Invariant measures for stochastic evolution equations with Hilbert space valued Lévy noise

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Abstract

Existence of invariant measures for semi-linear stochastic evolution equations in separable real Hilbert spaces is considered, where the noise is generated by Hilbert space valued Lévy processes. It is shown that if the Lévy process has locally bounded second moments, if the semigroup generated by the linear part is hyperbolic, and if the Lipschitz constants of the nonlinearities are sufficiently small, then existence of a mean square bounded solution implies existence of an invariant measure. In case the semigroup is exponentially stable, each solution is mean square bounded and there exists a unique invariant measure with finite second moment whenever the Lipschitz constants of the nonlinearities are sufficiently small. The stochastic integral with respect to the Hilbert space valued Lévy process is constructed as a series by means of a decomposition of the process into scalar processes. The existence of the invariant measure is proved by a coupling argument, which depends on weak uniqueness of solutions of the equation

Keywords: coupling, hyperbolic semigroup, infinite dimensional noise, Lévy process, semi-linear stochastic evolution equation, weak uniqueness

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1 Introduction

There appears to be a still growing interest in stochastic evolution equations with noise driven by more general processes than Wiener processes (Brownian motions). Existence and uniqueness as well as many related aspects of finite dimensional equations have been studied in great detail, see for instance [11, 15] and references therein. The theory of infinite dimensional equations driven by more general than Wiener processes is not as well-developped. Especially concerning invariant measures much less is known than for the case of Gaussian noise. The purpose of this paper is to extend the recent result of [17] on existence of invariant measures for semilinear hyperbolic equations driven by infinite dimensional Wiener processes to the case of infinite dimensional Lévy noise. Stationary solutions of affine evolution equations in Hilbert spaces with noise driven by Hilbert space valued Lévy processes have been studied by Chojnowska-Michalik in [2, 3]. Long term behavior of stochastic delay differential equations with more general noise processes have for instance been studied in [10, 13]. Important references for infinite dimensional equations driven by Wiener processes are the books [5, 6].

Consider a semi-linear stochastic evolution equation in a separable real Hilbert space E, formally written as

$$dX(t) = [AX(t) + F(X(t))]dt + G(X(t))dZ(t), \quad t \geq 0, \tag{1}$$

where the noise is driven by a Hilbert space valued process $(Z(t))_{t\geq 0}$ with stationary and independent increments and locally bounded second moments. It will be shown that if the semigroup generated by the linear part A is hyperbolic and the Lipschitz constants of the nonlinearities F and G are sufficiently small, then existence of a mean square bounded solution implies existence of an invariant measure. Moreover, if the semigroup is exponentially stable, then there always exists an invariant measure if the Lipschitz constants are small enough and it is unique.

The proof is an extension of the proof given in [17] for Wiener noise to more general processes. The methods in [17] do not depend on continuity of paths nor on the noise process being gaussian. They do, however, depend heavily on estimates for the second moments of the occurring stochastic integrals. Therefore the theory on integration with respect to Hilbert space valued Lévy processes presented in [2] can not directly be applied. It turns out that if the process itself has finite second moments then the stochastic integral satisfies suitable second moment estimates as well. More than that, the stochastic integral can then conveniently be constructed by means of a decomposition into scalar processes and satisfies an Ito isometry just as in [16] for Wiener processes. Peszat and Zabczyk have taken a somewhat different approach to stochastic integration with respect to infinite dimensional Lévy processes, more in the spirit of [5].

Let us now state precisely the setting and the main results.

Situation (∗). Let H and E be separable real Hilbert spaces, let A be the generator of a strongly continuous semigroup $(S(t))_{t\geq 0}$ in E, let $F : E \to E$ and $G : E \to \mathcal{L}(H, E)$ be such that

$$\begin{aligned} &\|F(x) - F(y)\| \leq L_F\|x - y\|, \\ &|||G(x) - G(y)||| \leq L_G\|x - y\|, \quad \text{for all } x, y \in E, \end{aligned}$$

for some constants $L_F, L_G > 0$. Here $\mathcal{L}(H, E)$ denotes the space of all bounded linear operators from H to E and $|||\cdot|||$ its norm. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $(\mathcal{F}_t)_{t\geq 0}$ be a filtration in $\mathcal{F}$, let $(Z(t))_{t\geq 0}$ be a family of random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ such that

$$\begin{aligned} &Z(t) \text{ is } \mathcal{F}_t\text{-measurable}, \\ &Z(t + u) - Z(t) \text{ is independent of } \mathcal{F}_t, \\ &Z(t + u) - Z(t) \text{ and } Z(s + u) - Z(s) \text{ have the same distribution} \end{aligned}$$

for all $s, t, u \geq 0$, and such that $Z(0) = 0$ and $\sup_{0\leq t\leq T} \mathbb{E}\,\|Z(t)\|^2 < \infty$ for all $T \geq 0$.

We will consider *mild solutions* of (1), which are continuous $(\mathcal{F}_t)_{t\geq 0}$-adapted functions $X : [0, \infty) \to L^2(\Omega; E)$ such that

$$X(t) = S(t)X(0) + \int_0^t S(t - s)F(X(s))\,\mathrm{d}s + \int_0^t S(t - s)G(X(s))\,\mathrm{d}Z(s), \quad t \geq 0.$$

Here $L^2(\Omega; E)$ denotes the space of (equivalence classes of) square Bochner integrable functions from $(\Omega, \mathcal{F}, \mathbb{P})$ to E. The stochastic integral will be defined in Section 3. We will establish the following two theorems.

Theorem 1.1. *Assume Situation (∗). Assume that $E = E_1 \oplus E_2$ where E_1 and E_2 are closed subspaces of E that are invariant under $S(t)$ for all $t \geq 0$, such that $S(t)|_{E_2}$ extends to a strongly continuous group on E_2, and such that there exist $M \geq 0$, $\alpha_1 > 0$, $\alpha_2 > 0$ with*

$$\begin{aligned} &\|S(t)x\| \leq Me^{-\alpha_1 t}\|x\| \text{ for all } x \in E_1, \\ &\|S(-t)x\| \leq Me^{-\alpha_2 t}\|x\| \text{ for all } x \in E_2, \end{aligned}$$

for all $t \geq 0$. Let P_1 denote the projection on E_1 along E_2 and P_2 the projection on E_2 along E_1. Let $\sigma := (\mathbb{E}\,\|Z(1) - \mathbb{E}\,Z(1)\|^2)^{1/2}$ and

$$K_1 := 6M^2\|P_1\|^2(L_F^2/\alpha_1 + L_G^2\sigma^2),$$
$$K_2 := 6M^2\|P_2\|^2(L_F^2/\alpha_2 + L_G^2\sigma^2).$$

If L_F and L_G are so small that

$$K_1 < \alpha_1, \quad K_2 < \alpha_2, \text{ and } \alpha_1 K_2 + \alpha_2 K_1 < \alpha_1\alpha_2,$$

then existence of a mild solution X of (1) with $\sup_{t\geq 0}\mathbb{E}\,\|X(t)\|^2 < \infty$ implies existence of an invariant measure μ of (1) with $\int_E \|x\|^2\, d\mu(x) < \infty$.

Theorem 1.2. *Assume Situation $(*)$ and assume that there exist $M \geq 0$ and $\alpha > 0$ such that*

$$\|S(t)\| \leq Me^{-\alpha t} \text{ for all } t \geq 0.$$

Let $\sigma := (\mathbb{E}\,\|Z(1) - \mathbb{E}\,Z(1)\|^2)^{1/2}$. If L_F and L_G are so small that

$$6M^2(L_F^2/\alpha + L_G^2\sigma^2) < \alpha$$

then there exists a unique invariant measure μ of (1) with $\int_E \|x\|^2\, d\mu(x) < \infty$.

It turns out that in case $\mathbb{E}\,Z(t) = 0$ for all $t \geq 0$ the class of functions G that is allowed in Situation $(*)$ may be enlarged to functions that map into a space of Hilbert-Schmidt type operators. For more details, see Theorem 5.9. The major part of the construction of the stochastic integral is summarized in the next theorem.

Theorem 1.3. *Assume Situation (*) and let $\Phi : [0,\infty) \to L^2(\Omega;\mathcal{L}(H,E))$ be adapted and continuous. There exists a unique $Q \in \mathcal{L}(H,H)$ such that*

$$\langle Qx, y\rangle_H = \mathbb{E}\,\langle Z(1), x\rangle\langle Z(1), y\rangle \text{ for all } x, y \in H.$$

This operator Q is symmetric, positive semi-definite, and of trace class. If $(h_i)_{i\in N}$ ($N \subset \mathbb{N}$) is an orthonormal basis of H consisting of eigenvectors of Q and $(\lambda_i)_{i\in N}$ are the corresponding eigenvalues, then the series

$$\sum_{i\in N}\int_0^t \Phi(s)h_i\, d\tilde{Z}_i(s)$$

converges in $L^2(\Omega; E)$ and for each i the integral

$$\int_0^t \Phi(s)h_i\, d\tilde{Z}_i(s)$$

exists as an $L^2(\Omega;E)$-limit of Ito-Riemann-Stieltjes sums. Here

$$\tilde{Z}(t) = Z(t) - \mathbb{E}\,Z(t) \text{ and } \tilde{Z}_i(t) = \langle \tilde{Z}(t), h_i\rangle, \quad t \geq 0,\ i \in N.$$

Moreover, for every $t \geq 0$ one has

$$\mathbb{E}\,\|\sum_{i\in N}\int_0^t \Phi(s)h_i\, dZ_i(s)\|^2 = \sum_{i\in N}\lambda_i\int_0^t \mathbb{E}\,\|\Phi(s)\|^2\, ds.$$

The stochastic integral will be defined by

$$\int_0^t \Phi(s)\,\mathrm{d}\tilde{Z}(s) := \sum_{i\in N}\int_0^T \Phi(s)h_i\,\mathrm{d}\tilde{Z}_i(s).$$

The organization of the paper is as follows. Section 2 studies Hilbert space valued processes with stationary independent increments and finite second moments and their decomposition into scalar processes. In Section 3 the decomposition is used to construct the stochastic integral and prove Theorem 1.3. Section 4 discusses existence and uniqueness of solutions of equation (1). Finally in Section 5 the existence of stationary solutions of (1) is considered and Theorems 1.1 and 1.2 are proved.

2 Processes with stationary independent increments

Definition 2.1. Let H be a separable real Hilbert space, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $Z = (Z(t))_{t\geq 0}$ be a family of H-valued random variables defined on $(\Omega, \mathcal{F}, \mathbb{P})$. We say that Z is a process with *independent increments* if

(1) for every $t \geq s \geq 0$ the increment $Z(t) - Z(s)$ is independent of the σ-algebra generated by $\{Z(u) : 0 \leq u \leq s\}$,

and we say that Z has *stationary increments* if

(2) for every $s, t, u \geq 0$ the increments $Z(t + u) - Z(t)$ and $Z(s + u) - Z(s)$ have the same distribution.

