On R-boundedness of unions of sets of operators

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Abstract

It is shown that the union of a sequence $\mathcal{T}_1, \mathcal{T}_2, \ldots$ of R-bounded sets of operators from X into Y with R-bounds $\tau_1, \tau_2, \ldots$, respectively, is R-bounded if X is a Banach space of cotype q, Y a Banach space of type p, and $\sum_{k=1}^{\infty} \tau_k^r < \infty$, where $r = pq/(q-p)$ if $q < \infty$ and $r = p$ if $q = \infty$. Here $1 \leq p \leq 2 \leq q \leq \infty$ and $p \neq q$. The power r is sharp. Each Banach space that contains an isomorphic copy of c_0 admits operators $T_1, T_2, \ldots$ such that $\|T_k\| = 1/k$, $k \in \mathbb{N}$, and $\{T_1, T_2, \ldots\}$ is not R-bounded. Further it is shown that the set of positive linear contractions in an infinite dimensional L^p is R-bounded only if $p = 2$.

1 Introduction

During the past few years a theory of L^p-multipliers for operator valued functions has been developed by means of the notion of R-boundedness of sets of operators, see for example [1, 2, 3, 4, 5, 6, 7, 8, 10, 12, 13, 17, 18, 19]. This theory has been applied to study maximal regularity of certain abstract evolution equations. For instance, L. Weis has shown in [18] that maximal L^p-regularity of the abstract Cauchy problem

$$\begin{aligned} &u'(t) = Au(t) + f(t) \quad \text{for a.e. } t \geq 0, \\ &u(0) = 0, \end{aligned} \tag{1}$$

in a Banach space X is equivalent to R-boundedness of the operator set

$$\{\lambda^n (\lambda I - A)^{-n} : \lambda \in i\mathbb{R}\}$$

for some $n \in \mathbb{N} = \{1, 2, \ldots\}$, whenever A is the generator of a bounded analytic semigroup on X and X is a UMD-space. Similarly, Arendt and Bu have shown in [1] that maximal L^p-regularity of the problem

$$\begin{aligned} &u''(t) + Au(t) = f(t) \quad \text{for a.e. } t \in [0, \pi] \\ &u(0) = u(\pi) = 0 \end{aligned} \tag{2}$$

comes down to R-boundedness of

$$\{k^2 (k^2 I - A)^{-1} : k \in \mathbb{N}\}.$$

Whether or not these sets of operators are R-bounded depends on the space X and the operator A that are considered. In order to be able to establish

R-boundedness in specific cases, it seems useful to have a rich theory on manipulations with R-bounded sets at one's disposal.

This paper concerns unions of R-bounded sets in the Banach spaces $X = L^q(\mu)$, where $(A, \mathcal{A}, \mu)$ is an arbitrary measure space, and $1 \le q < \infty$. More generally, Banach spaces X and Y are considered where X has cotype q and Y has type p. It will be shown in Section 3 that the union of a sequence $\mathcal{T}_1, \mathcal{T}_2, \dots$ of R-bounded sets of operators in $\mathcal{L}(X, Y)$ with R_2-bounds $\tau_1, \tau_2, \dots$, respectively, is R-bounded if $\sum_{k=1}^{\infty} \tau_k^r < \infty$, where $r = pq/(q-p)$ and $1 \le p \le 2 \le q \le \infty$, $p \ne q$. In particular, a sequence of operators $\{T_1, T_2, \dots\}$ on $L^p(\mu)$, $1 \le p < \infty$, is R-bounded whenever $\sum_{k=1}^{\infty} \|T_k\|^2 < \infty$. In Section 4 it is shown that the power r in the union theorem of Section 3 is sharp for $X = \ell^q(\mathbb{N})$ and $Y = \ell^p(\mathbb{N})$. It is also shown that Banach spaces that contain a copy of c_0 admit operators $T_1, T_2, \dots$ such that $\|T_k\| = 1/k$, $k \in \mathbb{N}$, and $\{T_1, T_2, \dots\}$ is not R-bounded. The ideas of Section 4 yield a characterization of $L^2(\mu)$ among $L^p(\mu)$, $1 \le p \le \infty$, by means of R-boundedness of positive contractions and isometries. This is discussed in Section 6. Section 7, finally, presents two examples related to resolvent families.

2 Preparations

Let X and Y be a real vector spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively, and let $\mathcal{L}(X, Y)$ denote the space of bounded linear operators that map X into Y. Let $1 \le r < \infty$. A subset $\mathcal{T}$ of $\mathcal{L}(X, Y)$ is called *R_r-bounded* if there exists a number $\tau_r \ge 0$ such that

$$\left(2^{-n} \sum_{\varepsilon \in \{-1,1\}^n} \left\| \sum_{k=1}^{n} \varepsilon_k T_k x_k \right\|_Y^r \right)^{1/r} \le \tau_r \left(2^{-n} \sum_{\varepsilon \in \{-1,1\}^n} \left\| \sum_{k=1}^{n} \varepsilon_k x_k \right\|_X^r \right)^{1/r}$$

for every $n \in \mathbb{N}$, every $x_1, \dots, x_n \in X$ and every $T_1, \dots, T_n \in \mathcal{T}$. The number τ_r is called an R_r-bound of $\mathcal{T}$. We always assume that the spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are Banach spaces. Kahane's inequality (see [15, Theorem 1.e.13, p. 74]) says that for every $p, q \in [1, \infty)$ there exists a constant $K_{p,q}(X) \ge 0$ such that

$$\left(2^{-n} \sum_{\varepsilon \in \{-1,1\}^n} \left\| \sum_{k=1}^{n} \varepsilon_k x_k \right\|_X^p \right)^{1/p} \le K_{p,q}(X) \left(2^{-n} \sum_{\varepsilon \in \{-1,1\}^n} \left\| \sum_{k=1}^{n} \varepsilon_k x_k \right\|_X^q \right)^{1/q}$$

for every $n \in \mathbb{N}$ and every $x_1, \dots, x_n \in X$. Thus a set $\mathcal{T} \subset \mathcal{L}(X, Y)$ is R_r-bounded for every $r \in [1, \infty)$ as soon as it is R_r-bounded for some $r \in [1, \infty)$ and therefore it is then simply called *R-bounded*.

