be applied to more general norms on $\mathbf{R}^n$ than l_p-norms. Indeed we will see that the ideas also work for the following class of norms.

Definition 3.1. Let $\mathcal{N}^n$ be the set of all strictly convex C^2-norms ρ on $\mathbf{R}^n$ such that ρ^* is C^2 on $J_\rho(U)$, where $U = \{x \in \mathbf{R}^n : x_i \neq 0 \text{ for all } i\}$. Moreover, it is required that for every $x \neq 0$ and every sequence $(u_m)_m$ in U with $u_m \to x$, one has that

(a) $G_\rho(x)_{ij} = 0$ for every $i \notin S(x)$ and $1 \leq j \leq n$,

(b) $G_{\rho^*}(u_m^*)_{ij}$ converges for all $i, j \in S(x)$ as $m \to \infty$, and

(c) $G_{\rho^*}(u_m^*)_{ij}$ converges to 0 for every $i \in S(x)$ and $j \notin S(x)$ as $m \to \infty$.

It is not hard to verify that an l_p-norm is in $\mathcal{N}^n$ if $p > 2$. For more general norms, however, it can be rather difficult to verify the properties of the dual norm. In Section 5 we will see how to partly overcome this problem. We now state the main theorem of this section.

Theorem 3.2. *If ρ is a norm in $\mathcal{N}^n$ and R is a one-complemented subspace of $\mathbf{R}^n$ under ρ, then R has a block basis.*

The proof of the theorem is based on three lemmas, which will be discussed first.

Lemma 3.3. *If ρ is a C^1-norm on $\mathbf{R}^n$, and R is a one-complemented subspace of $\mathbf{R}^n$ under ρ, then the following assertions are true*:

(i) *$J_\rho(R)$ is a linear subspace*;

(ii) *if ρ is C^2 on an open set V, then $G_\rho(x)y \in J_\rho(R)$ for every $x \in V \cap R$ and $y \in R$.*

Proof. Let $P : \mathbf{R}^n \to \mathbf{R}^n$ be a projection of ρ-norm one and range R. We remark that the transpose $P^\top$ of P is a projection of ρ^*-norm one. Indeed, $\langle x, P^\top y\rangle = \langle Px, y\rangle \leq \rho(Px)\rho^*(y) \leq \rho(x)\rho^*(y)$, so that

$$(4) \qquad \rho^*(P^\top y) = \sup\{\langle x, P^\top y\rangle : \rho(x) \leq 1\} \leq \rho^*(y) \quad \text{for } y \in \mathbf{R}^n.$$

To prove the first assertion, we show that $J_\rho(R)$ is the range of $P^\top$. So let $x \in R$, and observe that

$$\rho(x)\rho^*(x^*) = \langle x, x^*\rangle = \langle Px, x^*\rangle = \langle x, P^\top x^*\rangle \leq \rho(x)\rho^*(P^\top x^*).$$

Combining this inequality with equation (4) yields $\rho^*(P^\top x^*) = \rho^*(x^*)$ and $\langle x, x^*\rangle = \rho(x)\rho^*(P^\top x^*)$. As ρ is a C^1-norm, this gives $P^\top x^* = x^*$. Thus, $J_\rho(R)$ is a subset of the range of $P^\top$.

Now let $\mathcal{R}(P^\top)$ denote the range of $P^\top$. By duality we have that $J_{\rho^*}(\mathcal{R}(P^\top))$ is contained in R. Now using (ii) in Proposition 2.2, we find that $\mathcal{R}(P^\top) \subset J_\rho(R)$, and hence $J_\rho(R)$ is the range of $P^\top$.

To prove the second assertion, let $x \in V \cap R$ and $y \in R$. As $J_\rho(R)$ is a linear subspace we see that

$$\begin{aligned} G_\rho(x)y &= \lim_{t\to 0} \frac{\rho(x+ty)\nabla\rho(x+ty) - \rho(x)\nabla\rho(x)}{t} \\ &= \lim_{t\to 0} \frac{J_\rho(x+ty) - J_\rho(x)}{t} \end{aligned}$$

is in $J_\rho(R)$, which completes the proof. □

We would like to mention that the converse of the first assertion in the previous lemma also holds, see Calvert [**5**].

Lemma 3.4. *Let ρ be a norm in $\mathcal{N}^n$ and let $x \in \mathbf{R}^n$ with $x \neq 0$. If $(u_m)_m$ is a sequence in $U = \{z \in \mathbf{R}^n : z_i \neq 0$ for all $i\}$ such that $u_m \to x$ as $m \to \infty$, then*

$$\lim_{m\to\infty} G_{\rho^*}(u_m^*)G_\rho(x) = \mathrm{Diag}\,(\chi(x)). \tag{5}$$

Proof. First let $i, j \in S(x)$. By property (b) in Definition 3.1, we can define numbers $a_{ij} = \lim_{m\to\infty} G_{\rho^*}(u_m^*)_{ij}$. From Lemma 2.3, it follows that

$$\sum_{l=1}^{n} G_{\rho^*}(u_m^*)_{il} G_\rho(u_m)_{lj} = \delta_{ij} \quad \text{for all } m.$$

By letting $m \to \infty$ in the previous equality, and using property (a) in Definition 3.1, we deduce that

$$\sum_{l\in S(x)} a_{il} G_\rho(x)_{lj} = \delta_{ij} \quad \text{for all } i, j \in S(x). \tag{6}$$