The family $(Z(t))_{t\geq 0}$ is called a *Lévy process* (cf. [15, Section I.4, p.20]) if in addition to (1) and (2) also

(3) $Z(0) = 0$ $\mathbb{P}$-a.s.;

(4) $t \mapsto Z(t)$ is continuous in probability, that is, if $t \mapsto t_0$ then $\mathbb{P}(\|Z(t)-Z(t_0)\| > \varepsilon) \to 0$ for all $\varepsilon > 0$;

(5) for $\mathbb{P}$-almost every $\omega \in \Omega$ the path $t \mapsto Z(t)(\omega)$ is cadlag (i.e. right continuous and the left hand limit exists at each point).

The process Z is called $(\mathcal{F}_t)_t$*-adapted* if $Z(t)$ if $\mathcal{F}_t$-measurable for every $t \geq 0$. To abbreviate, we say that Z is an $(\mathcal{F}_t)_t$*-stationary independent increments process* if Z is $(\mathcal{F}_t)_t$-adapted, Z has stationary increments, and for every $t \geq s \geq 0$ the increment $Z(t) - Z(s)$ is independent of $\mathcal{F}_s$.

We will assume that $\sup_{0\leq t\leq T} \mathbb{E}\, \|Z(t)\|^2 < \infty$ for all T and view $Z(t)$ as an element of $L^2(\Omega; H)$. The independence and stationarity of the increments yields then a specific structure of the mean and covariance. The main tool is the next lemma.

Lemma 2.2. *Let $(H, \|\cdot\|)$ be a normed space. Let $T > 0$ and let $u : [0, T] \to H$ be additive, that is, $u(s + t) = u(s) + u(t)$ for every $s, t \geq 0$ with $s + t \leq T$. If u is bounded on $[0, T]$, then $u(t) = (t/T)u(T)$ for all $t \in [0, T]$.*

Proof. We may consider the function $t \mapsto u(tT) : [0, 1] \to H$ instead of u, or rather assume that $T = 1$. Clearly, $u(0) = 0$ and if $n, m \in \mathbb{N} = \{1, 2, \ldots\}$ with $n/m \leq 1$ then $mu(n/m) = u(n) = nu(1)$, so $u(t) = tu(1)$ for all $t \in [0, 1] \cap \mathbb{Q}$. Suppose there is a $t \in [0, 1]$ such that $w := u(t) - tu(1) \neq 0$. Choose a sequence $(q_n)_n$ in $[0, 1] \cap \mathbb{Q}$ such that $q_n \uparrow t$. By passing to a subsequence we may assume that $\sum_{n=1}^{\infty}(t - q_n) \leq 1$. By additivity we have that

$$u(t - q_n) = u(t) - u(q_n) = u(t) - q_n u(1),$$

which converges to w if $n \to \infty$. By passing to a suitable subsequence we may assume that

$$\|w - u(t - q_n)\| < 2^{-n} \text{ for all } n.$$

For every $N \in \mathbb{N}$ we then find

$$\begin{aligned} \|u(\sum_{n=1}^{N}(t-q_n))\| &= \|\sum_{n=1}^{N} u(t-q_n)\| \\ &= \|Nw - \sum_{n=1}^{N}(w-u(t-q_n))\| \\ &\geq N\|w\|-1. \end{aligned}$$

The latter is impossible if u is bounded. □

Corollary 2.3. *If $v : [0,\infty) \to [0,\infty)$ is additive, that is, $v(s+t) = v(s) + v(t)$ for all $s,t \geq 0$, then $v(t) = tv(1)$ for all $t \geq 0$.*

Proof. Let $T > 0$ be arbitrary. We want to prove that $v(T) = Tv(1)$. The function v is additive on $[0,T]$. As $v(t) \geq 0$ for all t, it follows that v is increasing on $[0,T]$. Hence $0 = v(0) \leq v(t) \leq v(T)$ for all $t \in [0,T]$. By Lemma 2.2, $v(t) = tv(1)$ for all $t \in [0,T]$. □

In Lemma 2.4–Proposition 2.6 below, *let H be a separable real Hilbert space, let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space, let $(\mathcal{F}_t)_t$ be a filtration in $\mathcal{F}$, and let $Z = (Z(t))_{t\geq 0}$ be an $(\mathcal{F}_t)_t$-stationary independent increments process in H defined on $(\Omega,\mathcal{F},\mathbb{P})$ with $Z(0) = 0$.*

Lemma 2.4. *If $\sup_{0\leq t\leq T} \mathbb{E}\,\|Z(t)\| < \infty$ for all $T > 0$, then*

$$\mathbb{E}\,Z(t) = t\mathbb{E}\,Z(1) \text{ for all } t \geq 0.$$

If $\mathbb{E}\,\|Z(t)\|^2 < \infty$ for all $t \geq 0$, then

$$\mathbb{E}\,\|\tilde{Z}(t)\|^2 = t\mathbb{E}\,\|\tilde{Z}(1)\|^2 \text{ for all } t \geq 0,$$

where $\tilde{Z}(t) = Z(t) - \mathbb{E}\,Z(t)$, $t \geq 0$.

Proof. If $\mathbb{E}\,\|Z(t)\| < \infty$, then $Z(t)$ is Bochner integrable and therefore $\mathbb{E}\,Z(t) \in H$ is well defined. For $s,t \geq 0$ we have that

$$\begin{aligned} \mathbb{E}\,Z(s+t) &= \mathbb{E}\,[Z(s+t) - Z(t)] + \mathbb{E}\,Z(t) \\ &= \mathbb{E}\,[Z(s) - Z(0)] + \mathbb{E}\,Z(t) \\ &= \mathbb{E}\,Z(s) + \mathbb{E}\,Z(t). \end{aligned}$$

Hence the function $t \mapsto \mathbb{E}\,Z(t)$ is additive and since $\|\mathbb{E}\,Z(t)\| \leq \mathbb{E}\,\|Z(t)\|$, it is also bounded on $[0,T]$. By Lemma 2.2 we obtain that $\mathbb{E}\,Z(t) = t\mathbb{E}\,Z(1)$, $t \geq 0$.

If $\mathbb{E}\,\|Z(t)\|^2 < \infty$, then $\mathbb{E}\,\|Z(t)\| \leq (\mathbb{E}\,\|Z(t)\|^2)^{1/2} < \infty$ and therefore $\tilde{Z}(t)$ is well-defined and $\mathbb{E}\,\|\tilde{Z}(t)\|^2 < \infty$ for all $t \geq 0$. For $s,t \geq 0$ we have

$$\begin{aligned} \mathbb{E}\,\|\tilde{Z}(s+t)\|^2 &= \mathbb{E}\,[\|\tilde{Z}(s+t) - \tilde{Z}(t)\|^2 + 2\langle \tilde{Z}(s+t) - \tilde{Z}(t), \tilde{Z}(t)\rangle + \|\tilde{Z}(t)\|^2] \\ &= \mathbb{E}\,\|\tilde{Z}(s)\|^2 + \mathbb{E}\,\|\tilde{Z}(t)\|^2. \end{aligned}$$

Application of Corollary 2.3 completes the proof. □

Proposition 2.5. *Assume that $\mathbb{E}\,\|Z(t)\|^2 < \infty$ for all $t \geq 0$ and let*

$$m(t) := \mathbb{E}\,Z(t), \qquad \tilde{Z}(t) = Z(t) - m(t),\ t \geq 0.$$

The following statements are equivalent:

(a) $\sup_{0\leq t\leq T} \mathbb{E}\,\|Z(t)\|^2 < \infty$ *for all $T \geq 0$;*

(b) $\sup_{0\le t\le T} \mathbb{E}\,\|Z(t)\| < \infty$ *for all* $T \ge 0$*;*

(c) $m(t) = tm(1)$ *for all* $t \ge 0$*;*

(d) $\sup_{0\le t\le T} \|m(t)\| < \infty$ *for all* $T \ge 0$*;*

(e) $t \mapsto Z(t)$ *is continuous in probability.*

Proof. (a)⇒(b) is clear since $\mathbb{E}\,\|Z(t)\| \le (\mathbb{E}\,\|Z(t)\|^2)^{1/2}$ for all $t \ge 0$.

(b)⇒(c) is Lemma 2.4(1).

(c)⇒(d) is trivial.

To show that (d)⇒(a), observe that

$$\begin{aligned} \mathbb{E}\,\|\tilde{Z}(t)\|^2 &= \mathbb{E}\,\left[\|Z(t)\|^2 - 2\langle Z(t), m(t)\rangle + \|m(t)\|^2\right] \\ &= \mathbb{E}\,\left[\|Z(t)\|^2 - 2\langle \mathbb{E}\,Z(t), m(t)\rangle + \|m(t)\|^2\right] \\ &= \mathbb{E}\,\|Z(t)\|^2 - \|m(t)\|^2 \text{ for all } t \ge 0. \end{aligned}$$

Hence by Lemma 2.4,

$$\sup_{0\le t\le T} \mathbb{E}\,\|Z(t)\|^2 \le T\mathbb{E}\,\|\tilde{Z}(1)\|^2 + \Big(\sup_{0\le t\le T} \|m(t)\|\Big)^2.$$

To prove equivalence with (e), let $s, t \ge 0$, $\varepsilon > 0$ and observe that

$$\begin{aligned} \mathbb{P}(\|\tilde{Z}(t) - \tilde{Z}(s)\| > \varepsilon) &\le (1/\varepsilon^2)\mathbb{E}\,\|\tilde{Z}(t) - \tilde{Z}(s)\|^2 \\ &= (1/\varepsilon^2)|t-s|\mathbb{E}\,\|\tilde{Z}(1)\|^2. \end{aligned}$$

Hence $t \mapsto \tilde{Z}(t)$ is continuous in probability and we infer that $t \mapsto Z(t)$ is continuous in probability if and only if m is continuous. Now it is clear that (c)⇒(e)⇒(d). □

In particular, if $(Z(t))_{t\ge 0}$ is a Lévy process such that $\mathbb{E}\,\|Z(t)\|^2 < \infty$ for all $t \ge 0$, then $\mathbb{E}\,Z(t) = t\mathbb{E}\,Z(1)$ and $\mathbb{E}\,\|Z(t) - \mathbb{E}\,Z(t)\|^2 = t\mathbb{E}\,\|Z(1) - \mathbb{E}\,Z(1)\|^2$, $t \ge 0$.

We say that the process $Z = (Z(t))_{t\ge 0}$ has *locally bounded second moments* if $\sup_{0\le t\le T} \mathbb{E}\,\|Z(t)\|^2 < \infty$ for all $T \ge 0$. The process Z can be decomposed in a sequence of scalar processes in the following way.