We will focus on Banach spaces X that have *cotype* $q \in [2, \infty]$ and Banach spaces Y that have *type* $p \in [1, 2]$, that is, there exist constants $M_X, M_Y \ge 0$ such that for every $n \in \mathbb{N}$, $x_1, \dots, x_n \in X$, and $y_1, \dots, y_n \in Y$ the inequalities

$$\left(\sum_{k=1}^{n} \|x_k\|_X^q \right)^{1/q} \le M_X \left(2^{-n} \sum_{\varepsilon \in \{-1,1\}^n} \left\| \sum_{k=1}^{n} \varepsilon_k x_k \right\|_X^2 \right)^{1/2}$$

and

$$\left(2^{-n}\sum_{\varepsilon\in\{-1,1\}^n}\left\|\sum_{k=1}^n \varepsilon_k y_k\right\|_Y^2\right)^{1/2} \le M_Y\left(\sum_{k=1}^n \|y_k\|_Y^p\right)^{1/p}$$

hold, respectively. The expression $(\sum_{k=1}^n \|x_k\|_X^q)^{1/q}$ should be replaced by

$$\max_{1\le k\le n}\|x_k\|_X$$

if $q=\infty$. The constants M_X and M_Y are called a *cotype q constant* of X and a *type p constant* of Y, respectively. Each Banach space is of type 1 and cotype ∞ with both type 1 and cotype ∞ constant equal to 1. If a Banach space is of type p then it is of type r for any $r\in[1,p]$ and if it is of cotype q then it is of cotype r for any $r\in[q,\infty]$. For an abitrary measure space $(A,\mathcal{A},\mu)$ and $1\le p<\infty$, the Banach space $L^p(\mu)$ is of type $p\wedge 2=\min\{p,2\}$ and of cotype $q\vee 2=\max\{q,2\}$. See [15, p. 72–73] for proofs and more details.

Every R-bounded subset of $\mathcal{L}(X,Y)$ is bounded. The converse is true if and only if X is of cotype 2 and Y is of type 2 (see [1, Proposition 1.13]).

In order to abbreviate notations, we fix a probability space $(\Omega,\mathcal{F},\mathbb{P})$ and independent identically distributed (i.i.d) random variables $\varepsilon_1,\varepsilon_2,\dots$ defined on $(\Omega,\mathcal{F},\mathbb{P})$ with $\mathbb{P}(\varepsilon_1=-1)=\mathbb{P}(\varepsilon_1=1)=1/2$. Expectation is denoted by $\mathbb{E}$. Then, for instance,

$$\mathbb{E}\left\|\sum_{k=1}^n\varepsilon_k x_k\right\|_X^r = 2^{-n}\sum_{\delta\in\{-1,1\}^n}\left\|\sum_{k=1}^n\delta_k x_k\right\|_X^r = \left\|\sum_{k=1}^n\varepsilon_k x_k\right\|_{L^r(\mathbb{P},X)}^r.$$

The next proposition collects some preliminary results which are needed in the sequel (see also [5, 7, 12, 19]).

Proposition 2.1. *Let X, Y, and Z be Banach spaces, let $1\le r<\infty$, and let $\mathcal{S},\mathcal{T}\subset\mathcal{L}(X,Y)$ and $\mathcal{U}\subset\mathcal{L}(Y,Z)$ be R_r-bounded by σ, τ, and u, respectively.*

1. *$\{T\}$ is R_r-bounded by $\|T\|$, for every $T\in\mathcal{L}(X,Y)$.*
2. *If $\mathcal{S}\subset\mathcal{T}$, then $\mathcal{S}$ is R_r-bounded by τ.*
3. *$\mathcal{U}\mathcal{S}=\{US:U\in\mathcal{U},S\in\mathcal{S}\}$ is R_r-bounded by $u\sigma$.*
4. *$\mathcal{T}\cup\{0\}$ is R_r-bounded by τ.*
5. *$\mathcal{S}+\mathcal{T}=\{S+T:S\in\mathcal{S},T\in\mathcal{T}\}$ is R_r-bounded by $\sigma+\tau$.*
6. *$\mathcal{S}\cup\mathcal{T}$ is R_r-bounded by $\sigma+\tau$.*
7. *If Λ is a directed partially ordered set and $(\mathcal{T}_\lambda)_{\lambda\in\Lambda}$ is an increasing family of subsets of $\mathcal{L}(X,Y)$ such that $\mathcal{T}_\lambda$ is R_r-bounded by τ for every $\lambda\in\Lambda$, then $\bigcup_{\lambda\in\Lambda}\mathcal{T}_\lambda$ is R_r-bounded by τ.*

It follows from 2 and 6 of Proposition 2.1 that the notion of R-boundedness defines a bornology in $\mathcal{L}(X,Y)$ and thus the word 'boundedness' is justified (see [11]).

The R_r-bound for the union $\mathcal{S} \cup \mathcal{T}$ given in 6 of Proposition 2.1 may seem rather large. It turns out, however, that this bound is the best bound that holds in general. In particular, a countable set $\{T_1, T_2, \ldots\}$ need not be R-bounded if the norms $\|T_1\|, \|T_2\|, \ldots$ are not summable (see Theorem 5.1). On the other hand, in a Hilbert space a union of R-bounded sets is R-bounded if their R-bounds are bounded. It may be expected that for $X = Y = L^p$ the conditions providing R-boundedness of a union of R-bounded sets is somewhere between boundedness and summability of the R-bounds of the components. That is, more towards boundedness if p is close to 2 and more towards summability if p is large. Theorem 3.1 shows that this expectation is true.

3 R-boundedness of unions

The proof of the next theorem uses the following observation on type and cotype of Bochner spaces. If $(A, \mathcal{A}, \mu)$ is a measure space and X a Banach space of type p, then $L^s(\mu, X)$ is of type p for each $p \le s < \infty$. If X is of cotype q, then so is $L^s(\mu, X)$ for each $1 \le s \le q$ (see [16, Lemme 1.1, Lemme 2.1]). More specifically, if X is of type p with type p constant M, then $L^2(\mu, X)$ also has type p constant M, and if X is of cotype q with cotype q constant M, then also $L^2(\mu, X)$ has cotype q constant M. These facts can easily be proved by means of the inequality

$$\left(\sum_{k=1}^{n} \left(\int |f_k(a)| \, \mathrm{d}\mu(a) \right)^p \right)^{1/p} \le \int \left(\sum_{k=1}^{n} |f_k(a)|^p \right)^{1/p} \mathrm{d}\mu(a),$$

which holds for $p \ge 1$, and the reversed inequality, which holds for $0 < p \le 1$, $n \in \mathbb{N}$, $f_1, \ldots, f_n \in L^1(\mu)$.

Theorem 3.1. *Let $1 \le p \le 2 \le q \le \infty$ be such that $p \ne q$. Let $(X, \|\cdot\|_X)$ be a Banach space of cotype q and $(Y, \|\cdot\|_Y)$ a Banach space of type p. Let $\mathcal{T}_1, \mathcal{T}_2, \ldots$ be subsets of $\mathcal{L}(X,Y)$ such that for each $k \in \mathbb{N}$ the set $\mathcal{T}_k$ is R_2-bounded by a number $\tau_k > 0$. Let $\mathcal{T} := \bigcup_{k=1}^{\infty} \mathcal{T}_k$ and denote*

$$r := r_{p,q} := \frac{pq}{q-p} \text{ if } q < \infty \quad \text{and} \quad r_{p,q} := p \text{ if } q = \infty.$$

If $R := (\sum_{k=1}^{\infty} \tau_k^r)^{1/r}$ is finite, then $\mathcal{T}$ is R_2-bounded by $M_Y M_X R$, where M_X is a cotype q constant of X and M_Y a type p constant of Y.