Thus we find for every $i, j \in S(x)$ that

$$\begin{aligned}\lim_{m\to\infty}(G_{\rho^*}(u_m^*)G_\rho(x))_{ij} &= \lim_{m\to\infty}\sum_{l\in S(x)} G_{\rho^*}(u_m^*)_{il}G_\rho(x)_{lj} \\ &= \sum_{l\in S(x)} a_{il}G_\rho(x)_{lj} = \delta_{ij}.\end{aligned}$$

Now let $i \in S(x)$ and $j \notin S(x)$. We remark that $G_\rho(x)$ is symmetric, as ρ is a C^2-norm. Exploiting this fact and property (a) shows that

$$G_{\rho^*}(u_m^*)_{il}G_\rho(x)_{lj} = G_{\rho^*}(u_m^*)_{il}G_\rho(x)_{jl} = 0 \quad \text{for all } m.$$

Therefore $\lim_{m\to\infty}(G_{\rho^*}(u_m^*)G_\rho(x))_{ij} = 0$ for $i \in S(x)$ and $j \notin S(x)$.

Finally, let $i \notin S(x)$ and $1 \le j \le n$. As ρ^* is C^2 on $J_\rho(U)$ the matrix $G_{\rho^*}(u_m^*)$ is symmetric for all m. Therefore we can use properties (a) and (c) to obtain

$$\begin{aligned}&\lim_{m\to\infty}(G_{\rho^*}(u_m^*)G_\rho(x))_{ij} \\ &= \lim_{m\to\infty}\left(\sum_{l\in S(x)} G_{\rho^*}(u_m^*)_{li}G_\rho(x)_{lj} + \sum_{l\notin S(x)} G_{\rho^*}(u_m^*)_{il}G_\rho(x)_{lj}\right) \\ &= 0.\end{aligned} \tag{7}$$

By collecting the pieces we find that $\lim_{m\to\infty} G_{\rho^*}(u_m^*)G_\rho(x) = \mathrm{Diag}\,(\chi(x))$. □

The following technical lemma is used to reduce the dimension in the proof of Theorem 3.2.

Lemma 3.5. *Let ρ be a norm on $\mathbf{R}^n$, and let $1 \le k \le n$. Suppose $\eta : \mathbf{R}^k \to \mathbf{R}$ is defined by $\eta(x) = \rho(\bar{x})$ for all $x \in \mathbf{R}^k$, where $\bar{x}_i = x_i$ for $1 \le i \le k$, and $\bar{x}_i = 0$ otherwise. If $\rho \in \mathcal{N}^n$, then $\eta \in \mathcal{N}^k$.*

Proof. Since ρ is a strictly convex C^2-norm on $\mathbf{R}^n$, it follows directly from the definition of η that η is a strictly convex C^2-norm on $\mathbf{R}^k$. Moreover,

$$G_\eta(x)_{ij} = G_\rho(\bar{x})_{ij} \quad \text{for all } 1 \le i, j \le k \text{ and } x \in \mathbf{R}^k.$$

Let $W = \{x \in \mathbf{R}^k : x_i \neq 0 \text{ for all } i\}$. To show that η^* is C^2 on $J_\eta(W)$, we fix $v \in W$. By mixing sequences it follows from (b) in Definition 3.1 that, if $1 \leq i, j \leq k$, then $G_{\rho^*}(u_m^*)_{ij}$ converges to the same limit, say a_{ij}, for every sequence $(u_m)_m$ in $U = \{x \in \mathbf{R}^n : x_i \neq 0 \text{ for all } i\}$ with $u_m \to \bar{v}$.

Let $A(v) = (a_{ij})$ and put $B(v) = (G_\rho(\bar{v})_{ij})$, where $1 \leq i,\ j \leq k$. By property (a) we know that $G_\rho(\bar{v})_{ij} = 0$ for all $k < i \leq n$ and $1 \leq j \leq n$, and hence $A(v)B(v) = I$ by Lemma 3.4. This implies that $\det B(v) \neq 0$. We now remark that $B(v) = G_\eta(v)$, so that we can apply Lemma 2.3 to see that η^* is C^2 on $J_\eta(W)$. Moreover,

$$G_{\eta^*}(v^*) = G_\eta(v)^{-1} = B(v)^{-1} = A(v). \tag{8}$$

We conclude the proof by verifying the properties (a), (b), and (c) for η. Let $x \in \mathbf{R}^k$ and $x \neq 0$. Clearly $G_\eta(x)_{ij} = 0$, if $i \notin S(x)$ and $1 \leq j \leq k$, because $G_\eta(x)_{ij} = G_\rho(\bar{x})_{ij} = 0$ if $i \notin S(\bar{x}) = S(x)$ and $1 \leq j \leq n$.

Let $(w_m)_m$ be a sequence in W such that $w_m \to x$, where $x \neq 0$. Then we know from property (b) for ρ that $G_{\rho^*}((w_m, \varepsilon, \dots, \varepsilon)^*)_{ij}$ converges to $A(w_m)_{ij}$, as $\varepsilon \to 0$ for each $m \geq 1$ and $1 \leq i, j \leq k$. It follows from (8) that $G_{\eta^*}(w_m^*)_{ij} = A(w_m)_{ij}$ for all $1 \leq i, j \leq k$ and $m \geq 1$. Thus, for each $m \geq 1$, there exists $\varepsilon_m > 0$ such that

$$|G_{\rho^*}((w_m, \varepsilon_m, \dots, \varepsilon_m)^*)_{ij} - G_{\eta^*}(w_m^*)_{ij}| < 1/m \quad \text{for all } 1 \leq i, j \leq k.$$

Combining this inequality with (b) and (c) for ρ gives that $G_{\eta^*}(w_m^*)_{ij}$ converges for each $i, j \in S(x)$, and $G_{\eta^*}(w_m^*)_{ij}$ converges to 0 for all $i \in S(x)$ and $j \notin S(x)$. □

By applying the previous lemmas we can now prove Theorem 3.2.