Proposition 2.6. *Let* $(h_i)_{i\in N}$ *(*$N \subset \mathbb{N}$*) be an orthonormal basis of* H*. Let* $Z_i(t) := \langle Z(t), h_i\rangle$*,* $t \ge 0$*,* $i \in N$*. Then:*

(1) *for every* $i \in N$*,* $(Z_i(t))_{t\ge 0}$ *is a scalar* $(\mathcal{F}_t)_t$*-stationary independent increments process and* $Z_i(0) = 0$ $\mathbb{P}$*-a.s. for every* $i \in N$*;*

(2) $\sum_{i\in N} |Z_i(t)|^2 < \infty$ $\mathbb{P}$*-a.s. for every* $t \ge 0$*;*

(3) *for any* $s, t, u \ge 0$ *and any finite set* $F \subset N$*, the random vectors* $(Z_i(t+u) - Z_i(t) : i \in F)$ *and* $(Z_i(s+u) - Z_i(s) : i \in F)$ *have the same distribution;*

(4) *for every* $t \ge 0$ *the series* $\sum_{i\in N} Z_i(t)h_i$ *converges* $\mathbb{P}$*-a.s. to* $Z(t)$*. If in addition* $\mathbb{E}\,\|Z(t)\|^2 < \infty$*, then* $\sum_{i\in N} \mathbb{E}\,|Z_i(t)|^2 < \infty$ *and the series* $\sum_{i\in N} Z_i(t)h_i$ *converges in* $L^2(\Omega; H)$ *to* $Z(t)$*.*

Conversely, if we have (1), (2), and (3) above, then $Z(t) := \sum_{i\in N} Z_i(t)h_i$ *converges in* H $\mathbb{P}$*-a.s. for every* $t \ge 0$ *and* $(Z(t))_{t\ge 0}$ *is an* $(\mathcal{F}_t)_t$*-stationary independent increments process in* H*.*

Proof. (1) Because Z is adapted to $(\mathcal{F}_t)_t$, $Z_i(t)$ is $\mathcal{F}_t$-measurable for every $t \geq 0$. Further for $t \geq s \geq 0$, $Z(t) - Z(s)$ is independent of $\mathcal{F}_s$ and hence $Z_i(t) - Z_i(s)$ is independent of $\mathcal{F}_s$. If we have proved (3), then it follows that $(Z_i(t))_{t\geq 0}$ is an $(\mathcal{F}_t)_t$-stationary independent increments process. As $Z(0) = 0$ $\mathbb{P}$-a.s., $Z_i(0) = 0$ $\mathbb{P}$-a.s.

(2) For $t \geq 0$ and $\mathbb{P}$-a.e. $\omega \in \Omega$, $Z(t)(\omega)$ is an element of H and by Bessel's identity we have

$$\sum_{i\in N} |\langle Z(t)(\omega), h_i\rangle|^2 = \|Z(t)(\omega)\|^2 < \infty.$$

(3) Let $s, t, u \geq 0$ and $m \in \mathbb{N}$. Since $Z(t+u) - Z(t)$ and $Z(s+u) - Z(s)$ have the same distribution, it follows that $(Z_1(t+u) - Z_1(t), \dots, Z_m(t+u) - Z_m(t))$ and $(Z_1(s+u) - Z_1(s), \dots, Z_m(s+u) - Z_m(s))$ have the same distribution.

(4) This is obvious if N is finite. So we assume that $N = \mathbb{N}$. As remarked in (2), for each $t \geq 0$ and $\mathbb{P}$-a.e. $\omega \in \Omega$, $Z(t)(\omega) \in H$ and we have $Z(t)(\omega) = \sum_{i=1}^{\infty} \langle Z(t)(\omega), h_i\rangle h_i$ (convergent in H). If $\mathbb{E}\,\|Z(t)\|^2 < \infty$, then by Bessel's identity

$$\sum_{i=1}^{\infty} \mathbb{E}\,|Z_i(t)|^2 = \mathbb{E}\,\|Z(t)\|^2 < \infty,$$

so that $\mathbb{E}\,\|Z(t) - \sum_{i=1}^{m} Z_i(t)h_i\|^2 = \mathbb{E}\,\|\sum_{i=m+1}^{\infty} Z_i(t)h_i\|^2 = \sum_{i=m+1}^{\infty} \mathbb{E}\,|Z_i(t)|^2 \to 0$ as $m \to \infty$.

Let us now consider the converse statement. Let $(Z_i(t))_{t\geq 0}$, $i \in N$, be $(\mathcal{F}_t)_t$-stationary independent increments processes in $\mathbb{R}$ with $Z_i(0) = 0$ satisfying (2) and (3). By (2), for $t \geq 0$, $Z(t) := \sum_{i\in N} Z_i(t)h_i$ converges in H $\mathbb{P}$-a.s., and then $Z(t)$ is $\mathcal{F}_t$-measurable. For $t \geq s \geq 0$, $Z_i(t) - Z_i(s)$ is independent of $\mathcal{F}_s$ for all $i \in N$ and hence $Z(t) - Z(s)$ is independent of $\mathcal{F}_s$. Further, (3) yields that for $s, t, u \geq 0$ the random variables $\sum_{i\in F}(Z_i(t+u) - Z_i(t))h_i$ and $\sum_{i\in F}(Z_i(s+u) - Z_i(s))h_i$ have the same distribution for any finite $F \subset \mathbb{N}$. Therefore $Z(t+u) - Z(t)$ and $Z(s+u) - Z(s)$ have the same distribution. □

In case of a Wiener process a special choice of the basis $(h_i)_i$ in H leads to a decomposition into independent processes. For $(\mathcal{F}_t)_t$-stationary independent increments processes we can achieve in a similar fashion a decomposition into uncorrelated processes. The key object to consider is the *covariance operator* associated with a Lévy process with locally bounded second moments. Its construction and properties are very similar to the Gaussian case and therefore we omit proofs. See also [14].

In Theorem 2.7–Corollary 2.9 below, *let H be a separable real Hilbert space, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $(\mathcal{F}_t)_{t\geq 0}$ be a filtration in $\mathcal{F}$, and let $Z = (Z(t))_{t\geq 0}$ be an $(\mathcal{F}_t)_t$-stationary independent increments process in H defined on $(\Omega, \mathcal{F}, \mathbb{P})$ with $Z(0) = 0$ and such that*

$$\mathbb{E}\,\|Z(t)\|^2 < \infty \text{ for all } \geq 0.$$

Let

$$\tilde{Z}(t) := Z(t) - \mathbb{E}\,Z(t) \text{ and } m := \mathbb{E}\,Z(1).$$

Theorem 2.7. (1) *There exists a bounded linear operator $Q : H \to H$ which is uniquely defined by*

$$\langle Qx, y\rangle = \mathbb{E}\,\langle \tilde{Z}(1), x\rangle\langle \tilde{Z}(1), y\rangle, \quad x, y \in H.$$

This operator Q is symmetric, positive semi-definite, and of trace class. Moreover, $\operatorname{trace} Q = \mathbb{E}\,\|\tilde{Z}(1)\|^2$ *and*

$$\langle tQx, y\rangle = \mathbb{E}\,\langle \tilde{Z}(t), x\rangle\langle \tilde{Z}(t), y\rangle, \quad x, y \in H,$$

for every $t \geq 0$.

(2) *There exists an orthonormal basis $(h_i)_{i\in N}$ of H consisting of eigenvectors of Q. If $(\lambda_i)_{i\in N}$ are the corresponding eigenvalues and*

$$\tilde{Z}_i(t) := \langle \tilde{Z}(t), h_i\rangle, \quad t \geq 0,\ i \in N,$$

then

$$\lambda_i = \mathbb{E}\,|\tilde{Z}_i(1)|^2,\ \textit{for all } i \in N, \quad \textit{and } \sum_{i\in N} \lambda_i = \operatorname{trace} Q.$$

Moreover, for every $s, t \geq 0$ and $i, j \in N$ one has

$$\mathbb{E}\,\tilde{Z}_i(s)\tilde{Z}_j(t) = 0 \quad \textit{if } i \neq j.$$

Proof. (1) Very similar to the case of Wiener processes.

(2) It follows from standard theory of trace class operators that Q is compact (see [4, §18]) and hence there exists an orthonormal basis $(h_i)_{i\in N}$ of H consisting of eigenvectors of Q. Clearly,

$$\mathbb{E}\,|\tilde{Z}(1)|^2 = \mathbb{E}\,\langle \tilde{Z}(1), h_i\rangle^2 = \langle Qh_i, h_i\rangle = \lambda_i,\ i \in N.$$

By Bessel's identity,

$$\mathbb{E}\,\|\tilde{Z}(1)\|^2 = \mathbb{E}\sum_{i\in N}\langle \tilde{Z}(1), h_i\rangle^2 = \sum_{i\in N}\lambda_i = \operatorname{trace} Q.$$

Let $t \geq s \geq 0$ and $i, j \in N$ with $i \neq j$. By Proposition 2.6(3), for $s \geq r \geq 0$ the random vectors $(\tilde{Z}_i(s) - \tilde{Z}_i(r), \tilde{Z}_j(s) - \tilde{Z}_j(r))$ and $(\tilde{Z}_i(s-r), \tilde{Z}_j(s-r))$ have the same distribution, so

$$\begin{aligned}
\mathbb{E}\,(\tilde{Z}_i(s) - \tilde{Z}_i(r))(\tilde{Z}_j(s) - \tilde{Z}_j(r)) &= \mathbb{E}\,\tilde{Z}_i(s-r)\tilde{Z}_j(s-r) \\
&= \mathbb{E}\,\langle \tilde{Z}(s-r), h_i\rangle\langle \tilde{Z}(s-r), h_j\rangle \\
&= (s-r)\langle Qh_i, h_j\rangle = 0.
\end{aligned}$$

Proposition 2.6(1) yields that $\tilde{Z}_k(t) - \tilde{Z}_k(s)$ and $\tilde{Z}_\ell(s)$ are independent for all $k, \ell \in N$, so we conclude

$$\begin{aligned}
\mathbb{E}\,\tilde{Z}_i(s)\tilde{Z}_j(t) &= \mathbb{E}\,\tilde{Z}_i(s)\left(\tilde{Z}_j(t) - \tilde{Z}_j(s)\right) + \mathbb{E}\,\left(\tilde{Z}_i(s) - \tilde{Z}_i(0)\right)\left(\tilde{Z}_j(s) - \tilde{Z}_j(0)\right) \\
&= 0.
\end{aligned}$$

□

Definition 2.8. The bounded linear operator $Q : H \to H$ that satisfies

$$\langle Qx, y\rangle = \mathbb{E}\,\langle \tilde{Z}(1), x\rangle\langle \tilde{Z}(1), y\rangle \text{ for all } x, y \in H$$

is called the *covariance operator* of the process $(Z(t))_{t\geq 0}$.