Proof. Let $n \in \mathbb{N}$, $T_1, \ldots, T_n \in \mathcal{T}$ and let $x_1, \ldots, x_n \in X$. Choose a partition $I_1, \ldots I_K$ of $\{1, \ldots, n\}$ satisfying

$$T_i \in \mathcal{T}_k \text{ for all } i \in I_k, \; k = 1, \ldots, K.$$

Here K is a finite number and some of the sets I_k may be empty. Let $(\Omega', \mathcal{F}', \mathbb{P}')$ be a probability space and let $\varepsilon'_{k,i}$, $i \in I_k$, $k = 1, \ldots, K$, be i.i.d. random variables on $(\Omega', \mathcal{F}', \mathbb{P}')$ with $\mathbb{P}'(\varepsilon'_{k,i} = -1) = \mathbb{P}'(\varepsilon'_{k,i} = 1) = 1/2$. Then also the products $\varepsilon_k \varepsilon'_{k,i}$, $i \in I_k$, $k = 1, \ldots, K$, (in the usual way extended to $\Omega \times \Omega'$) are i.i.d. random variables with uniform distribution on $\{-1, 1\}$. Denote expectation with respect to $\mathbb{P}'$ by $\mathbb{E}'$.

Let us start the estimations. In the next sequence of inequalities we use consecutively type p of $L^p(\mathbb{P}', Y)$, R-boundedness, Hölder's inequality, and cotype q of $L^q(\mathbb{P}', X)$. We obtain for $q < \infty$,

$$\begin{aligned}
\left(\mathbb{E} \left\| \sum_{k=1}^{n} \varepsilon_k T_k x_k \right\|_Y^2 \right)^{1/2} &= \left(\mathbb{E}\,\mathbb{E}' \left\| \sum_{k=1}^{K} \sum_{i \in I_k} \varepsilon_k \varepsilon'_{k,i} T_i x_i \right\|_Y^2 \right)^{1/2} \\
&= \left(\mathbb{E} \left\| \sum_{k=1}^{K} \varepsilon_k \sum_{i \in I_k} \varepsilon'_{k,i} T_i x_i \right\|_{L^2(\mathbb{P}', Y)}^2 \right)^{1/2} \\
&\le M_Y \left(\sum_{k=1}^{K} \left\| \sum_{i \in I_k} \varepsilon'_{k,i} T_i x_i \right\|_{L^2(\mathbb{P}', Y)}^p \right)^{1/p} \\
&\le M_Y \left(\sum_{k=1}^{K} \tau_k^p \left\| \sum_{i \in I_k} \varepsilon'_{k,i} x_i \right\|_{L^2(\mathbb{P}', X)}^p \right)^{1/p} \\
&\le M_Y R \left(\sum_{k=1}^{K} \left\| \sum_{i \in I_k} \varepsilon'_{k,i} x_i \right\|_{L^2(\mathbb{P}', X)}^q \right)^{1/q} \\
&\le M_Y R M_X \left(\mathbb{E} \left\| \sum_{k=1}^{K} \varepsilon_k \sum_{i \in I_k} \varepsilon'_{k,i} x_i \right\|_{L^2(\mathbb{P}', X)}^2 \right)^{1/2} \\
&= M_Y R M_X \left(\mathbb{E}\,\mathbb{E}' \left\| \sum_{k=1}^{K} \sum_{i \in I_k} \varepsilon_k \varepsilon'_{k,i} x_i \right\|_X^2 \right)^{1/2} \\
&= M_Y R M_X \left(\mathbb{E} \left\| \sum_{k=1}^{n} \varepsilon_k x_k \right\|_X^2 \right)^{1/2}.
\end{aligned}$$

If $q = \infty$, we only have to replace

$$\left(\sum_{k=1}^{K} \left\| \sum_{i \in I_k} \varepsilon'_{k,i} x_i \right\|_{L^2(\mathbb{P}', X)}^q \right)^{1/q}$$

by $\max_{1 \le k \le K} \| \sum_{i \in I_k} \varepsilon'_{k,i} x_i \|_{L^2(\mathbb{P}', X)}$ and recall that any Banach space is of cotype ∞. □

The case $p = q = 2$, which is not treated in the previous theorem follows directly from the definitions. If X is of cotype 2 and Y is of type 2, then every bounded subset of $\mathcal{L}(X,Y)$ is R-bounded. Moreover, if M_X is a cotype 2 constant of X and M_Y is a type 2 constant of Y and if $\mathcal{T} \subset \mathcal{L}(X,Y)$ and $\tau \in \mathbb{R}$ are such that $\|T\| \leq \tau$ for all $T \in \mathcal{T}$, then $M_X M_Y \tau$ is an R_2-bound of $\mathcal{T}$. Further notice that the cases $p > 2$ and $q < 2$ are not of interest, since the only Banach space with type $p > 2$ or cotype $q < 2$ is $\{0\}$ (see [15, p. 73]).

Corollary 3.2. *Let X be a Banach space of type 2 or cotype 2. Let $\mathcal{T}_1, \mathcal{T}_2, \ldots$ be subsets of $\mathcal{L}(X)$ such that for each k the set $\mathcal{T}_k$ is R-bounded by $\tau_k > 0$. If $\sum_{k=1}^{\infty} \tau_k^2 < \infty$, then $\bigcup_{k=1}^{\infty} \mathcal{T}_k$ is R-bounded.*

Corollary 3.3. *Let $1 \leq p \leq 2 \leq q \leq \infty$ be such that $p \neq q$. Let X be a Banach space of cotype q and let Y be a Banach space of type p. Let $T_1, T_2, \ldots \in \mathcal{L}(X,Y)$. If $\sum_{k=1}^{\infty} \|T_k\|^{r_{p,q}} < \infty$, then $\{T_1, T_2, \ldots\}$ is R-bounded and $M_Y M_X (\sum_{k=1}^{\infty} \|T_k\|^{r_{p,q}})^{1/r_{p,q}}$ is an R_2-bound of this set, where $r_{p,q}$, M_Y, and M_X are as in Theorem 3.1.*

Example 3.4. (Cesaro means) Let $1 \leq p \leq 2 \leq q \leq \infty$, $p \neq q$, and either $p > 1$ or $q < \infty$, and let X be a Banach space of type p and cotype q. Let M_1 be a type p constant and M_2 be a cotype q constant. Let $T \in \mathcal{L}(X)$ be such that $\|T\| \leq 1$ and $1 \in \rho(T)$. Let

$$S_m := \frac{1}{m} \sum_{k=1}^{m} T^k, \quad m = 1, 2, \ldots.$$

Then the set $\{S_1, S_2, \ldots\}$ is R-bounded. More specifically, for each $n \in \mathbb{N}$ the set $\{S_1, \ldots, S_n\}$ is R_2-bounded by

$$C \sup_{1 \leq m \leq n} \|T(I - T^m)(I - T)^{-1}\|,$$

where $C = M_1 M_2 (\sum_{m=1}^{\infty} (1/m)^{r_{p,q}})^{1/r_{p,q}}$. Indeed, it is clear that

$$\frac{1}{m}(T + \cdots + T^m) = \frac{1}{m} T(I - T^m)(I - T)^{-1}.$$

Corollary 3.3 yields that the set $\{S_1, \ldots, S_n\}$ is R_2-bounded by

$$C \sup_{1 \leq m \leq n} \|T(I - T^m)(I - T)^{-1}\|.$$

Since $p > 1$ or $q < \infty$, we have $r_{p,q} > 1$ and C is finite. The proof is completed by the observation that the sets $\{S_1, \ldots, S_n\}$ are increasing in n and that for all m

$$\|T(I - T^m)(I - T)^{-1}\| \leq \|T\|(\|I\| + \|T\|^m)\|(I - T)^{-1}\| \leq 2\|(I - T)^{-1}\|.$$