Proof of Theorem 3.2. Let R be a one-complemented subspace of $\mathbf{R}^n$ under ρ. Then there exists a projection $P : \mathbf{R}^n \to \mathbf{R}^n$ of ρ-norm one and range R. Let $I = \{i : x_i \neq 0 \text{ for some } x \in R\}$. By relabeling we may assume that $I = \{1, \dots, k\}$ for a certain $1 \leq k \leq n$. Define $\eta : \mathbf{R}^k \to \mathbf{R}$ by $\eta(x) = \rho(\bar{x})$, where $\bar{x}_i = x_i$ for $1 \leq i \leq k$, and $\bar{x}_i = 0$ otherwise. Further, let $S = \{x \in \mathbf{R}^k : \bar{x} \in R\}$. It is easy to see that S is a one-complemented subspace of $\mathbf{R}^k$ under η.

Indeed define $Q : \mathbf{R}^k \to \mathbf{R}^k$ by $(Qx)_i = (P\bar{x})_i$ for $1 \leq i \leq k$ and $x \in \mathbf{R}^k$. Observe that $\overline{Qx} = P\bar{x}$, as $(P\bar{x})_i = 0$ for all $i > k$. Therefore, $Qx = x$ if and only if $P\bar{x} = \bar{x}$, so that S is the set of fixed points of Q. Moreover, $(Q(Qx))_i = (PP\bar{x})_i = (P\bar{x})_i = (Qx)_i$ for $1 \leq i \leq k$, and $\eta(Qx) = \rho(P\bar{x}) \leq \rho(\bar{x}) = \eta(x)$ for $x \in \mathbf{R}^k$. Thus, we conclude that Q is a projection of η-norm at most one and range S.

Now let $x, y \in S$. By taking a suitable linear combination of elements of R we can find a vector in R with support $\{1, \ldots, k\}$, so that S contains a vector with all its entries nonzero. Therefore we can find a sequence $(w_m)_m$ in the intersection of S with $W = \{z \in \mathbf{R}^k : z_i \neq 0 \text{ for all } 1 \leq i \leq k\}$ such that $w_m \to x$ as $m \to \infty$. We know by Lemma 3.5 that $\eta \in \mathcal{N}^k$, and hence $G_\eta(x)y \in J_\eta(S)$ by Lemma 3.3. Applying Lemma 3.3 again for η^* and recalling that J_{η^*} is the inverse of J_η gives $G_{\eta^*}(w_m^*)G_\eta(x)y \in S$ for all m. Consequently, Lemma 3.4 yields that

$$\lim_{m\to\infty} G_{\eta^*}(w_m^*)G_\eta(x)y = \chi(x)y \in S.$$

Using Lemma 2.1 we find that S has a block basis, and from this it follows that R has a block basis. □

If ρ is a Riesz norm, the assertion in Theorem 3.2 is also true if the conditions are satisfied by ρ^* instead of ρ. Recall that a norm ρ on $\mathbf{R}^n$ is a *Riesz norm* if $\rho(x) \leq \rho(y)$ for all $x, y \in \mathbf{R}^n$ with $|x| \leq |y|$. The proof of the corollary uses the fact that the dual norm of a Riesz norm is again a Riesz norm.

Corollary 3.6. *If ρ is a Riesz norm on $\mathbf{R}^n$ and $\rho^* \in \mathcal{N}^n$, then every one-complemented subspace of $\mathbf{R}^n$ under ρ has a block basis.*

Proof. Let $P : \mathbf{R}^n \to \mathbf{R}^n$ be a projection of ρ-norm one and range R. Then $P^\top : \mathbf{R}^n \to \mathbf{R}^n$ is a projection with ρ^*-norm one and range $J_\rho(R)$. From Theorem 3.2 it follows that $J_\rho(R)$ has a block basis.

Now let $x, y \in R$. We will show that there exists $z \in R$ such that $S(z) = S(x) \cap S(y)$. As $J_\rho(R)$ has a block basis we know by Lemma 2.1 that there exists $w \in J_\rho(R)$ such that $S(w) = S(x^*) \cap S(y^*)$. Put $z = J_{\rho^*}(w)$ and remark that $z \in R$. Now it suffices to prove that $S(v) = S(v^*)$ for all $v \in \mathbf{R}^n$. So let $v \in \mathbf{R}^n$. Put $u = \operatorname{sgn}(v)|v^*|$, where $\operatorname{sgn}(v)_i = 1$ if $v_i > 0$, $\operatorname{sgn}(v)_i = -1$ if $v_i < 0$, and $\operatorname{sgn}(v)_i = 0$ otherwise. Since ρ is a Riesz norm, so is ρ^*, and hence $\rho^*(u) \leq \rho^*(v^*)$. This implies that

$$\langle v, u\rangle = \langle |v|, |v^*|\rangle \geq \langle v, v^*\rangle = \rho(v)\rho^*(v^*) \geq \rho(v)\rho^*(u).$$

Therefore $v^* = u$ and hence $S(v^*)$ is contained in $S(v)$. The other inclusion is obtained by using duality, and this completes the proof. □

4. A matrix lemma. To apply Theorem 3.2 one needs to decide if ρ satisfies the properties in Definition 3.1. A major difficulty is to verify the properties for the dual of ρ. For instance, if ρ is a positive linear combination of ℓ_p-norms, then it is not clear what the dual of ρ is, and hence there is no direct way to verify the properties. It is therefore useful to find assumptions for ρ that yield the properties for ρ^*. In this section a matrix lemma is presented that will help to formulate such conditions for ρ. This lemma is of a purely linear algebraic nature and is more or less independent of the main issue of the paper.