Corollary 2.9. *There exists an orthonormal basis $(h_i)_{i\in N}$ of H such that the decomposition*

$$Z_i(t) := \langle Z(t), h_i\rangle \quad t \geq 0, i \in N,$$

yields scalar $(\mathcal{F}_t)_t$-stationary independent increments processes $(Z_i(t))_{t\geq 0}$, $i \in N$, that are mutually uncorrelated, that is,

$$\mathbb{E}\,[(Z_i(t) - \mathbb{E}\,Z_i(t))(Z_j(s) - \mathbb{E}\,Z_j(s))] = 0$$

for all $s, t \geq 0$, $i, j \in N$ with $i \neq j$.

Example 2.10. Let us show that uncorrelatedness of the components is in general not sufficient for (3) of Proposition 2.6. Let P_1 and P_2 be two scalar Poisson processes with rate 1 and let B_1 and B_2 be two normalized Brownian motions, all of them defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and adapted and with independent increments with respect to a filtration $(\mathcal{F}_t)_{t\geq 0}$ in $\mathcal{F}$. Assume moreover that P_1, P_2, B_1, and B_2 are mutually independent. Let

$$P_2(t) := \begin{cases} P_1(t), & 0 \leq t \leq 1, \\ P_1(1) + P_2(t) - P_2(1), & t > 0, \end{cases}$$

and

$$B_3(t) := \begin{cases} B_1(t), & 0 \leq t \leq 1, \\ B_1(1) + B_2(t) - B_2(1), & t > 0. \end{cases}$$

Then P_3 is a Poisson process with rate 1 and B_3 is a normalized Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$ with respect to $(\mathcal{F}_t)_{t\geq 0}$. Let

$$Z_1(t) := P_1(t) + B_1(t), \quad Z_2(t) := P_3(t) - B_3(t), \; t \geq 0.$$

The processes Z_1 and Z_2 are stationary independent increments processes, $\mathbb{E}\, Z_1(t) = \mathbb{E}\, Z_2(t) = 0$ for all $t \geq 0$, and $\mathbb{E}\, Z_1(s)Z_2(t) = 0$ for all $s, t \geq 0$. However, the process $(Z_1(t), Z_2(t))_{t\geq 0}$ in $\mathbb{R}^2$ does not have stationary increments. For instance, the distributions of $Z_1(1) - Z_2(1)$ and $(Z_1(2) - Z_1(1)) - (Z_2(2) - Z_2(1))$ are not equal.

3 Stochastic integral

If $Z = (Z(t))_{t\geq 0}$ is an $(\mathcal{F}_t)_t$-stationary independent increments process in a separable real Hilbert space H defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $(\mathcal{F}_t)_t$ in $\mathcal{F}$, and $Z(0) = 0$ and $\sup_{0\leq t\leq T} \mathbb{E}\, \|Z(t)\|^2 < \infty$ for all $T \geq 0$, then Z can be splitted into a deterministic process and a process with zero means. Indeed,

$$Z(t) = mt + \tilde{Z}(t), \quad t \geq 0,$$

where

$$m := \mathbb{E}\, Z(1), \qquad \tilde{Z}(t) = Z(t) - mt, \quad t \geq 0.$$

Accordingly, a stochastic integral with respect to Z will be the sum of a Riemann (or Bochner) integral and a stochastic integral with respect to $(\tilde{Z}(t))_{t\geq 0}$, which has zero means. We first construct the latter integral, which we will do via decomposition into scalar processes.

In Definition 3.1–Corollary 3.4 below, *let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space, let* $(\mathcal{F}_t)_{t\geq 0}$ *be a filtration in* $\mathcal{F}$*, let* $N \subset \mathbb{N}$*, and let* $(Z_i(t))_{t\geq 0}$*,* $i \in N$*, be mutually uncorrelated* $(\mathcal{F}_t)_t$*-stationary independent increments processes in* $\mathbb{R}$ *defined on* $(\Omega, \mathcal{F}, \mathbb{P})$ *with* $Z_i(0) = 0$ *for all* i*,* $\sum_{i\in N} \mathbb{E}\, |Z_i(t)|^2 < \infty$ *and* $\mathbb{E}\, Z_i(t) = 0$ *for all* $t \geq 0$ *and* $i \in N$*. Let* $\lambda_i := \mathbb{E}\, |Z_i(1)|^2$*.*

The class of integrands that we consider contains both adapted continuous functions and adapted simple functions.

Definition 3.1. Let X be a Banach space and let $a, b \in \mathbb{R}$ with $a < b$. A function $F : [a, b] \to X$ is called *piecewise uniformly continuous* (PUC) if there exist $a = a_0 < a_1 < \cdots < a_n = b$ such that F is uniformly continuous on (a_{k-1}, a_k) for $k = 1, \ldots, n$. A function $F : [0, \infty) \to X$ is called PUC if $F|_{[0,T]}$ is PUC for every $T > 0$.

Proposition 3.2. *Let* E *be a separable real Hilbert space and let* $T > 0$*. Let* $i, j \in N$ *and let* $\Phi_i, \Phi_j : [0, T] \to L^2(\Omega; E)$ *be adapted PUC functions.*

(1) *There exists an element in* $L^2(\Omega; E)$ *denoted by* $\int_0^T \Phi_i(t)\, dZ_i(t)$ *such that for every* $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that*

$$\mathbb{E}\left\|\int_0^T \Phi_i(t)\, dZ_i(t) - \sum_{k=0}^{n-1} \Phi_i(t_k)(Z_i(t_{k+1}) - Z_i(t_k))\right\|^2 < \varepsilon$$

for every partition $0 = t_0 < t_1 < \cdots < t_n = T$ *with* $\max_k(t_{k+1} - t_k) < \delta$. *In other words, the Ito-Riemann-Stieltjes sums*

$$\sum_{k=0}^{n-1} \Phi(t_k)\left(Z_i(t_{k+1}) - Z_i(t_k)\right)$$

converge to $\int_0^T \Phi_i(t)\, dZ_i(t)$ *in* $L^2(\Omega; E)$.

(2) *One has* $\mathbb{E}\int_0^T \Phi_i(t)\, dZ_i(t) = 0$ *and*

$$\mathbb{E}\left\|\int_0^T \Phi_i(t)\, dZ_i(t)\right\|^2 = \lambda_i \int_0^T \mathbb{E}\,\|\Phi_i(t)\|^2\, dt.$$

(3) *If* $i \neq j$ *one has*

$$\mathbb{E}\left\langle \int_0^T \Phi_i(t)\, dZ_i(t), \int_0^T \Phi_j(t)\, dZ_j(t)\right\rangle = 0.$$

Proof. (1) Let $0 = a_0 < a_1 < \cdots < a_K = T$ be such that Φ_i is uniformly continuous on each interval (a_{k-1}, a_k). Let $M := \sup_{0\le s\le T} \mathbb{E}\,\|\Phi_i(s)\|^2$ and let $\varepsilon > 0$. Take $\delta > 0$ such that $8MN\delta\lambda_i < \varepsilon$ and such that for each k one has $\mathbb{E}\,\|\Phi_i(r) - \Phi_i(s)\|^2 < \varepsilon/(2T(\lambda_i + 1))$ whenever $r, s \in (a_{k-1}, a_k)$ and $|r - s| < \delta$. Let $0 = t_0 < t_1 < \cdots < t_n = T$ be a partition of $[0, T]$ with $\max_k(t_{k+1} - t_k) < \delta$ and let $0 = s_0 < s_1 < \cdots < s_m = T$ be a refinement. Define the simple function

$$\Psi := \sum_{k=0}^{n-1} \Phi_i(t_k)\mathbb{1}_{[t_k, t_{k+1})}.$$

Then the Ito-Riemann-Stieltjes sum of Φ_i corresponding to the partition $t_1, \ldots, t_n$ is

$$\sum_{k=0}^{n-1} \Phi_i(t_k)\left(Z_i(t_{k+1}) - Z_i(t_k)\right) = \sum_{k=0}^{m-1} \Psi(s_k)\left(Z_i(s_{k+1}) - Z_i(s_k)\right).$$

By adaptedness, independent increments, and the assumption $\mathbb{E}\, Z_i(t) = 0$ for all t,

$$
\begin{aligned}
&\mathbb{E}\left\| \sum_{k=0}^{n-1} \Phi_i(t_k)\left(Z_i(t_{k+1}) - Z_i(t_k)\right) - \sum_{k=0}^{m-1} \Phi(s_k)\left(Z_i(s_{k+1}) - Z_i(s_k)\right) \right\|^2 \\
&\quad = \mathbb{E} \sum_{k=0}^{m-1}\sum_{\ell=0}^{m-1} \Big\langle \Psi(s_k) - \Phi_i(s_k), \Psi(s_\ell) - \Phi_i(s_\ell) \Big\rangle \\
&\qquad\qquad\qquad \cdot \left(Z_i(s_{k+1}) - Z_i(s_k)\right)\left(Z_i(s_{\ell+1}) - Z_i(s_\ell)\right) \\
&\quad = \sum_{k=0}^{m-1} \mathbb{E}\, \|\Psi(s_k) - \Phi_i(s_k)\|^2 \mathbb{E}\, \left(Z_i(s_{k+1}) - Z_i(s_k)\right)^2 \\
&\qquad + 2\sum_{k=0}^{m-1}\sum_{\ell=0}^{k-1} \mathbb{E}\left[\Big\langle \Psi(s_k) - \Phi_i(s_k), \Psi(s_\ell) - \Phi_i(s_\ell) \Big\rangle \left(Z_i(s_{\ell+1}) - Z_i(s_\ell)\right) \right] \\
&\qquad\qquad\qquad \cdot \mathbb{E}\, \left(Z_i(s_{k+1}) - Z_i(s_k)\right) \\
&\quad = \sum_{k=0}^{m-1} \mathbb{E}\, \|\Psi(s_k) - \Phi_i(s_k)\|^2 \lambda_i (s_{k+1} - s_k).
\end{aligned}
$$

Denote by $L \subset \{1, \ldots, n\}$ the set of indices ℓ for which the interval $[t_{\ell-1}, t_\ell)$ contains one or more of the points a_k (the jump points of Φ_i). Note that L has at most K elements. Hence

$$
\begin{aligned}
&\sum_{k=0}^{m-1} \mathbb{E}\, \|\Psi(s_k) - \Phi_i(s_k)\|^2 \lambda_i (s_{k+1} - s_k) \\
&\quad = \left(\sum_{\ell\in L} \sum_{k: s_k \in [t_{\ell-1}, t_\ell)} + \sum_{\ell\notin L} \sum_{k: s_k \in [t_{\ell-1}, t_\ell)} \right) \mathbb{E}\, \|\Psi(s_k) - \Phi_i(s_k)\|^2 \lambda_i (s_{k+1} - s_k) \\
&\quad \le \sum_{\ell\in L} \sum_{k: s_k \in [t_{\ell-1}, t_\ell)} \mathbb{E}\, \Big(\|\Psi(s_k)\| + \|\Phi_i(s_k)\| \Big)^2 \lambda_i (s_{k+1} - s_k) \\
&\qquad + \sum_{\ell\notin L} \sum_{k: s_k \in [t_{\ell-1}, t_\ell)} \frac{\varepsilon}{2T(\lambda_i + 1)} \lambda_i (s_{k+1} - s_k) \\
&\quad \le \sum_{\ell\in L} 4M\lambda_i (s_{k+1} - s_k) + \frac{\varepsilon}{2T(\lambda_i + 1)} \lambda_i T \\
&\quad \le N4M\lambda_i\delta + \varepsilon/2 < \varepsilon.
\end{aligned}
$$

Since $L^2(\Omega; E)$ is complete, it follows that the limit of the Ito-Riemann-Stieltjes sums $\sum_{k=0}^{n-1} \Phi_i(t_k)\left(Z_i(t_{k+1}) - Z_i(t_k)\right)$ over partitions $0 = t_0 < t_1 < \cdots < t_n = T$ with mesh sizes tending to zero converge in $L^2(\Omega; E)$.