4 Sharpness of the powers

We will show next that the power $r_{p,q}$ in Theorem 3.1 is sharp. That means, if $1 \le p \le 2 \le q \le \infty$, $p \ne q$, and $s > r_{p,q}$, then there are Banach spaces X and Y such that X is of cotype q and Y of type p and there are sets $\mathcal{T}_1, \mathcal{T}_2, \ldots$ in $\mathcal{L}(X,Y)$ with R$_2$-bounds $\tau_1, \tau_2, \ldots$, respectively, such that $\sum_{k=1}^{\infty} \tau_k^s < \infty$ but $\bigcup_{k=1}^{\infty} \mathcal{T}_k$ is not R-bounded.

In fact we show more. If $1 \le p \le \infty$, $1 \le q \le \infty$, not $q \le 2 \le p$, and $s > r_{p,q}$ where

$$r_{p,q} := \frac{(p \wedge 2)(q \vee 2)}{(q \vee 2) - (p \wedge 2)} \text{ if } q < \infty \quad \text{and} \quad r_{p,q} := p \wedge 2 \text{ if } q = \infty,$$

then there exist $T_1, T_2, \ldots \in \mathcal{L}(\ell^q(\mathbb{N}), \ell^p(\mathbb{N}))$ such that $\sum_{k=1}^{\infty} \|T_k\|^s < \infty$ and $\{T_1, T_2, \ldots\}$ not R-bounded.

The analysis is based on Khintchine's inequality (see [14, Theorem 2.b.3, p. 66]), which says that for any $1 \le r < \infty$ there exist constants c_r and C_r in $(0, \infty)$ such that

$$c_r \left(\sum_{k=1}^{n} |\alpha_k|^2 \right)^{1/2} \le \left(\mathbb{E} \left| \sum_{k=1}^{n} \varepsilon_k \alpha_k \right|^r \right)^{1/r} \le C_r \left(\sum_{k=1}^{n} |\alpha_k|^2 \right)^{1/2}, \tag{3}$$

for all $n \in \mathbb{N}$ and all $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$.

Lemma 4.1. *Let $(A, \mathcal{A}, \mu)$ be a measure space, let $1 \le p \le \infty$ and $1 \le q \le \infty$, let $1 \le r < \infty$, $n \in \mathbb{N}$, and $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$. Suppose that there exist $e_1, \ldots, e_n \in L^p(\mu) \cap L^q(\mu)$ with*

$$e_k \ge 0, \;\; e_k \ne 0, \;\; e_k \wedge e_\ell = 0 \textit{ for all } \ell \ne k \textit{ and } k = 1, \ldots, n.$$

Let $T_1, \ldots, T_n \in \mathcal{L}(L^q(\mu), L^p(\mu))$ and let τ be an R_r-bound of $\{\alpha_1 T_1, \ldots, \alpha_n T_n\}$.

1. *If $T_k e_k = e_1$ and $\|e_k\|_q = 1$ for $k = 1, \ldots, n$, then*

$$\tau \ge c_r \|e_1\|_p n^{-1/q} \|(\alpha_1, \ldots, \alpha_n)\|_2$$

if $q < \infty$ and $\tau \ge c_r \|e_1\|_p \|(\alpha_1, \ldots, \alpha_n)\|_2$ if $q = \infty$.

2. *If $T_k e_1 = e_k$ and $\|e_k\|_p = 1$ for $k = 1, \ldots, n$, then*

$$\tau \ge C_r^{-1} \|e_1\|_q^{-1} n^{-1/2} \|(\alpha_1, \ldots, \alpha_n)\|_p.$$

3. *If $T_k e_k = e_k$ and $\|e_k\|_q = 1$ for $k = 1, \ldots, n$, then*

$$\tau \ge n^{-1/q} \|(\alpha_1 \|e_1\|_p, \ldots, \alpha_n \|e_n\|_p)\|_p$$

if $q < \infty$ and $\tau \ge \|(\alpha_1 \|e_1\|_p, \ldots, \alpha_n \|e_n\|_p)\|_p$ if $q = \infty$.

Here c_r and C_r are constants that satisfy (3).

Proof. For any $\beta_1,\dots\beta_n \in \mathbb{R}$ and $s=p$ or $s=q$, disjointness of the e_k yields

$$\left(\mathbb{E}\left\|\sum_{k=1}^n \varepsilon_k\beta_k e_k\right\|_s^r\right)^{1/r} = \begin{cases} (\sum_{k=1}^n |\beta_k|^s\|e_k\|_s^s)^{1/s} & \text{if } s<\infty, \\ \max_{1\le k\le n}|\beta_k|\|e_k\|_s & \text{if } s=\infty. \end{cases}$$

By (3), for any $\gamma_1,\dots,\gamma_n \in \mathbb{R}$ and $s=p$ or $s=q$,

$$\begin{aligned}\left(\mathbb{E}\left\|\sum_{k=1}^n \varepsilon_k\gamma_k e_1\right\|_s^r\right)^{1/r} &= \left(\mathbb{E}\left|\sum_{k=1}^n \varepsilon_k\gamma_k\right|^r \|e_1\|_s^r\right)^{1/r} \\ &\le C_r\left(\sum_{k=1}^n \gamma_k^2\right)^{1/2}\|e_1\|_s\end{aligned}$$

and $(\mathbb{E}\|\sum_{k=1}^n \varepsilon_k\gamma_k e_1\|_s^r)^{1/r} \ge c_r(\sum_{k=1}^n\gamma_k^2)^{1/2}\|e_1\|_s$.

To show 1., choose $x_k=e_k$, $k=1,\dots,n$. Then $T_kx_k=e_1$, so

$$\left(\mathbb{E}\left\|\sum_{k=1}^n \varepsilon_k\alpha_k T_k x_k\right\|_p^r\right)^{1/r} \ge c_r\left(\sum_{k=1}^n\alpha_k^2\right)^{1/2}\|e_1\|_p.$$

Hence

$$c_r\|e_1\|_p\|(\alpha_1,\dots,\alpha_n)\|_2 \le \tau\left(\mathbb{E}\left\|\sum_{k=1}^n\varepsilon_k x_k\right\|_q^r\right)^{1/r} = \begin{cases} n^{1/q} & \text{if } q<\infty \\ 1 & \text{if } q=\infty\end{cases}$$

and the lower bound for τ follows.