Before we give the lemma it is convenient to introduce the following technical definition.

Definition 4.1. A sequence of $n \times n$ matrices $(A(m))_m$, where $A(m) = (a(m)_{ij})$, is said to *behave well relative to* $S \subset \{1, \dots, n\}$ if

(1) $(a(m)_{ij})_{i,j\in S}$ converges to an invertible matrix,

(2) $a(m)_{ii} \neq 0$ for all $i \notin S$ and m large,

(3) $a(m)_{ij}/a(m)_{ii} \to 0$ as $m \to \infty$ for all $i \notin S$ and $j \neq i$,

(4) $a(m)_{ij} \to 0$ as $m \to \infty$ for all $i \in S$ and $j \notin S$.

The matrix lemma can now be stated as follows.

Lemma 4.2. *Let $(A(m))_m$ and $(B(m))_m$ be two sequences of $n \times n$ matrices, and let $S \subset \{1, \dots, n\}$. If $B(m)A(m) = I$ for all m, then $(A(m))_m$ behaves well relative to S if and only if $(B(m)^\top)_m$ behaves well relative to S. In that case $b(m)_{ii}a(m)_{ii} \to 1$ for all $i \notin S$ and $\lim_{m\to\infty}(B(m)A(m))_{i,j\in S} = I$.*

Proof. As $A(m)^\top B(m)^\top = (B(m)A(m))^\top = I$, it suffices to prove one implication of the equivalence. So suppose that $(A(m))_m$ behaves well relative to S. Without loss of generality, we may assume that $S = \{1, \dots, k\}$. Next we divide the matrices in blocks in the following manner:

$$A(m) = \begin{pmatrix} A_{11}(m) & A_{12}(m) \\ A_{21}(m) & A_{22}(m) \end{pmatrix}$$

and

$$B(m) = \begin{pmatrix} B_{11}(m) & B_{12}(m) \\ B_{21}(m) & B_{22}(m) \end{pmatrix},$$

where $A_{11}(m)$ and $B_{11}(m)$ are $k \times k$ matrices, and $A_{22}(m)$ and $B_{22}(m)$ are $(n-k) \times (n-k)$ matrices. Define for each $m \geq 1$ the matrix

$$C(m) = \begin{pmatrix} A_{11}(m) & 0 \\ 0 & C_{22}(m) \end{pmatrix},$$

where $C_{22}(m) = \operatorname{Diag}(a(m)_{(n-k)(n-k)}, \dots, a(m)_{nn})$. From property (1) in Definition 4.1 it follows that $A = \lim_{m\to\infty} A_{11}(m)$ is invertible. Therefore $A_{11}(m)$ is invertible for all m sufficiently large, and hence $A_{11}(m)^{-1} \to A^{-1}$ as $m \to \infty$. Consequently $A_{11}(m)^{-1}$ is bounded. Since

$$\begin{aligned} &C(m)^{-1}(A(m) - C(m)) \\ &\qquad = \begin{pmatrix} 0 & A(m)_{11}^{-1} A_{12}(m) \\ C_{22}(m)^{-1} A_{21}(m) & C(m)_{22}^{-1}(A_{22}(m) - C_{22}(m)) \end{pmatrix} \end{aligned}$$

and $(A(m))_m$ behaves well relative to S, we deduce that $(C(m)^{-1}(A(m) - C(m)))_{ij} \to 0$ as $m \to \infty$ for all $1 \leq i, j \leq n$. Indeed, $A_{12}(m) \to 0$ by property (4) and the $n-k$ bottom rows converge to 0 by property (3). Thus, $\|I - C(m)^{-1}A(m)\| < 1$ for all m sufficiently large, so that $C(m)^{-1}A(m)$ is invertible and

$$\begin{aligned} \|I - A(m)^{-1}C(m)\| &= \left\| \sum_{i=1}^{\infty} (I - C(m)^{-1}A(m))^i \right\| \\ &\leq \frac{\|I - C(m)^{-1}A(m)\|}{1 - \|I - C(m)^{-1}A(m)\|}. \end{aligned}$$

As the righthand side converges to 0 as $m \to \infty$, and $B(m)A(m) = I$ for all m, we find that $\|I - B(m)C(m)\| \to 0$. From this, we obtain the following four relations as $m \to \infty$:

$$B_{11}(m)A_{11}(m) \longrightarrow I, \quad B_{22}(m)C_{22}(m) \longrightarrow I,$$
$$B_{12}(m)C_{22}(m) \longrightarrow 0, \quad B_{21}(m)A_{11}(m) \longrightarrow 0.$$

The first relation implies that $B_{11}(m) \to A^{-1}$, and hence $(B(m)^{\top}_{ij})_{i,j \in S}$ converges to an invertible matrix. Moreover $b(m)_{ii}a(m)_{ii} \to 1$ for all $i \notin S$ by the second relation, so that $b(m)_{ii} \neq 0$ for all m large.