(2)–(3) For any partition $0 = t_0 < t_1 < \cdots < t_n = T$,

$$
\begin{aligned}
\mathbb{E} \sum_{k=0}^{n-1} \Phi_i(t_k)\left(Z_i(t_{k+1}) - Z_i(t_k)\right) &= \sum_{k=0}^{n-1} \mathbb{E}\, \Phi_i(t_k) \mathbb{E}\, \left(Z_i(t_{k+1}) - Z_i(t_k)\right) \\
&= 0
\end{aligned}
$$

and hence $\mathbb{E} \int_0^T \Phi_i(t)\, \mathrm{d}Z_i(t) = 0$.

Because of (1) and continuity of the inner product in $L^2(\Omega; E)$, it suffices to show for (3) that

$$
\mathbb{E} \left\langle \sum_{k=0}^{n-1} \Phi_i(t_k)\left(Z_i(t_{k+1}) - Z_i(t_k)\right), \sum_{\ell=0}^{n-1} \Phi_j(t_\ell)\left(Z_j(t_{\ell+1}) - Z_j(t_\ell)\right) \right\rangle = 0
$$

for each partition $0 = t_0 < T_1 < \cdots < t_n = T$. For $0 \leq k \leq n-1$ and $0 \leq \ell \leq k-1$, $\Phi_i(t_k), \Phi_j(t_\ell)$ and $Z_j(t_{\ell+1}) - Z_j(t_\ell)$ are $\mathcal{F}_{t_k}$-measurable and therefore independent of $Z_i(t_{k+1}) - Z_i(t_k)$. For $0 \leq k \leq n-1$ and $\ell \geq k+1$, $\Phi_i(t_k), \Phi_j(t_\ell)$, and $Z_i(t_{k+1}) - Z_i(t_k)$ are $\mathcal{F}_{t_\ell}$-measurable and therefore independent of $Z_j(t_{\ell+1}) - Z_j(t_\ell)$. Further, for $0 \leq k \leq n-1$ is $\langle \Phi_i(t_k), \Phi_j(t_k) \rangle$ independent of the increments $Z_p(t_{k+1}) - Z_p(t_k)$ for all p. Recall also that $\mathbb{E}\left(Z_p(t_{k+1}) - Z_p(t_k)\right) = 0$ for all p. Thus we find

$$
\begin{aligned}
&\mathbb{E} \left\langle \sum_{k=0}^{n-1} \Phi_i(t_k) \left(Z_i(t_{k+1}) - Z_i(t_k)\right), \sum_{\ell=0}^{n-1} \Phi_j(t_\ell) \left(Z_j(t_{\ell+1}) - Z_j(t_\ell)\right) \right\rangle \\
&\quad = \sum_{k=0}^{n-1} \mathbb{E}\left[\langle \Phi_i(t_k), \Phi_j(t_k) \rangle \left(Z_i(t_{k+1}) - Z_i(t_k)\right)\left(Z_j(t_{k+1}) - Z_j(t_k)\right)\right] \\
&\quad\quad + \sum_{k=0}^{n-1} \sum_{\ell=0}^{k-1} \mathbb{E}\left[\langle \Phi_i(t_k), \Phi_j(t_\ell) \rangle \left(Z_i(t_{k+1}) - Z_i(t_k)\right)\left(Z_j(t_{\ell+1}) - Z_j(t_\ell)\right)\right] \\
&\quad\quad + \sum_{k=0}^{n-1} \sum_{\ell=k+1}^{n-1} \mathbb{E}\left[\langle \Phi_i(t_k), \Phi_j(t_\ell) \rangle \left(Z_i(t_{k+1}) - Z_i(t_k)\right)\left(Z_j(t_{\ell+1}) - Z_j(t_\ell)\right)\right] \\
&\quad = \sum_{k=0}^{n-1} \mathbb{E}\left[\langle \Phi_i(t_k), \Phi_j(t_k) \rangle\right] \mathbb{E}\left[\left(Z_i(t_{k+1}) - Z_i(t_k)\right)\left(Z_j(t_{k+1}) - Z_j(t_k)\right)\right].
\end{aligned}
$$

If $i \neq j$, then Z_i and Z_j are uncorrelated (see Corollary 2.9) and we can conclude (3) since Z_i and Z_j have zero means. If $i = j$ then we find

$$
\mathbb{E} \left\| \sum_{k=0}^{n-1} \Phi_i(t_k)(Z_i(t_{k+1}) - Z_i(t_k)) \right\|^2 = \sum_{k=0}^{n-1} \mathbb{E} \left\| \Phi_i(t_k) \right\|^2 \lambda_i (t_{k+1} - t_k),
$$

and (2) follows. □

Theorem 3.3. *Let $T > 0$. For each $i \in N$ let $\Phi_i : [0,T] \to L^2(\Omega; E)$ be an adapted PUC function and assume that*

$$
\sum_{i \in N} \lambda_i \int_0^T \mathbb{E} \left\| \Phi_i(t) \right\|^2 dt < \infty.
$$

Then:

(1) *for each $t \in [0,T]$ the series $\sum_{i \in N} \int_0^t \Phi_i(s)\, dZ_i(s)$ converges in $L^2(\Omega; E)$ and*

$$
\mathbb{E} \left\| \sum_{i \in N} \int_0^t \Phi_i(s)\, dZ_i(s) \right\|^2 = \sum_{i \in N} \lambda_i \int_0^t \mathbb{E} \left\| \Phi_i(s) \right\|^2 ds;
$$

(2) *the function $t \mapsto \sum_{i \in N} \int_0^t \Phi_i(s)\, dZ_i(s) : [0,T] \to L^2(\Omega; E)$ is $(\mathcal{F}_t)_t$-adapted and continuous;*

(3) *if D is a separable real Hilbert space and $A : E \to D$ is a bounded linear operator then for each $t \in [0,T]$*

$$
\sum_{i \in N} \int_0^t A\Phi_i(s)\, dZ_i(s) = A \left(\sum_{i \in N} \int_0^t \Phi_i(s)\, dZ_i(s) \right).
$$

Proof. (1) follows from Proposition 3.2(2)–(3).

(2) By construction, for each $t \in [0,T]$ the integral $\int_0^t \Phi_i(s)\,\mathrm{d}Z_i(s)$ is $\mathcal{F}_t$-measurable. In order to prove the continuity, let $t \in [0,T]$ and $h \in [0, T-t]$. We have

$$\begin{aligned}
&\mathbb{E}\Big\| \sum_{i\in N} \int_0^{t+h} \Phi_i(s)\,\mathrm{d}Z_i(s) - \sum_{i\in N} \int_0^{t} \Phi_i(s)\,\mathrm{d}Z_i(s) \Big\|^2 \\
&\quad = \mathbb{E}\Big\| \sum_{i\in N} \int_0^{T} \mathbb{1}_{[t,t+h)}(s)\Phi_i(s)\,\mathrm{d}Z_i(s) \Big\|^2 \\
&\quad = \sum_{i\in N} \lambda_i \int_0^T \mathbb{E}\,\|\mathbb{1}_{[t,t+h)}(s)\Phi_i(s)\|^2\,\mathrm{d}s \\
&\quad = \int_0^T \mathbb{1}_{[t,t+h)}(s) \sum_{i\in N} \lambda_i \mathbb{E}\,\|\Phi_i(s)\|^2\,\mathrm{d}s,
\end{aligned}$$

which tends to zero as $h \downarrow 0$, by Lebesgue's dominated convergence theorem and the assumption that $\int_0^T \sum_{i\in N} \lambda_i \mathbb{E}\,\|\Phi_i(s)\|^2\,\mathrm{d}s < \infty$. Left continuity is proved in the same fashion.

(3) follows from the very construction of the integral and the fact that the map $f \mapsto Af$ from $L^2(\Omega; E)$ to $L^2(\Omega; D)$ is linear, bounded, and preserves adaptedness. □

If E is a separable real Hilbert space, denote by $\mathcal{L}(H,E)$ the Banach space of bounded linear operators from H to E and denote its norm by $|||\cdot|||$.

Corollary 3.4. *Let E be a separable real Hilbert space, let $T > 0$ and let $\Phi : [0,T] \to L^2(\Omega; \mathcal{L}(H,E))$ be an adapted PUC function. Let $(h_i)_{i\in N}$ be elements of H with $\|h_i\| = 1$ for all i and denote $\Phi_i(t) := \Phi(t)h_i$, $t \geq 0$, $i \in N$. Then*

$$\sum_{i\in N} \int_0^T \Phi_i(t)\, dZ_i(t)$$

converges in $L^2(\Omega; E)$ and

$$\mathbb{E}\,\| \sum_{i\in N} \int_0^T \Phi_i(t)\, dZ_i(t) \|^2 \leq \big(\sum_{i\in N} \lambda_i\big) \int_0^T \mathbb{E}\,|||\Phi(t)|||^2\, dt.$$

Proof. Every $\Phi_i : [0,T] \to L^2(\Omega; E)$ is PUC and adapted and

$$\sum_{i\in N} \lambda_i \int_0^T \mathbb{E}\,\|\Phi_i(t)\|^2\,\mathrm{d}t \leq \big(\sum_{i\in N} \lambda_i\big) \int_0^T \mathbb{E}\,|||\Phi(t)|||^2\,\mathrm{d}t.$$

The conclusion follows from Theorem 3.3(1). □

Let us apply the above results to processes $(Z_i(t))_{t\geq 0}$ that arize as components of an H-valued process.