Similarly, 2. and 3. are shown by choosing $x_k=e_1$ and $x_k=e_k$ for $k=1,\dots,n$, respectively. □

Proposition 4.2. *Let $1\le p\le\infty$ and $1\le q\le\infty$ be such that not $p=q=2$. Define for $k\in\mathbb{N}$ operators $S_k,T_k,U_k\in\mathcal{L}(\ell^q(\mathbb{N}),\ell^p(\mathbb{N}))$ by*

$$S_kx := x(k)\mathbb{1}_{\{1\}},\quad T_kx := x(1)\mathbb{1}_{\{k\}},\quad U_k := x(k)\mathbb{1}_{\{k\}},\quad x\in\ell^q(\mathbb{N}).$$

1. *If $p\ge 2$, $q>2$, and $s>r_{p,q}$, then $\{(1/k)^{1/s}S_k\colon k\in\mathbb{N}\}$ is not R-bounded.*
2. *If $p<2$, $q\le 2$, and $s>r_{p,q}$, then $\{(1/k)^{1/s}T_k\colon k\in\mathbb{N}\}$ is not R-bounded.*
3. *If $p\le 2\le q$ and $s>r_{p,q}$, then $\{(1/k)^{1/s}U_k\colon k\in\mathbb{N}\}]$ is not R-bounded.*

Proof. 1. For an $n\in\mathbb{N}$, choose $x_k=\mathbb{1}_{\{k\}}$, $k=1,\dots,n$. According to Lemma 4.1.1, an R_r-bound of $\{(1/k)^{1/s}S_k\colon k\in\mathbb{N}\}$ would be greater than $c_rn^{-1/q}(\sum_{k=1}^n k^{-2/s})^{1/2}$ if $q<\infty$ and greater than $c_r(\sum_{k=1}^n k^{-2/s})^{1/2}$ if $q=\infty$, for every $n\in\mathbb{N}$. As $(\sum_{k=1}^n k^{-2/s})^{1/2}\ge n^{1/2-1/s}$, and $1/2-1/s-1/q>0$, this is impossible.

2. For $n\in\mathbb{N}$, choose $x_k=\mathbb{1}_{\{1\}}$, $k=1,\dots,n$. According to Lemma 4.1.2, an R_r-bound of $\{(1/k)^{1/s}T_k\colon k\in\mathbb{N}\}$ would be at least $C_r^{-1}n^{-1/2}(\sum_{k=1}^n k^{-p/s})^{1/p}$,

which is greater than $C_r^{-1} n^{-1/2+1/p-1/s}$. This is impossible, as $-1/2 + 1/p - 1/s > 0$.

3. For $n \in \mathbb{N}$, choose $x_k = \mathbb{1}_{\{k\}}$, $k = 1, \dots, n$. Lemma 4.1.3 yields that an R_r-bound of $\{(1/k)^{1/s} U_k \colon k \in \mathbb{N}\}$ would majorize $n^{-1/q} (\sum_{k=1}^n k^{-p/s})^{1/p}$ if $q < \infty$ and $(\sum_{k=1}^n k^{-p/s})^{1/p}$ if $q = \infty$, for all n. As $(\sum_{k=1}^n k^{-p/s})^{1/p} \geq n^{1/p-1/s}$ and $-1/q + 1/p - 1/s > 0$, such an R_r-bound cannot exist. $\square$

It follows from the previous proposition that the power $r_{p,q}$ is sharp in Theorem 3.1 for $X = \ell^q(\mathbb{N})$ and $Y = \ell^p(\mathbb{N})$. Indeed, observe first that the operators S_k, T_k, and U_k are contractive from X into Y. In the case $p \geq 2$ and $q > 2$, we can choose for each $s > r_{p,q}$ a $t \in (r_{p,q}, s)$ and then $(1/k)^{1/t} S_k$ are operators in $\mathcal{L}(\ell^q, \ell^p)$ such that $\sum_{k=1}^\infty \|(1/k)^{1/t} S_k\|^s < \infty$ and $\{(1/k)^{1/t} S_k \colon k \in \mathbb{N}\}$ is not R-bounded, by Proposition 4.2.1. In the case $p < 2$ and $q \leq 2$ we consider $(1/k)^{1/t} T_k$, $k \in \mathbb{N}$, and use Proposition 4.2.2. If $p \leq 2 \leq q$ we use $(1/k)^{1/t} U_k$ and Proposition 4.2.3. Observe that in the remaining case $q \leq 2 \leq p$ the space $X = \ell^q(\mathbb{N})$ has cotype 2 and $Y = \ell^p(\mathbb{N})$ has type 2, so that every bounded subset of $\mathcal{L}(X, Y)$ is bounded.

5 Banach spaces containing $\mathbf{c_0}$

Theorem 3.1 might raise questions about converse implications. For instance, for which Banach spaces X is the following statement true: if $T_1, T_2, \dots$ are bounded linear operators on X and $(\|T_k\|_k)_k \in \ell^r(\mathbb{N})$ for some $r > 1$, then $\{T_k \colon k \in \mathbb{N}\}$ is R-bounded. Corollary 3.3 yields that this property is true if X is of type $p > 1$ or of cotype $q < \infty$. We will show next that the above property does not hold for Banach spaces containing an isomorphic copy of c_0.

Theorem 5.1. *Let X be a Banach space that contains an isomorphic copy of c_0. Then there are $T_1, T_2, \dots \in \mathcal{L}(X)$ such that $\|T_k\| = 1/k$ and such that $\{T_1, T_2, \dots\}$ is not R-bounded. More specifically, one can arrange that there exists a constant $C > 0$ such that the R-bound of $\{T_1, \dots, T_n\}$ is $\geq C \sum_{k=1}^n 1/k$ for every n.*

Proof. For clarity of the argument we first consider the case that $X = c_0$. Define for $n \geq 1$ and $0 \leq m \leq n$ the blocks of indices

$$I_{m,\ell}^n := \{i \in \mathbb{N} \colon 2^n + (\ell - 1)2^{n-m} \leq i < 2^n + \ell 2^{n-m}\}, \quad \ell = 1, \dots, 2^m,$$

and the elements

$$y_m^n := \sum_{\ell=1}^{2^m} (-1)^\ell \mathbb{1}_{I_{m,\ell}^n}.$$

Define for $k = 1, 2, \dots$ the operators

$$T_k x := \frac{1}{k} \sum_{n=0}^\infty x(2^n + k) y_k^n, \quad x \in c_0.$$

Since the elements y_k^n, $n \in \mathbb{N}$, have disjoint supports, we obtain $T_k \in \mathcal{L}(c_0)$. Further, for each $n \in \mathbb{N}$, $T_k \mathbb{1}_{\{2^n+k\}} = (1/k) y_k^n$, and $\|T_k\| = 1/k$, $k = 1, 2, \dots$.