Recall that $(A_{11}(m)^{-1})_m$ is bounded. Therefore it follows from the last relation that $B_{21}(m) = B_{21}(m)A_{11}(m)A_{11}(m)^{-1} \to 0$, and thus $b(m)_{ij} \to 0$ for all $i \notin S$ and $j \in S$. Hence $(B(m)^{\top})_m$ satisfies property (4) in Definition 4.1.

To prove property (3) in Definition 4.1, we remark that $b(m)_{ij} \times a(m)_{jj} \to 0$ for all $i, j \notin S$ and $i \neq j$ by the second relation. As $b(m)_{jj}a(m)_{jj} \to 1$ for all $j \notin S$, we see that $b(m)_{ij}/b(m)_{jj} \to 0$ for all $i, j \notin S$ and $i \neq j$. Furthermore, it follows from the third relation that $b(m)_{ij}a(m)_{jj} \to 0$ for $i \in S$ and $j \notin S$, and hence we find that $b(m)_{ij}/b(m)_{jj} \to 0$ for $i \in S$ and $j \notin S$. Thus $(B(m)^{\top})_m$ satisfies property (3) in Definition 4.1, and hence $(B(m)^{\top})_m$ behaves well relative to S. □

To conclude this section we remark that, if both $(A(m))_m$ and $(B(m))_m$ are sequences of symmetric matrices, and $B(m)A(m) = I$ for each $m \geq 1$, then $(A(m))_m$ behaves well relative to S if and only if $(B(m))_m$ behaves well relative to S.

5. Sufficient conditions to be in $\mathcal{N}^n$. The main objective of this section is to give conditions for a norm ρ to be in $\mathcal{N}^n$, which can be verified without any knowledge of its dual norm. In fact, we define another class of norms, and show that this class is contained in $\mathcal{N}^n$. Let $\mathcal{N}_0^n$ be the collection of strictly convex C^1-norms ρ on $\mathbf{R}^n$ such that ρ is a Riesz norm, ρ is C^2 on $U = \{x \in \mathbf{R}^n : x_i \neq 0 \text{ for all } i\}$, and $\det G_\rho(x) \neq 0$ for all $x \in U$.

Definition 5.1. A norm $\rho \in \mathcal{N}_0^n$ is said to be in $\mathcal{N}_1^n$ if for every $x \in \mathbf{R}^n \setminus \{0\}$ and every sequence $(u_m)_m$ in U, with $u_m \to x$, the sequence $(G_\rho(u_m))_m$ behaves well relative to $S(x)$, and $G_\rho(u_m)_{ii} \to 0$ for all $i \notin S(x)$. It is said to be in $\mathcal{N}_2^n$ if, for every $x \in \mathbf{R}^n \setminus \{0\}$ and every sequence $(u_m)_m$ in U, with $u_m \to x$, the sequence $(G_\rho(u_m))_m$ behaves well relative to $S(x)$, and $G_\rho(u_m)_{ii} \to \infty$ for all $i \notin S(x)$.

We will see that $\mathcal{N}_1^n$ is contained in $\mathcal{N}^n$. But before we discuss this inclusion we first show that $\mathcal{N}_1^n$ is dual to $\mathcal{N}_2^n$.

Lemma 5.2. *A norm ρ belongs to $\mathcal{N}_1^n$ if and only if ρ^* belongs to $\mathcal{N}_2^n$.*

Proof. Let ρ be in $\mathcal{N}_0^n$. Then we know from general theory that ρ^* is a strictly convex C^1-norm on $\mathbf{R}^n$, and that ρ^* is a Riesz norm. Since $\det G_\rho(x) \neq 0$ for all $x \in U$, Lemma 2.3 implies that ρ^* is a C^2 function on $J_\rho(U)$. As ρ^* is a Riesz norm, $J_\rho(U) = U$, so that ρ^* is C^2 on U.

Further, $G_{\rho^*}(u^*)G_\rho(u) = I$ for all $u \in U$ by Lemma 2.3. This implies that $G_{\rho^*}(u)$ is invertible for all $u \in U$, as $J_\rho(U) = U$. Therefore $\det G_{\rho^*}(u) \neq 0$ for all $u \in U$. Thus, $\rho^* \in \mathcal{N}_0^n$ and by duality we conclude that $\rho \in \mathcal{N}_0^n$ if and only if $\rho^* \in \mathcal{N}_0^n$.

Now let $\rho \in \mathcal{N}_1^n$. From the previous paragraph it follows that $\rho^* \in \mathcal{N}_0^n$. Let $y \in \mathbf{R}^n \setminus \{0\}$ and $(v_m)_m$ be a sequence in U with $v_m \to y$. Let $x = J_\rho^{-1}(y)$ and, for each $m \geq 1$, let $u_m = J_\rho^{-1}(v_m)$. Clearly, $(u_m)_m$ in U and $u_m \to x$. Furthermore, $S(x) = S(y)$, since ρ is a Riesz norm. By Lemma 2.3 we have that $G_{\rho^*}(v_m)G_\rho(u_m) = I$ for all $m \geq 1$. We remark that both $G_{\rho^*}(v_m)$ and $G_\rho(u_m)$ are symmetric for all $m \geq 1$. Therefore, Lemma 4.2 implies that $(G_{\rho^*}(v_m))_m$ behaves well relative to $S(x) = S(y)$. Moreover,

$$G_{\rho^*}(v_m)_{ii}G_\rho(u_m)_{ii} \longrightarrow 1 \quad \text{for all } i \notin S(y). \tag{10}$$