In proposition 3.5–Proposition 3.8, *let H and E be separable real Hilbert spaces, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a filtration $(\mathcal{F}_t)_{t\geq 0}$ in $\mathcal{F}$, and let $Z = (Z(t))_{t\geq 0}$ be an $(\mathcal{F}_t)_t$-stationary independent increments process in H defined on $(\Omega, \mathcal{F}, \mathbb{P})$ with $Z(0) = 0$ and such that*

$$\sup_{0\leq t\leq T} \mathbb{E}\,\|Z(t)\|^2 < \infty \textit{ for all } T \geq 0.$$

Let

$$m := \mathbb{E}\, Z(1), \qquad \tilde{Z}(t) = Z(t) - mt, \quad t \geq 0.$$

Let Q be the covariance operator of Z, let $(h_i)_{i\in N}$ ($N \subset \mathbb{N}$) be an orthonormal basis of H consisting of eigenvectors of Q, and let $(\lambda_i)_{i\in N}$ be the corresponding eigenvalues. Let

$$Z_i(t) := \langle Z(t), h_i\rangle \text{ and } \tilde{Z}_i(t) := \langle \tilde{Z}(t), h_i\rangle, \quad t \geq 0,\ i \in N.$$

We need to show that the definition of the stochastic integral becomes independent of the choice of the eigenvector basis $(h_i)_i$.

Proposition 3.5. *Let $T > 0$ and let $\Phi : [0,T] \to L^2(\Omega; \mathcal{L}(H,E))$ be PUC and adapted. Denote $\Phi_i(t) := \Phi(t)h_i$, $t \geq 0$, $i \in N$. Then:*

(1) *$\sum_{i\in N} \int_0^T \Phi_i(t)\, dZ_i(t)$ converges in $L^2(\Omega : E)$ and*

$$\mathbb{E}\left\|\sum_{i\in N}\int_0^T \Phi_i(t)\, dZ_i(t)\right\|^2 \leq (\text{trace } Q)\int_0^T \mathbb{E}\, |||\Phi(t)|||^2\, dt;$$

(2) *for every $y \in E$ and every $\varepsilon > 0$ there exists a $\delta > 0$ such that for every $0 = t_0 < t_1 < \cdots < t_n = T$ with $\max_k |t_{k+1} - t_k| < \delta$ one has*

$$\mathbb{E}\left|\left\langle \sum_{k=0}^{n-1} \Phi(t_k)\left(\tilde{Z}(t_{k+1}) - \tilde{Z}(t_k)\right), y\right\rangle - \left\langle \sum_{i\in N}\int_0^T \Phi_i(t)\, d\tilde{Z}_i(t), y\right\rangle\right|^2 < \varepsilon;$$

(3) *if $(g_i)_{i\in N}$ is another orthonormal basis of H consisting of eigenvectors of Q, then*

$$\sum_{i\in N}\int_0^T \Phi(t)g_i\, d\langle \tilde{Z}(t), g_i\rangle = \sum_{i\in N}\int_0^T \Phi_i(t)\, d\tilde{Z}_i(t).$$

Proof. (1) follows from Corollary 3.4 and Theorem 2.7.

(2) Let $y \in E$ and $\varepsilon > 0$. Choose $\delta > 0$ such that

$$(\text{trace } Q)\|y\|^2 \int_0^T \mathbb{E}\, |||\sum_{k=0}^{n-1} \Phi(t_k)\mathbb{1}_{[t_k,t_{k+1})}(t) - \Phi(t)|||^2\, \mathrm{d}t < \varepsilon$$

for every partition $0 = t_0 < t_1 < \cdots < t_n = T$ with $\max_k |t_{k+1} - t_k| < \delta$. Then for any such a partition, let $\Psi := \sum_{k=0}^{n-1} \Phi(t_k)\mathbb{1}_{[t_k,t_{k+1})}$. By Parseval's identity and

Theorem 3.3(1),

$$\begin{aligned}
&\mathbb{E}\Big|\Big\langle\sum_{k=0}^{n-1}\Phi(t_k)\left(\tilde Z(t_{k+1})-\tilde Z(t_k)\right),y\Big\rangle-\sum_{i\in N}\int_0^T\langle\Phi_i(t),y\rangle\,\mathrm{d}\tilde Z_i(t)\Big|^2\\
&\quad=\mathbb{E}\Big|\sum_{k=0}^{n-1}\Big\langle\Phi(t_k)^*y,\tilde Z(t_{k+1})-\tilde Z(t_k)\Big\rangle-\sum_{i\in N}\int_0^T\langle\Phi_i(t),y\rangle\,\mathrm{d}\tilde Z_i(t)\Big|^2\\
&\quad=\mathbb{E}\Big|\sum_{i\in N}\sum_{k=0}^{n-1}\langle\Phi_i(t_k)^*y,h_i\rangle\langle\tilde Z(t_{k+1})-\tilde Z(t_k),h_i\rangle\\
&\qquad-\sum_{i\in N}\int_0^T\langle\Phi_i(t),y\rangle\,\mathrm{d}\tilde Z_i(t)\Big|^2\\
&\quad=\mathbb{E}\Big|\sum_{i\in N}\int_0^T\langle(\Psi(t)-\Phi(t))h_i,y\rangle\,\mathrm{d}\tilde Z_i(t)\Big|^2\\
&\quad=\sum_{i\in N}\lambda_i\int_0^T\mathbb{E}\,|\langle(\Psi(t)-\Phi(t))h_i,y\rangle|^2\,\mathrm{d}t\\
&\quad\le\text{trace }Q\int_0^T\mathbb{E}\,|||\Psi(t)-\Phi(t)|||^2\|y\|^2\,\mathrm{d}t<\varepsilon.
\end{aligned}$$

(3) From (2) we infer

$$\Big\langle\sum_{i\in N}\int_0^T\Phi(t)g_i\,\mathrm{d}\langle\tilde Z(t),g_i\rangle,y\Big\rangle=\Big\langle\sum_{i\in N}\int_0^T\Phi_i(t)\,\mathrm{d}\tilde Z_i(t),y\Big\rangle\quad\mathbb{P}\text{-a.s.}$$

for all $y\in E$. Since E is separable the assertion follows. ☐

Let us now address the integral with respect to the deterministic mean process and combine the two integrals.

Lemma 3.6. *Let $T>0$ and let $\Phi:[0,T]\to L^2(\Omega;\mathcal{L}(H;E))$ be PUC. Then $\int_0^T\Phi(t)m\,dt$ exists as a Riemann integral and*

$$\mathbb{E}\left\|\int_0^T\Phi(t)m\,dt\right\|^2\le T\int_0^T\mathbb{E}\,\|\Phi(t)m\|^2\,dt.$$

Proof. The first assertion is standard, the second follows from the Cauchy-Schwarz inequality. ☐

Theorem 3.3, Corollary 3.4, Proposition 3.5 and Lemma 3.6 allow us to give the next definition.

Definition 3.7. Let $T>0$ and let $\Phi:[0,T]\to L^2(\Omega;\mathcal{L}(H;E))$ be PUC and adapted. We define

$$\int_0^T\Phi(t)\,\mathrm{d}Z(t):=\int_0^T\Phi(t)m\,\mathrm{d}t+\sum_{i\in N}\int_0^T\Phi_i(t)\,\mathrm{d}\tilde Z_i(t),$$

where $\Phi_i(t):=\Phi(t)h_i$, $t\ge0$, $i\in N$. Then $\int_0^T\Phi(t)\,\mathrm{d}Z(t)\in L^2(\Omega;E)$ and

$$\mathbb{E}\left\|\int_0^T\Phi(t)\,\mathrm{d}Z(t)\right\|^2\le2(\|m\|T+\text{trace }Q)\int_0^T\mathbb{E}\,|||\Phi_i(t)|||^2\,\mathrm{d}t.$$

If $m = 0$ we have

$$\mathbb{E} \int_0^T \Phi(t)\,\mathrm{d}Z(t) = 0$$

and

$$\mathbb{E} \left\| \int_0^T \Phi(t)\,\mathrm{d}Z(t) \right\|^2 = \sum_{i \in N} \lambda_i \int_0^T \mathbb{E}\,\|\Phi_i(t)\|^2\,\mathrm{d}t.$$

The following observation relates the integral of Definition 3.7 to the integral of [2].

Proposition 3.8. *Let $T > 0$ and let $\Phi : [0,T] \to L^2(\Omega; \mathcal{L}(H,E))$ be adapted and simple, that is,*

$$\Phi(t) = \sum_{k=0}^{n-1} \Phi(t_k) \mathbb{1}_{[t_k, t_{k+1})}(t)$$

for all t and for some $0 < t_0 < \cdots < t_n = T$. Then

$$\int_0^T \Phi(t)\;dZ(t) = \sum_{k=0}^{n-1} \Phi(t_k)\left(Z(t_{k+1}) - Z(t_k)\right).$$

Proof. Denote $\Phi_i(t) := \langle \Phi(t), h_i \rangle$, $t \geq 0$, $i \in N$. If we consider a limit over partitions that are refinements of $t_0, \ldots, t_n$, then Proposition 3.5(2) yields

$$\left\langle \sum_{i \in N} \int_0^T \Phi_i(t)\,\mathrm{d}\tilde{Z}_i(t), y \right\rangle = \left\langle \sum_{k=0}^{n-1} \Phi(t_k) \left(\tilde{Z}(t_{k+1}) - \tilde{Z}(t_k) \right), y \right\rangle \quad \mathbb{P}\text{-a.s.}$$

for every $y \in E$. Hence, as E is separable,

$$\sum_{i \in N} \int_0^T \Phi_i(t)\,\mathrm{d}\tilde{Z}_i(t) = \sum_{k=0}^{n-1} \Phi(t_k) \left(\tilde{Z}(t_{k+1}) - \tilde{Z}(t_k) \right).$$

Clearly,

$$\int_0^T \Phi(t) m\,\mathrm{d}t = \sum_{k=0}^{n-1} \Phi(t_k) m (t_{k+1} - t_k)$$

and the assertion follows. □

Example 3.9. Let $(Z(t))_{t \geq 0}$ be a Poisson process with rate 1 defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mathcal{F}_t$ be the σ-algebra generated by $\{Z(s) : 0 \leq s \leq t\}$, $t \geq 0$. Then $(Z(t))_{t \geq 0}$ is an $(\mathcal{F}_t)_t$-independent increments process with $Z(0) = 0$ and

$$\mathbb{P}(Z(t) = k) = e^{-t} t^k / k!,\ k \in \mathbb{N},\ t > 0.$$

We will compute that

$$\int_0^t Z(s)\,\mathrm{d}Z(s) = \frac{1}{2} Z(t)^2 - \frac{1}{2} Z(t), \quad t \geq 0.$$

First notice that $\mathbb{E}\,Z(t)^{m+1} = t \sum_{\ell=0}^{m} \binom{m}{\ell} \mathbb{E}\,Z(t)^\ell$ for $t \geq 0$ and $m \in \mathbb{N}$, so that

$$\begin{aligned} &\mathbb{E}\,Z(t) = t, && \mathbb{E}\,Z(t)^2 = t^2 + t, \\ &\mathbb{E}\,Z(t)^3 = t^3 + 3t^2 + t, && \mathbb{E}\,Z(t)^4 = t^4 + 6t^3 + 7t^2 + t. \end{aligned}$$