We claim that $\{T_1, T_2, \ldots\}$ is not R-bounded. Indeed, let $n \in \mathbb{N}$ and let $x_k = \mathbb{1}_{\{2^n+k\}}$, $k = 1, \ldots, n$. As the x_k are disjointly supported and $\|x_k\|_\infty = 1$, we have

$$2^{-n} \sum_{\varepsilon \in \{-1,1\}^n} \left\| \sum_{k=1}^{n} \varepsilon_k x_k \right\|_\infty = 1.$$

On the other hand, if $\varepsilon \in \{-1,1\}^n$, then we can choose ℓ_1 such that $\varepsilon_1 y_1^n$ equals 1 on I^n_{1,ℓ_1}, and we can choose inductively ℓ_m such that $\varepsilon_m y_m^n$ equals 1 on I^n_{m,ℓ_m} and $I^n_{m,\ell_m} \subset I^n_{m-1,\ell_{m-1}}$, for $m = 2, \ldots, n$. With $i \in I^n_{n,\ell_n}$ we find that

$$\left\| \sum_{m=1}^{n} \varepsilon_m (1/m) y_m \right\|_\infty \geq \sum_{m=1}^{n} \varepsilon_m (1/m) y_m(i) = \sum_{m=1}^{n} (1/m).$$

Thus,

$$2^{-n} \sum_{\varepsilon \in \{-1,1\}^n} \left\| \sum_{k=1}^{n} \varepsilon_k T_k x_k \right\|_\infty \geq \sum_{k=1}^{n} 1/k.$$

It follows that $\{T_1, T_2, \ldots\}$ is not R-bounded. Actually we have shown that the smallest R_1-bound of $\{T_1, \ldots, T_n\}$ equals $\sum_{k=1}^n 1/k = \sum_{k=1}^n \|T_k\|$.

Next, let X be a Banach space such that there exists a linear injection $i\colon c_0 \to X$ and constants $c, C > 0$ such that

$$c\|i(x)\|_X \leq \|x\|_\infty \leq C\|i(x)\|_X, \quad x \in c_0.$$

For $n, k \in \mathbb{N}$ the map $x \mapsto (i^{-1}(x))(2^n + k)$ is a bounded linear function on $i(c_0)$ with norm $\leq C$ and according to Hahn-Banach's theorem it can be extended to a bounded linear functional $\varphi_{n,k}$ on X with $\|\varphi_{n,k}\| \leq C$. Now define the operators

$$T_k x := (1/k) i \left(\sum_{n=0}^{\infty} \varphi_{n,k}(x) y_k^n \right), \quad x \in X,\ k = 1, 2, \ldots.$$

Then $T_k \in \mathcal{L}(X)$ and $\|T_k\| \leq k^{-1} c^{-1} C$ for all k. Further, $T_k(i(\mathbb{1}_{\{2^n+k\}})) = (1/k) i(y_k^n)$ for all $k, n \in \mathbb{N}$. Rescale T_k to $S_k := (1/k)T_k/\|T_k\|$, so that $\|S_k\| = 1/k$ for all k. If we fix $n \in \mathbb{N}$ and let

$$x_k := i(\mathbb{1}_{\{2^n+k\}}), \quad k = 1, \ldots, n,$$

then

$$\left\| \sum_{k=1}^{n} \varepsilon_k S_k x_k \right\|_X \geq c^{-1} \sum_{k=1}^{n} k^{-2} \|T_k\|^{-1} \geq c C^{-2} \sum_{k=1}^{n} 1/k.$$

The assertions now follow easily. □

6 Characterization of $\mathbf{L^2}$

Lemma 4.1 leads to a characterization of L^2 among L^p by means of R-boundedness of positive linear contractions and isometries. By a linear *contraction* on a Banach space $(X, \|\cdot\|)$ we mean a linear map $T \in \mathcal{L}(X)$ with $\|T\| \leq 1$. The map $T \in \mathcal{L}(X)$ is an *isometry* if $\|Tx\| = \|x\|$ for all $x \in X$. If X is a Banach lattice, then we call a linear map $T : X \to X$ *positive* if $Tx \geq 0$ for all $x \geq 0$. If $(\Omega, \mathcal{F}, \mu)$ is a measure space, $1 \leq p \leq \infty$, and $\Omega_1 \in \mathcal{F}$, then $L^p(\mu)$ decomposes into $L^p(\mu) = L^p(\mu_1) \oplus L^p(\mu_2)$, where $\Omega_2 = \Omega \setminus \Omega_1$, $\mathcal{F}_i = \{A \cap \Omega_i \colon A \in \mathcal{F}\}$, $\mu_i(A) = \mu(A)$, $A \in \mathcal{F}_i$, $i = 1, 2$, and the norm of $L^p(\mu_1) \oplus L^p(\mu_2)$ is given by $(\|f|_{\Omega_1}\|^p + \|f|_{\Omega_2}\|^p)^{1/p}$ if $p < \infty$ and $\|f|_{\Omega_1}\|_\infty \vee \|f|_{\Omega_2}\|_\infty$ if $p = \infty$. We further recall that an *atom* in a σ-finite measure space $(\Omega, \mathcal{F}, \mu)$ is a set $A \in \mathcal{F}$ with $\mu(A) > 0$ such that for every $E \in \mathcal{F}$ with $\mu(E \setminus A) = 0$ either $\mu(E) = \mu(A)$ or $\mu(E) = 0$.

Theorem 6.1. *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, let $1 \leq p \leq \infty$, and let $X = L^p(\mu)$. Assume that X is infinite dimensional.*

1. *The set of all positive linear contractions on X is R-bounded if and only if $p = 2$.*
2. *If X is separable, then the set of all positive linear surjective isometries on X is R-bounded if and only if $p = 2$.*

Proof. 1. Since in a Hilbert space every uniformly bounded set of operators is R-bounded, the 'if' part is clear. Suppose that $p \neq 2$. Let $n \in \mathbb{N}$ be arbitrary. Since $L^p(\mu)$ is infinite dimensional, there exist mutually disjoint sets $A_1, \ldots, A_n \in \mathcal{F}$ with $\mu(A_i) > 0$ for $i = 1, \ldots, n$. Define $\alpha_i := \mu(A_i)^{-1/p}$ if $p < \infty$ and $\alpha_i := 1$ if $p = \infty$, for $i = 1, \ldots, n$. Further define

$$e_i := \alpha_i \mathbb{1}_{A_i} \text{ and } \varphi_i(f) := \alpha_i^{-1} \mu(A_i)^{-1} \int f \mathbb{1}_{A_i} \, \mathrm{d}\mu, \; f \in X,$$

$i = 1, \ldots, n$, and

$$Tf := \varphi_1(f) e_n + \varphi_2(f) e_1 + \varphi_3(f) e_2 + \cdots + \varphi_n(f) e_{n-1},$$

$f \in X$. It is easily checked that T is a positive linear contraction from X into X. Further, $Te_i = e_{i-1}$ for $i = 2, \ldots, n$ and $Te_1 = e_n$. Let $T_i := T^{i-1}$, $i = 1, \ldots, n$. Then Lemma 4.1 yields that any R_r-bound of $\{T_1, \ldots, T_n\}$ is at least $c_r n^{1/2 - 1/p} \vee C_r^{-1} n^{1/p - 1/2}$ if $p < \infty$ and at least $c_r n^{1/2}$ if $p = \infty$. It follows that the set of positive linear contractions is not R-bounded.