To infer that $G_{\rho^*}(v_m)_{ii} \to \infty$ for all $i \notin S(y)$, we remark that $G_\rho(u_m)$ is invertible for all $m \geq 1$. Combining this with the fact that $G_\rho(u_m)$ is positive semi-definite implies that $G_\rho(u_m)$ is positive definite for all $m \geq 1$. Therefore $G_\rho(u_m)_{ii} > 0$ for all $i \notin S(x)$ and $m \geq 1$. Now (10) and the fact that $G_\rho(u_m)_{ii} \to 0$ for $i \notin S(x)$ yields that $G_{\rho^*}(v_m)_{ii} \to \infty$ for all $i \notin S(y)$. Thus we find that $\rho^* \in \mathcal{N}_2^n$. The other implication can be proved in a similar fashion. □

Next we show that every norm in $\mathcal{N}_1^n$ belongs to $\mathcal{N}^n$. The greater part of the proof of this inclusion consists of showing that a norm in $\mathcal{N}_1^n$ is a C^2-norm. To establish this fact some arguments from analysis are required and for the reader's convenience we prove it separately in Section 7.

Lemma 5.3. *Every norm in $\mathcal{N}_1^n$ is a C^2-norm.*

Using this lemma we now show that every norm in $\mathcal{N}_1^n$ is in $\mathcal{N}^n$.

Proposition 5.4. *Every norm in $\mathcal{N}_1^n$ belongs to $\mathcal{N}^n$.*

Proof. Let $\rho \in \mathcal{N}_1^n$. It follows from Lemma 5.3 that ρ is a C^2-norm. Hence, $G_\rho(x)$ is continuous. Furthermore Lemma 5.2 implies that $\rho^* \in \mathcal{N}_2^n$, so that ρ^* is C^2 on $U = \{x \in \mathbf{R}^n : x_i \neq 0 \text{ for all } i\}$. Now let $x \neq 0$ and $(u_m)_m$ be a sequence in U with $u_m \to x$. From property (3) in Definition 4.1 it follows that $G_\rho(u_m)_{ij} \to 0$ for all $i \notin S(x)$ and $j \neq i$, because $G_\rho(u_m)_{ii} \to 0$ if $i \notin S(x)$. Thus $G_\rho(x)_{ij} = \lim_{m\to\infty} G_\rho(u_m)_{ij} = 0$ for all $i \notin S(x)$ and $1 \leq j \leq n$. This proves the first property in Definition 3.1.

To establish the second property, we remark that $u_m^* \in U$ for all m and $u_m^* \to x^*$, so that by property (1) in Definition 4.1 $(G_{\rho^*}(u_m^*))_{i,j\in S(x^*)}$ converges to an invertible matrix. Since $S(x) = S(x^*)$, this implies that $(G_{\rho^*}(u_m^*))_{ij}$ converges for all $i, j \in S(x)$. The third property is an immediate consequence of property (4) in Definition 4.1 for $G_{\rho^*}(u_m^*)$, and this completes the proof. □

A combination of Theorem 3.2, Corollary 3.6 and Proposition 5.4 yields the following corollary.

Corollary 5.5. *If R is a one-complemented subspace in $\mathbf{R}^n$ under ρ, and ρ is in $\mathcal{N}_1^n$ or $\mathcal{N}_2^n$, then R has a block basis.*

6. Sums of ℓ_p-norms. In this section the previous results are applied to positive linear combinations of ℓ_p-norms. In particular, the following theorem is proved.

Theorem 6.1. *Let* $\alpha_1, \ldots, \alpha_r > 0$ *and let* $p_1, \ldots, p_r \in (1, \infty)$. *Suppose that* $\eta(x) = \sum_{k=1}^{r} \alpha_k \rho_k(x)$ *for all* $x \in \mathbf{R}^n$, *where* ρ_k *is the* ℓ_p*-norm on* $\mathbf{R}^n$ *with* $p = p_k$. *If* $\min\{p_1, \ldots, p_r\} \neq 2$, *then every one-complemented subspace of* $\mathbf{R}^n$ *under* η *has a block basis. Moreover,* $\eta \in \mathcal{N}_1^n$ *if* $\min\{p_1, \ldots, p_r\} > 2$, *and* $\eta \in \mathcal{N}_2^n$ *if* $\min\{p_1, \ldots, p_r\} < 2$.

Proof. By Corollary 5.5 it suffices to show the second assertion. We remark that $\rho_k \in \mathcal{N}_0^n$ for $1 \leq k \leq r$. Indeed, as ρ_k is an ℓ_p-norm on $\mathbf{R}^n$ with $1 < p < \infty$, it is clear that ρ_k is a strictly convex C^1-norm, and ρ_k is C^2 on $U = \{x \in \mathbf{R}^n : x_i \neq 0 \text{ for all } i\}$. Moreover, ρ_k is Riesz, so that $J_{\rho_k}(U) = U$. As ρ_k^* is again an ℓ_p-norm with $1 < p < \infty$, we can apply Lemma 2.3 to see that $\det G_{\rho_k}(u) \neq 0$ for all $u \in U$.