It is straightforward to infer that

$$\mathbb{E}\,[Z(t)^2 - Z(t) - t^2]^2 = t^4 + 4t^3 + 2t^2, \quad t \geq 0. \tag{2}$$

Let $T > 0$ and let $\varepsilon > 0$. Take $\delta := \min\{T^{-1}\varepsilon/7, T^{-2}\varepsilon, 1\}$. Let $0 = t_0 < t_1 < \cdots < t_n = T$ be any partition with $\max_k(t_{k+1} - t_k) < \delta$. We show that $\mathbb{E}\,|\sum_{k=0}^{n-1} Z(t_k)(Z(t_{k+1}) - Z(t_k)) - (\frac{1}{2}Z(T)^2 - \frac{1}{2}Z(T))|^2 < \varepsilon$. We have that

$$\sum_{k=0}^{n-1} Z(t_k)(Z(t_{k+1}) - Z(t_k)) = \frac{1}{2}\sum_{k=0}^{n-1}(Z(t_{k+1}) + Z(t_k))(Z(t_{k+1}) - Z(t_k)) - \frac{1}{2}\sum_{k=0}^{n-1}(Z(t_{k+1}) - Z(t_k))^2$$

and

$$\sum_{k=0}^{n-1}(Z(t_{k+1})^2 - Z(t_k)^2) = Z(T)^2 - Z(0)^2 = Z(T)^2.$$

Further,

$$\sum_{k=0}^{n-1}(Z(t_{k+1}) - Z(t_k))^2 = \sum_{k=0}^{n-1}[(Z(t_{k+1}) - Z(t_k))^2 - (Z(t_{k+1}) - Z(t_k)) - (t_{k+1} - t_k)^2] + Z(T) + \sum_{k=0}^{n-1}(t_{k+1} - t_k)^2.$$

Hence

$$\begin{aligned}
&\mathbb{E}\Big|\sum_{k=0}^{n-1} Z(t_k)(Z(t_{k+1}) - Z(t_k)) - (\frac{1}{2}Z(T)^2 - \frac{1}{2}Z(T))\Big|^2 \\
&= \mathbb{E}\Big|\frac{1}{2}\sum_{k=0}^{n-1}[(Z(t_{k+1}) - Z(t_k))^2 - (Z(t_{k+1}) - Z(t_k)) - (t_{k+1} - t_k)^2] \\
&\quad + \frac{1}{2}\sum_{k=0}^{n-1}(t_{k+1} - t_k)^2\Big|^2 \\
&\le 2\mathbb{E}\Big|\frac{1}{2}\sum_{k=0}^{n-1}[(Z(t_{k+1}) - Z(t_k))^2 - (Z(t_{k+1}) - Z(t_k)) - (t_{k+1} - t_k)^2]\Big|^2 \\
&\quad + 2\mathbb{E}\Big|\frac{1}{2}\sum_{k=0}^{n-1}(t_{k+1} - t_k)^2\Big|^2 \\
&= \frac{1}{2}\sum_{k=0}^{n-1}\mathbb{E}\left|(Z(t_{k+1}) - Z(t_k))^2 - (Z(t_{k+1}) - Z(t_k)) - (t_{k+1} - t_k)^2\right|^2 \\
&\quad + \frac{1}{2}\Big(\sum_{k=0}^{n-1}(t_{k+1} - t_k)^2\Big)^2,
\end{aligned}$$

where in the last step we have used that $(Z(t))_{t\ge 0}$ has independent increments and $\mathbb{E}\,[(Z(t_{k+1}) - Z(t_k))^2 - (Z(t_{k+1}) - Z(t_k)) - (t_{k+1} - t_k)^2] = 0$ for all k. Since the increments of $(Z(t))_{t\ge 0}$ are stationary, (2) yields that

$$\begin{aligned}
&\sum_{k=0}^{n-1}\mathbb{E}\left|(Z(t_{k+1}) - Z(t_k))^2 - (Z(t_{k+1}) - Z(t_k)) - (t_{k+1} - t_k)^2\right|^2 \\
&\quad = \sum_{k=0}^{n-1}(t_{k+1} - t_k)^2[(t_{k+1} - t_k)^2 + 4(t_{k+1} - t_k) + 2] \le T\delta(\delta^2 + 4\delta + 2) < \varepsilon.
\end{aligned}$$

Clearly, $(\sum_{k=0}^{n-1}(t_{k+1} - t_k)^2)^2 \le (T\delta)^2 < \varepsilon$, and the proof is complete.

Notice that Theorem 1.3 is contained in Theorem 2.7, Corollary 3.4, and Proposition 3.2.

4 Stochastic differential equations

By a simple transformation in equation (1) we can arrange that $(Z(t))_{t\geq 0}$ is a process with zero means and, if desirable, with $\mathbb{E}\,\|Z(1)\|^2 = 1$. The main result concerning existence and uniqueness of mild solutions of (1) is Theorem 4.1 below. It considers semi-linear equations of the form

$$dX(t) = [AX(t) + F(t, X(t))]dt + G(t, X(t))dZ(t), \quad t \geq 0,$$

and the slightly more general form

$$dX(t) = [AX(t) + F(t, X(t))]dt + \sum_{i\in N} G_i(t, X(t))dZ_i(t), \quad t \geq 0.$$

In Theorem 4.1–Corollary 4.2 below, *let H and E be separable real Hilbert spaces, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a filtration $(\mathcal{F}_t)_{t\geq 0}$ in $\mathcal{F}$, and let $Z = (Z(t))_{t\geq 0}$ be an $(\mathcal{F}_t)_t$-stationary independent increments process in H defined on $(\Omega, \mathcal{F}, \mathbb{P})$ with $Z(0) = 0$ and $\sup_{0\leq t\leq T} \mathbb{E}\,\|Z(t)\|^2 < \infty$ for all $T \geq 0$. Let Q be the covariance operator of Z, let $(h_i)_{i\in N}$ ($N \subset \mathbb{N}$) be an orthonormal basis of H consisting of eigenvectors of Q and let $(\lambda_i)_{i\in N}$ be the corresponding eigenvalues. Let*

$$m := \mathbb{E}\,Z(t), \qquad \tilde{Z}(t) := Z(t) - mt,$$
$$\tilde{Z}_i(t) := \langle \tilde{Z}(t), h_i\rangle, \quad t \geq 0,\ i \in N.$$

Let $(S(t))_{t\geq 0}$ be a strongly continuous semigroup on E and let $F : [0, \infty) \times E \to E$ be such that there exists $L_F > 0$ with

$$\|F(t, x) - F(t, y)\| \leq L_F\|x - y\|, \quad t \geq 0,\ x, y \in E.$$

Theorem 4.1. *Let $G_i : [0, \infty) \times E \to E$ be continuous for every $i \in N$ and such that there exists a constant $L_G > 0$ with*

$$\sum_{i\in N} \lambda_i\|G_i(t, x) - G_i(t, y)\|^2 \leq L_G^2\|x - y\|^2,$$
$$\int_0^t \sum_{i\in N} \lambda_i\|G_i(s, 0)\|^2\, ds < \infty, \quad t \geq 0,\ x, y \in E.$$

Then:

(1) *for every $X_0 \in L^2(\Omega; E)$ that is $\mathcal{F}_0$-measurable there exists a unique adapted continuous function $X : [0, \infty) \to L^2(\Omega; E)$ such that for all $t \geq 0$*

$$X(t) = S(t)X_0 + \int_0^t S(t - s)F(s, X(s))\, ds + \sum_{i\in N} \int_0^t S(t - s)G_i(s, X(s))\, dZ_i(s);$$

(2) *the solution X of statement (1) is also weakly unique (unique in distribution). That means, if $(\Omega', \mathcal{F}', \mathbb{P}')$ is another probability space with a filtration $(\mathcal{F}'_t)_t$ in $\mathcal{F}'$, $Z' = (Z'(t))_{t\geq 0}$ is an $(\mathcal{F}'_t)_t$-stationary independent increments process in H defined on $(\Omega', \mathcal{F}', \mathbb{P}')$ such that for every $0 \leq t_1 \leq t_2 \leq \cdots \leq t_n$ the distributions of $(\tilde{Z}(t_1), \ldots, \tilde{Z}(t_n))$ and $(Z'(t_1), \ldots, Z'(t_n))$ are the same, and if $X'_0 \in L^2(\Omega'; E)$ is $\mathcal{F}'_0$-measurable and X'_0 and X_0 have the same distribution, then the unique adapted continuous function $X' : [0, \infty) \to L^2(\Omega'; E)$ that satisfies*

$$X'(t) = S(t)X'_0 + \int_0^t S(t-s)F(t, X'(s)\, ds + \sum_{i\in N} \int_0^T S(t-s)G_i(s, X'(s))\, dZ'_i(s),$$

for all $t \geq 0$*, has the property that for every* $0 \leq t_1 \leq t_2 \leq \cdots \leq t_n$ *the random vectors*

$$(X(t_1), \dots, X(t_n), \tilde{Z}(t_1), \dots, \tilde{Z}(t_n)) \text{ and}$$
$$(X'(t_1), \dots, X'(t_n), Z'(t_1), \dots, Z'(t_n))$$

have the same distribution. Here

$$Z'_i(t) = \langle Z'(t), h_i \rangle, \quad i \in N.$$

For a proof of Theorem 4.1 one can follow almost literally the proofs of Theorem 4.1 and Theorem 4.2 in [16], due to Proposition 3.2 and Theorem 3.3.

Corollary 4.2. *Let* $G : [0, \infty) \times E \to \mathcal{L}(H, E)$ *be continuous and such that there exists a constant* L_G *with*

$$|||G(t,x) - G(t,y)||| \leq L_G\|x - y\|, \quad t \geq 0, \ x, y \in E.$$

Then for every $X_0 \in L^2(\Omega; E)$ *that is* $\mathcal{F}_0$*-measurable there exists a unique adapted continuous function* $X : [0, \infty) \to L^2(\Omega; E)$ *such that*

$$X(t) = S(t)X_0 + \int_0^t S(t-s)F(s, X(s))\, ds + \int_0^t S(t-s)G(s, X(s))\, dZ(s), \tag{3}$$

$t \geq 0$*. Moreover, this soluton* X *is also weakly unique.*

Proof. Let

$$\tilde{F}(t,x) := F(t,x) + G(t,x)m, \text{ and}$$
$$G_i(t,x) := G(t,x)h_i, \quad t \geq 0, \ x \in E.$$

Then (3) is equivalent to

$$X(t) = S(t)X_0 + \int_0^t S(t-s)\tilde{F}(s, X(s))\, \mathrm{d}s + \sum_{i\in N} \int_0^t S(t-s)G_i(s, X(s))\, \mathrm{d}\tilde{Z}_i(s),$$

$t \geq 0$. Further,

$$\sum_{i\in N} \lambda_i \|G_i(t,x) - G_i(t,y)\|^2 \leq (\text{trace } Q)|||G(t,x) - G(t,y)||| \leq (\text{trace } Q)L_G\|x - y\|$$

for all $t \geq 0$, $x, y \in E$, and

$$\int_0^t \sum_{i\in N} \lambda_i \|G_i(s,0)\|^2\, \mathrm{d}s \leq (\text{trace } Q) \int_0^t |||G(s,0)|||^2\, \mathrm{d}s < \infty$$

for all $t \geq 0$ by continuity of G. Hence Theorem 4.1 may be applied. □

5 Invariant measures

We are now in a position to extend the results of [17] about existence of invariant measures to evolution equations with more general noise processes than Wiener processes. We will first consider processes with zero means, which can be obtained in the more general situation by a simple transformation of the equation.