2. As before, the 'if' part is well known. Suppose that $p \neq 2$. Since $L^p(\mu)$ is infinite dimensional, there are mutually disjoint sets $A_1, A_2, \ldots \in \mathcal{F}$ with $0 < \mu(A_i) < \infty$ for all i. Then $\Omega_1 := \bigcup_{i=1}^\infty A_i$ with $\mathcal{F}_1 = \{A \cap \Omega_1 \colon A \in \mathcal{F}\}$ and $\mu_1(A) = \mu(A)$, $A \in \mathcal{F}_1$, is σ-finite. Consider the induced decomposition $L^p(\mu) = L^p(\mu_1) \oplus L^p(\mu_2)$. Any positive linear surjective isometry on $L^p(\mu_1)$ extends (by identity on $L^p(\mu_2)$) to a positive linear surjective isometry on $L^p(\mu)$. It therefore suffices to show that the positive linear surjective isometries on $L^p(\mu_1)$ are not R-bounded.

It will be convenient and ligitimate to interpret inclusions and equalities in $(\Omega_1, \mathcal{F}_1, \mu_1)$ modulo sets of measure zero. We adopt the formalism of [9] to view $(\Omega_1, \mathcal{F}_1, \mu_1)$ as a measure algebra rather than a measure space. If A and B are two distinct atoms in the measure algebra, then $A \cap B = \emptyset$. As $(\Omega_1, \mathcal{F}_1, \mu_1)$

is σ-finite, it contains at most countably many atoms, say B_i, $i \in N$, where $N \subset \mathbb{N}$. The sets $\Omega_3 = \bigcup_{i \in N} B_i$ and $\Omega_4 = \Omega_1 \setminus \Omega_3$ induce a decomposition $L^p(\mu_1) = L^p(\mu_3) \oplus L^p(\mu_4)$. The space $L^p(\mu_3)$ is isometrically isomorphic to $\ell^p(N)$ as a Banach lattice and the measure μ_4 has no atoms. If N is infinite, it is clear how to use Lemma 4.1 to show that the set of positive linear surjective isometries on $L^p(\mu_1)$ is not R-bounded.

If N is finite, then $\mu_4(\Omega_4) > 0$, as $L^p(\mu_1)$ is infinite dimensional. Choose an $\Omega_5 \in \mathcal{F}_4$ with $0 < \mu_4(\Omega_5) < \infty$ and consider the induced decomposition $L^p(\mu_4) = L^p(\mu_5) \oplus L^p(\mu_6)$. Since $L^p(\mu)$ is separable, also $L^p(\mu_5)$ is separable and hence the measure space $(\Omega_5, \mathcal{F}_5, \mu_5)$ is separable (see for the definition [9, VIII.40, p. 168] and for the implication [9, VIII.42, p. 177]). From [9, VIII.41 Theorem C, p. 173] it follows that $(\Omega_5, \mathcal{F}_5, \mu_5)$ is as a measure algebra isomorphic to the interval $[0, \mu_5(\Omega_5)]$ with its Borel σ-algebra and Lebesgue measure. Thus, $L^p(\mu_5)$ is as a Banach lattice isometrically isomorphic to $L^p[0, \mu_5(\Omega_5)])$, and the proof can easily be completed with aid of Lemma 4.1. □

7 Two examples

The purpose of this section is to indicate how the previous results can be used for manipulations with resolvent families. We do not establish R-boundedness of the resolvent families mentioned in the Introduction and the examples serve as mere illustration. Let X be a Banach space of type $p > 1$ or of cotype $q < \infty$. For instance, $X = L^p(\mu)$, where $1 \le p < \infty$ and $(\Omega, \mathcal{F}, \mu)$ is a measure space. Let $(T(t))_{t \ge 0}$ be a strongly continuous semigroup on X with generator A such that $(0, \infty) \in \rho(A)$.

Example 7.1. The following two statements are equivalent:

(a) $\{\lambda(\lambda I - A)^{-1} \colon \lambda \ge 1\}$ is R-bounded,

(b) $\{k^n(k^n I - A)^{-1} \colon k \in \mathbb{N},\ k \ge m\}$ is R-bounded for some $m, n \in \mathbb{N}$.

If we show the implication (b)⇒(a), then the equivalence is clear. We may assume that $m = 1$. We use that for a map $\lambda \mapsto S(\lambda) \colon [a, b] \to \mathcal{L}(X)$ with $\|S(\lambda) - S(\mu)\| \le L|\lambda - \mu|$ for all $\lambda, \mu \in [a, b]$ and some constant L the set $\{S(\lambda) - S(a) \colon \lambda \in [a, b]\}$ is R_r-bounded by $L(b - a)$ for every $1 \le r < \infty$. This follows readily from Proposition 2.1.1, 4, 5, and 7 and the observation that $\sum_{i=1}^n \|S(\lambda_i) - S(\lambda_{i-1})\| \le L(b - a)$, so that the vector sum of the sets $\{0, S(\lambda_i) - S(\lambda_{i-1})\}$, $0 \le i \le n$, is R_r-bounded by $L(b - a)$, for any $a = \lambda_0 \le \lambda_1 \le \cdots \le \lambda_n \le b$.

Since there exists a constant $M \ge 0$ such that

$$\|\mu(\mu I - A)^{-1} - \lambda(\lambda I - A)^{-1}\| \le (M/\lambda)|\lambda - \mu|, \quad \lambda, \mu \ge 1,$$

it follows that for each $k \in \mathbb{N}$,

$$\mathcal{T}_k := \{\lambda(\lambda I - A)^{-1} - k^n(k^n I - A)^{-1} \colon \lambda \in [k^n, k^{n+1}]\}$$

is R_r-bounded by $(M/k^n)((k+1)^n - k^n) \le Mn((k+1)/k)^{n-1}/k$. By Theorem 3.1, it follows that $\bigcup_{k=1}^\infty \mathcal{T}_k$ is R-bounded, so that $\{\lambda(\lambda I - A)^{-1} \colon \lambda \ge 1\} \subset \{k^n(k^n I - A)^{-1} : k \in \mathbb{N}\} + \bigcup_{k=1}^\infty \mathcal{T}_k$ is R-bounded.

Example 7.2. If $\|T(t)\| \leq 1$ for all $t \geq 0$ and if the set of Cesaro means $\{n^{-1}(T(1) + \cdots T(n))\colon n \in \mathbb{N}\}$ is R-bounded, then the set $\{\int_1^\infty \lambda e^{-\lambda t} T(t)\,\mathrm{d}t\colon \lambda \geq 0\}$ is R-bounded. Here $\int_1^\infty \lambda e^{-\lambda t} T(t)\,\mathrm{d}t$ denotes the bounded linear operator $x \mapsto \int_1^\infty \lambda e^{-\lambda t} T(t) x\,\mathrm{d}t$. For a proof, denote $S_n := n^{-1} \sum_{k=1}^n T(1)^k$, $n \in \mathbb{N}$.