Thus, to prove that $\eta \in \mathcal{N}_0^n$, it suffices to show that if $\rho, \gamma \in \mathcal{N}_0^n$, then $\rho + \gamma \in \mathcal{N}_0^n$. Clearly $\rho + \gamma$ is a C^1-norm and it is C^2 on U. It is also straightforward to verify that $\rho + \gamma$ is a strictly convex Riesz norm. To see that $\det G_{\rho+\gamma}(u) \neq 0$ for all $u \in U$, observe that

$$G_{\rho+\gamma}(u) = G_\rho(u) + G_\gamma(u) + H_{\rho\gamma}(u) \quad \text{for all } u \in U. \tag{11}$$

The matrices $G_\rho(u)$ and $G_\gamma(u)$ are positive semi-definite and invertible for each $u \in U$, and hence both positive definite. As ρ and γ are both nonnegative convex functions, the matrix $H_{\rho\gamma}(u)$ is positive semi-definite for $u \in U$. Therefore, $G_{\rho+\gamma}(u)$ is positive definite and hence $\det G_{\rho+\gamma}(u) \neq 0$ for all $u \in U$, which proves that $\rho + \gamma \in \mathcal{N}_0^n$.

Now, let $x \neq 0$, and let $(u_m)_m$ be a sequence in U such that $u_m \to x$. Further, let S denote $S(x)$. We show that if ρ and γ are norms in $\mathcal{N}_0^n$ such that the matrices $(G_\rho(u_m))_{i,j\in S}$ and $(G_\gamma(u_m))_{i,j\in S}$ converge to an invertible matrix, then $(G_{\rho+\gamma}(u_m))_{i,j\in S}$ converges to an invertible matrix. Since $\nabla\rho$ and $\nabla\gamma$ are continuous, and

$$G_\rho(u) = \nabla\rho(u)\nabla\rho(u)^\top + \rho(u)H_\rho(u)$$

and

$$G_\gamma(u) = \nabla\gamma(u)\nabla\gamma(u)^\top + \gamma(u)H_\gamma(u) \quad \text{for all } u \in U,$$

it follows that $(H_\rho(u_m))_{i,j\in S}$ and $(H_\gamma(u_m))_{i,j\in S}$ are convergent. As

$$H_{\rho\gamma}(u) = \nabla\rho(u)\nabla\gamma(u)^\top + \nabla\gamma(u)\nabla\rho(u)^\top + \rho(u)H_\gamma(u) + \gamma(u)H_\rho(u)$$

for all $u \in U$, this implies that $(H_{\rho\gamma}(u_m))_{i,j\in S}$ converges. We remark that, for each $m \geq 1$, the matrix $H_{\rho\gamma}(u_m)$ is positive semi-definite, and therefore $(H_{\rho\gamma}(u_m))_{i,j\in S}$ converges to a positive semi-definite matrix. Thus, it follows from (11) that $(G_{\rho+\gamma}(u_m))_{i,j\in S}$ converges to a positive definite matrix and hence its limit is invertible.

Now, if ρ is an ℓ_p-norm with $p \in (1,\infty)$ and $(u_m)_m$ a sequence in U with $u_m \to x$, then $(G_\rho(u_m))_{i,j\in S(x)}$ converges to an invertible matrix, and thus we conclude that $(G_\eta(u_m))_m$ satisfies property (1) in Definition 4.1.

The other three properties in Definition 4.1 can be verified straightforwardly by using the following identity:

$$G_\eta(u) = \sum_{k,l} \alpha_k\alpha_l \nabla\rho_k(u)\nabla\rho_l(u)^\top + \sum_{k,l} \alpha_k\alpha_l\rho_k(u)H_{\rho_l}(u) \tag{12}$$

for $u \in U$, and remarking that for ρ an ℓ_p-norm:

$$(D_i\rho)(u) = \operatorname{sgn}(u_i)|u_i|^{p-1}\|u\|_p^{1-p}, \tag{13}$$

$$(D_iD_j\rho)(u) = (1-p)\operatorname{sgn}(u_i)\operatorname{sgn}(u_j)|u_iu_j|^{p-1}\|u\|_p^{1-2p}, \quad \text{for } i \neq j, \tag{14}$$

and

$$(D_iD_i\rho)(u) = (p-1)|u_i|^{p-2}\|u\|_p^{1-p}(1 - |u_i|^p\|u\|_p^{-p}). \tag{15}$$

Thus, we conclude that $(G_\eta(u_m))_m$ behaves well relative to $S(x)$.

From (15) we see that $G_\rho(u_m)_{ii} \to 0$ for all $i \notin S(x)$, if ρ is an ℓ_p-norm with $p \in (2,\infty)$. On the other hand, if $p \in (1,2)$, then $G_\rho(u_m)_{ii} \to \infty$ for all $i \notin S(x)$. Thus, $\eta \in \mathcal{N}_1^n$ if $\min\{p_1,\dots,p_r\} > 2$, and $\eta \in \mathcal{N}_2^n$ if $\min\{p_1,\dots,p_r\} < 2$. $\square$

Using similar ideas it can be shown that if ρ is a norm in $\mathcal{N}_2^n$ and γ is a norm either in $\mathcal{N}_1^n$ or in $\mathcal{N}_2^n$, then the sum $\rho+\gamma$ is again in $\mathcal{N}_2^n$. By using this observation and the duality relation in Lemma 5.2, one can find many other norms for which each one-complemented subspace has a block basis. For instance, $(\|\cdot\|_{2\sqrt{2}} + \|\cdot\|_\pi)^* + \|\cdot\|_{37}$. However, we do not know whether each one-complemented subspace admits a block basis, if the norm is given by $\|\cdot\|_2 + \|\cdot\|_3$.

7. Proof of Lemma 5.3.

To prove Lemma 5.3 we use the following observation.