We first show that the set $\{\sum_{k=1}^n \lambda e^{-\lambda k} T(k)\colon \lambda \geq 0,\ n \in \mathbb{N}\}$ is R-bounded. Indeed, according to [19, Lemma 2.2.6], the convex hull of the set of Cesaro means and the zero operator $\mathrm{co}\{0, S_k\colon k \in \mathbb{N}\}$ is also R-bounded. Now use that $\{\sum_{k=1}^n \alpha_k T(k)\colon n \in \mathbb{N},\ \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n \geq 0,\ \sum_{k=1}^n \alpha_k \leq 1\} = \mathrm{co}\{0, S_k\colon k \in \mathbb{N}\}$. For a proof, notice for the less obvious inclusion that for $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n \geq 0$ with $\alpha_1 + \cdots + \alpha_n \leq 1$ the choice $\beta_k := k(\alpha_k - \alpha_{k+1})$, $1 \leq k \leq n$, ($\alpha_{n+1} := 0$) yields

$$\sum_{k=1}^n \beta_k \sum_{\ell=1}^k (1/k) T(\ell) = \sum_{\ell=1}^n \left(\sum_{k=\ell}^n (\alpha_k - \alpha_{k+1}) \right) T(\ell) = \sum_{\ell=1}^n \alpha_\ell T(\ell)$$

and $\sum_{k=1}^n \beta_k \leq 1$.

Next, we show that $\{\int_1^n \lambda e^{-\lambda t} T(t)\,\mathrm{d}t\colon \lambda \geq 0,\ n \in \mathbb{N}\}$ is R-bounded. For a proof, observe first that $\lambda \mapsto \int_0^1 e^{-\lambda t} T(t)\,\mathrm{d}t$ is continuous and that for any $0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{m+1}$,

$$\begin{aligned}
&\sum_{k=1}^m \left\| \int_0^1 e^{-\lambda_{k+1} t} T(t)\,\mathrm{d}t - \int_0^1 e^{-\lambda_k t} T(t)\,\mathrm{d}t \right\| \\
&\qquad \leq \sum_{k=1}^m \int_0^1 (e^{-\lambda_k t} - e^{-\lambda_{k+1} t})\,\mathrm{d}t \\
&\qquad \leq \int_0^1 (e^{-\lambda_1} - e^{-\lambda_{m+1} t})\,\mathrm{d}t \leq 1.
\end{aligned}$$

Hence $\lambda \mapsto \int_0^1 e^{-\lambda t} T(t)\,\mathrm{d}t$ is of bounded variation and thus $\{\int_0^1 e^{-\lambda t} T(t)\,\mathrm{d}t\colon \lambda \geq 0\}$ is R-bounded (use, e.g., [19, Theorem 2.2.8] or [7]). Further,

$$\begin{aligned}
&\sum_{k=1}^{n-1} \lambda e^{-\lambda k} T(k) \int_0^1 e^{-\lambda t} T(t)\,\mathrm{d}t \\
&\qquad = \sum_{k=1}^{n-1} \int_0^1 \lambda e^{-\lambda (t+k)} T(t+k)\,\mathrm{d}t \\
&\qquad = \int_1^n \lambda e^{-\lambda t} T(t)\,\mathrm{d}t, \quad \lambda \geq 0,\ n \in \mathbb{N}.
\end{aligned}$$

Due to Proposition 2.1.3 we obtain the asserted R-boundedness.

Finally, notice that $\{\int_1^\infty \lambda e^{-\lambda t} T(t)\,\mathrm{d}t\colon \lambda \geq 0\}$ is in the strong closure of $\{\int_1^n \lambda e^{-\lambda t} T(t)\,\mathrm{d}t\colon \lambda \geq 0,\ n \in \mathbb{N}\}$, so that it is an R-bounded set (see [19, Theorem 2.2.8] or [7]).

If $1 \in \rho(T(1))$, then Example 3.4 yields that $\{S_k\colon k \in \mathbb{N}\}$ is R-bounded.

It remains to investigate whether the set $\{\int_0^1 \lambda e^{-\lambda t} T(t)\,\mathrm{d}t\colon \lambda \geq 0\}$ is R-bounded, in order to establish R-boundedness of the set $\{\lambda(\lambda I - A)^{-1} = \int_0^\infty \lambda e^{-\lambda t} T(t)\,\mathrm{d}t\colon \lambda \geq 1\}$.

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References

[1] W. Arendt and S. Bu, The operator-valued Marcinkiewicz multiplier theorem and maximal regularity, *Math. Z.* **240** (2002), no. 2, 311–343.

[2] W. Arendt and S. Bu, Tools for maximal regularity, *Math. Proc. Cambridge Philos. Soc.* **134** (2003), no. 2, 317–336.

[3] S. Blunck, Maximal regularity of discrete and continuous time evolution equations, *Studia Math.* **146** (2001), no. 2, 157–176.

[4] S. Bu, Some remarks about the R-boundedness, *Chinese Ann. Math. Ser. B* **25** (2004), no. 3, 421-432.

[5] Ph. Clément, B. de Pagter, F.A. Sukochev, and H. Witvliet, Schauder decompositions and multiplier theorems, *Studia Math.* **138** (2000), 135–163.

[6] Ph. Clément and J. Prüss, An operator-valued tranference principle and maximal regularity on vector-valued L_p-spaces, in: Evolution Equations and Their Applications in Physics and Life Sciences, Lumer and Weis (eds.), Marcel Dekker (2000), 67–87.

[7] R. Denk, M. Hieber, and Jan Prüss, R-boundedness, Fourier multipliers and problems of elliptic and parabolic type, *Mem. Amer. Math. Soc.* **166** (2003), no. 788.

[8] M. Girardi and L. Weis, Operator-valued Fourier multiplier theorems on $L_p(X)$ and geometry of Banach spaces, *J. Funct. Anal.* **204** (2003), no. 2, 320–354.

[9] P. Halmos, *Measure theory*, Graduate Texts in Mathematics 18, Springer-Verlag, New York, 1974.

[10] M. Hoffmann, N. Kalton, and T. Kucherenko, R-bounded approximating sequences and applications to semigroups, *J. Math. Anal. Appl.* **294** (2004), no. 2, 373–386.

[11] H. Hogbe-Nlend, *Théorie des Bornologies et Applications*, Lecture Notes in Mathematics 213, Springer-Verlag, Berlin, 1971.

[12] P. Kunstmann and L. Weis, Maximal L_p-regularity for parabolic equations, Fourier multiplier theorems and H^∞-functional calculus, *Functional analytic methods for evolution equations*, Lecture Notes in Mathematics 1855, p. 65–311, Springer-Verlag, Berlin, 2004

[13] C. Le Merdy and A. Simard, A factorization property of R-bounded sets of operators on L^p-spaces, *Math. Nachr.* **243** (2002), 146–155.

[14] J. Lindenstrauss and L. Tzafriri, Classical Banach Spaces I, Sequence Spaces, Springer-Verlag, Berlin, 1977.

[15] J. Lindenstrauss and L. Tzafriri, Classical Banach Spaces II, Function Spaces, Springer-Verlag, Berlin, 1979.

[16] B. Maurey and G. Pisier, Séries de variables aléatoires vectorielles indépendantes et propriétes géométriques des espaces de Banach, *Studia Math.* **58** (1976), no. 1, 45–90.

[17] J. Prüss, Maximal regularity for evolution equations in L_p-spaces, *Conf. Semin. Mat. Univ. Bari* no. 285 (2002), 1–39 (2003).

[18] L. Weis, Operator-valued Fourier multiplier theorems and maximal L_p-regularity, *Math. Ann.* **319** (2001), 735–758.

[19] H. Witvliet, Unconditional Schauder decompositions and multiplier theorems, PhD-thesis, Delft University of Technology, Delft, 2000.

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