Lemma 7.1. *Let* $g : \mathbf{R}^n \to \mathbf{R}^n$ *be a continuous map, and let* $U \subset \mathbf{R}^n$ *be such that, for each* $x, y \in \mathbf{R}^n$ *and every* $\epsilon > 0$*, there exist* $x', y' \in U$ *with* $\|x - x'\| < \epsilon$*,* $\|y - y'\| < \epsilon$*, and the line-segment between* x' *and* y' *contains at most finitely many points in* $\mathbf{R}^n \setminus U$*. If* $g : \mathbf{R}^n \to \mathbf{R}^n$ *is differentiable on* U *and there exists a continuous map* $B : \mathbf{R}^n \to \mathcal{L}(\mathbf{R}^n, \mathbf{R}^n)$ *such that* $(Dg)(u) = B(u)$ *for all* $u \in U$*, then* g *is* C^1 *on* $\mathbf{R}^n$*, and moreover* $(Dg)(x) = B(x)$ *for all* $x \in \mathbf{R}^n$*.*

Proof. The proof is based on the following claim.

Claim. *For each* $x, h \in \mathbf{R}^n$*, we have that*

$$g(x+h) - g(x) = \int_0^1 B(x+th)h\,dt. \tag{16}$$

Indeed, if we assume the claim for a moment and we let $x, h \in \mathbf{R}^n$, then

$$\begin{aligned}\frac{\|g(x+h) - g(x) - B(x)h\|}{\|h\|} &= \frac{1}{\|h\|}\left\| \int_0^1 B(x+th)h\,dt - \int_0^1 B(x)h\,dt \right\| \\ &\le \sup_{0 \le t \le 1} \|B(x+th) - B(x)\|.\end{aligned}$$

The right-hand side goes to 0 as $h \to 0$, since B is continuous. Therefore, g is differentiable on $\mathbf{R}^n$ and $(Dg)x = B(x)$ for each $x \in \mathbf{R}^n$. Moreover, g is C^1 on $\mathbf{R}^n$, since B is continuous.

To prove the claim we first assume that $x + th \in U$ for all $t \in (0,1)$. Put $r(t) = g(x+th)$ for $t \in [0,1]$. Then r is differentiable in each $t \in (0,1)$, and continuous on $[0,1]$. This implies that $r(1) - r(0) = \int_0^1 r'(t)\,dt$, and hence

$$g(x+h) - g(x) = \int_0^1 B(x+th)h\,dt.$$

Now if the line-segment $\{x + th : 0 \le t \le 1\}$ contains only finitely many points in $\mathbf{R}^n \setminus U$, then we can break it into finitely many pieces and apply the previous observation for each piece. Therefore the equality (16) is also true for this case.

Finally, we consider the general case. From the assumptions it follows that there exist $(x_n)_n \in U$ and $(h_n)_n \in \mathbf{R}^n$, with $x_n + h_n \in U$, such that $x_n \to x$, $x_n + h_n \to x + h$, and for each $n \geq 1$ the intersection of the line-segment $\{x_n + th_n : 0 \leq t \leq 1\}$ with $\mathbf{R}^n \setminus U$ finite. We know that

$$g(x_n + h_n) - g(x_n) = \int_0^1 B(x_n + th_n)h_n \, dt \quad \text{for each } n \geq 1.$$

Since g is continuous and B is uniformly continuous on compact sets, we can take limits on both sides and deduce that

$$g(x + h) - g(x) = \int_0^1 B(x + th)h \, dt.$$

This completes the proof of the claim. □

Proof of Lemma 5.3. Let ρ be in $\mathcal{N}_1^n$, and let $U = \{x \in \mathbf{R}^n : x_i \neq 0 \text{ for all } i\}$. Clearly there exist for each $x, y \in \mathbf{R}^n$ and every $\epsilon > 0$ points $x', y' \in U$ such that $\|x - x'\| < \epsilon$, $\|y - y'\| < \epsilon$, and the intersection of the line-segment between x' and y' with $\mathbf{R}^n \setminus U$ finite. It follows from Proposition 2.2 that $J_\rho : \mathbf{R}^n \to \mathbf{R}^n$ is a continuous map. Since ρ is C^2 on U, the map J_ρ is differentiable on U.

Now define $B : \mathbf{R}^n \to \mathcal{L}(\mathbf{R}^n, \mathbf{R}^n)$ by $B(x) = (DJ_\rho)x = G_\rho(x)$ for each $x \in U$ and $B(x) = \lim_{m\to\infty} G_\rho(u_m)$, if $x \in \mathbf{R}^n \setminus U$, where $(u_m)_m$ in U is such that $u_m \to x$. We remark that B is well defined. Indeed $\rho \in \mathcal{N}_1^n$ implies that for every $x \in \mathbf{R}^n$ and every sequence $(u_m)_m$ in U with $u_m \to x$ the matrix $(G_\rho(u_m))_{i,j\in S(x)}$ converges to an invertible matrix, $G_\rho(u_m)_{ij} \to 0$ for $i \in S(x)$ and $j \notin S(x)$, and $G_\rho(u_m)_{ij} \to 0$ for $i \notin S(x)$ and $1 \leq j \leq n$.

The map B is continuous by construction, and hence Lemma 7.1 implies that J_ρ is C^1 on $\mathbf{R}^n$. Therefore, G_ρ is continuous on $\mathbf{R}^n$, and hence ρ is C^2 on $\mathbf{R}^n \setminus \{0\}$, because $\nabla\rho(x) = J_\rho(x)/\rho(x)$ for $x \neq 0$. □

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