In Lemma 5.1–Theorem 5.9 below, *let* H *and* E *be separable real Hilbert spaces, let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space with a filtration* $(\mathcal{F}_t)_t$ *in* $\mathcal{F}$*, and let* $Z = (Z(t))_{t\geq 0}$

be an $(\mathcal{F}_t)_t$*-stationary independent increments process in* H *defined on* $(\Omega, \mathcal{F}, \mathbb{P})$ *with* $Z(0) = 0$, $\mathbb{E}\,\|Z(t)\|^2 < \infty$ *and* $\mathbb{E}\,Z(t) = 0$ *for all* $t \geq 0$*. Let* Q *be the covariance operator of* Z*, let* $(h_i)_{i\in N}$ *(*$N \subset \mathbb{N}$*) be an orthonormal basis of* H *consisting of eigenvectors of* Q*, and let* $(\lambda_i)_{i\in N}$ *be the corresponding eigenvalues. Let*

$$Z_i(t) := \langle Z(t), h_i\rangle, \quad t \geq 0,\ i \in N.$$

Let $(S(t))_{t\geq 0}$ *be a strongly continuous semigroup on* H *with generator* A*. Let* $F : H \to H$ *be such that*

$$\|F(x) - F(y)\| \leq L_F\|x - y\| \text{ for all } x, y \in H,$$

for some $L_F > 0$ *and let* $G_i : H \to H$*,* $i \in N$*, be such that*

$$\sum_{i\in N} \lambda_i\|G_i(x) - G_i(y)\|^2 \leq L_G\|x - y\|^2 \text{ for all } x, y \in H,$$
$$\sum_{i\in N} \lambda_i\|G_i(0)\|^2 < \infty,$$

for some $L_G > 0$*.*

We consider mild solutions of the equation

$$dX(t) = [AX(t) + F(X(t))]dt + \sum_{i\in N} G_i(X(t))dZ_i(t),\ t \geq 0. \tag{4}$$

Recall that for every $X_0 \in L^2(\Omega; E)$ that is $\mathcal{F}_0$-measurable equation (4) has a unique mild solution, which is also weakly unique, by Theorem 4.1.

Lemma 5.1. *Let* $\Phi_i : [0, \infty) \to L^2(\Omega; E)$*,* $i \in N$*, be adapted PUC functions such that*

$$\sum_{i\in N} \lambda_i \int_0^T \mathbb{E}\,\|\Phi_i(t)\|^2\, dt < \infty \text{ for all } T \geq 0.$$

Let $T \geq 0$ *and* $\tau \geq 0$ *and let*

$$Z^\tau(t) := Z(t + \tau) - Z(\tau),$$
$$Z_i^\tau(t) := \langle Z^\tau(t), h_i\rangle,\ t \geq 0,\ i \in N.$$

Then $Z^\tau = (Z^\tau(t))_{t\geq 0}$ *is an* $(\mathcal{F}_{t+\tau})_{t\geq 0}$*-stationary independent increments process with* $Z^\tau(0) = 0$, $\mathbb{E}\,\|Z^\tau(t)\|^2 < \infty$ *and* $\mathbb{E}\,Z^\tau(t) = 0$ *for all* $t \geq 0$*,* Z^τ *has covariance operator* Q*, and*

$$\sum_{i\in N} \int_0^T \Phi_i(t + \tau)\, dZ_i^\tau(t) = \sum_{i\in N} \int_\tau^{T+\tau} \Phi_i(t)\, dZ_i(t).$$

Here

$$\int_\tau^{T+\tau} \Phi_i(t)\, dZ_i(t) := \int_0^{T+\tau} \Phi_i(t)\mathbb{1}_{[\tau,T+\tau]}(t)dZ_i(t), \qquad i \in N.$$

Definition 5.2. The semigroup $(S(t))_{t\geq 0}$ is called *hyperbolic* if $E = E_1 \oplus E_2$ where E_1 and E_2 are closed subspaces of E that are invariant under $S(t)$ for all $t \geq 0$, such that $S(t)|_{E_2}$ extends to a strongly continuous group on E_2, and such that there exist $M \geq 0$, $\alpha_1 > 0$, $\alpha_2 > 0$ with

$$\|S(t)x\| \leq Me^{-\alpha_1 t}\|x\| \text{ for all } x \in E_1,$$
$$\|S(-t)x\| \leq Me^{-\alpha_2 t}\|x\| \text{ for all } x \in E_2,$$

for all $t \geq 0$. We let P_1 denote the projection on E_1 along E_2 and P_2 the projection on E_2 along E_1.

Proposition 5.3. *Assume that $(S(t))_{t\geq 0}$ is hyperbolic, and let E_1, E_2, M, α_1, α_2, P_1, and P_2 be as in Defintion 5.2. Let $\tau > 0$. Let $X_0 \in L^2(\Omega; E)$ be $\mathcal{F}_0$-measurable, let X be the mild solution of (4) with $X(0) = X_0$, and let Y be the mild solution of*

$$dY(t) = [AY(t) + F(Y(t))]dt + \sum_{i\in N} G_i(Y(t))dZ_i^\tau(t),\ t \geq 0,$$

with $Y(0) = X_0$, where

$$Z^\tau(t) := Z(t+\tau) - Z(t), \quad Z_i^\tau(t) := \langle Z^\tau(t), h_i\rangle,\ t \geq 0,\ i \in N.$$

Define

$$\begin{aligned} z_1(t) &:= \mathbb{E}\,\|P_1[X(t+\tau) - Y(t)]\|^2,\\ z_2(t) &:= \mathbb{E}\,\|P_2[X(t+\tau) - Y(t)]\|^2,\ t \geq 0. \end{aligned}$$

(1) *If $\sup_{t\geq 0} \mathbb{E}\,\|X(t)\|^2 < \infty$, then*

$$\begin{aligned} z_1(t) &\leq 3M^2 e^{-2\alpha_1 t} z_1(0) + 2L_1 \int_0^t e^{-\alpha_1(t-s)}(z_1(s) + z_2(s))\,ds\\ z_2(t) &\leq 2L_2 \int_t^\infty e^{-\alpha_2(s-t)}(z_1(s) + z_2(s))\,ds, \end{aligned}$$

for all $t \geq 0$, where

$$L_1 = 3M^2\|P_1\|^2(L_F^2/\alpha_1 + L_G^2) \quad \text{and} \quad L_2 = 3M^2\|P_2\|^2(L_F^2/\alpha_2 + L_G^2).$$

(2) *If $(S(t))_{t\geq 0}$ is exponentially stable, that is, $E_2 = \{0\}$, then $z_2 = 0$ and*

$$z_1(t) \leq 3M^2 e^{-2\alpha_1 t} z_1(0) + 2K_1 \int_0^t e^{-\alpha_1(t-s)} z_1(s)\,ds$$

for all $t \geq 0$, where $K_1 = 3M^2(L_F^2/\alpha_1 + L_G^2)$.

Proof. Let $t \geq 0$. We have

$$\begin{aligned} X(t) &= S(t)X_0 + \int_0^t S(t-s)F(X(s))\,\mathrm{d}s + \sum_{i\in N}\int_0^t S(t-s)G_i(X(s))\,\mathrm{d}Z_i(s),\\ Y(t) &= S(t)X_0 + \int_0^t S(t-s)F(Y(s))\,\mathrm{d}s + \sum_{i\in N}\int_0^T S(t-s)G_i(Y(s))\,\mathrm{d}Z_i^\tau(s). \end{aligned}$$

Hence by Theorem 3.3(3),

$$\begin{aligned} X(t+\tau) &= S(t)S(\tau)X_0 + S(t)\int_0^\tau S(\tau-s)F(X(s))\,\mathrm{d}s\\ &\quad + S(t)\sum_{i\in N}\int_0^\tau S(\tau-s)G_i(X(s))\,\mathrm{d}Z_i(s)\\ &\quad + \int_\tau^{t+\tau} S(t+\tau-s)F(X(s))\,\mathrm{d}s\\ &\quad + \sum_{i\in N}\int_\tau^{t+\tau} S(t+\tau-s)G_i(X(s))\,\mathrm{d}Z_i(s)\\ &= S(t)X(\tau) + \int_0^t S(t-s)F(X(s+\tau))\,\mathrm{d}s\\ &\quad + \sum_{i\in N}\int_0^t S(t-s)G_i(X(s+\tau))\,\mathrm{d}Z_i^\tau(s), \end{aligned}$$

by Lemma 5.1. So

$$\begin{aligned}P_1[X(t+\tau)-Y(t)] &= S(t)P_1(X(\tau)-X_0)\\ &\quad+\int_0^t S(t-s)P_1[F(X(s+\tau))-F(Y(s))]\,\mathrm{d}s\\ &\quad+\sum_{i\in N}\int_0^t S(t-s)P_1[G_i(X(s+\tau))-G_i(Y(s))]\,\mathrm{d}Z_i^\tau(s)\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}\,\|P_1[X(t+\tau)-Y(\tau)]\|^2 &\le 3\mathbb{E}\,\|S(t)P_1[X(\tau)-X_0]\|^2\\ &\quad+3\mathbb{E}\,\|\int_0^t S(t-s)P_1[F(X(s+\tau))-F(Y(s))]\,\mathrm{d}s\|^2\\ &\quad+3\mathbb{E}\,\|\sum_{i\in N}\int_0^t S(t-s)P_1[G_i(X(s+\tau))-G_i(Y(s))]\,\mathrm{d}Z_i^\tau(s)\|^2.\end{aligned}$$

We estimate

$$\begin{aligned}&\mathbb{E}\,\|\sum_{i\in N}\int_0^t S(t-s)P_1[G_i(X(s+\tau)-G_i(Y(s))]\,\mathrm{d}Z_i^\tau(s)\|^2\\ &\quad\le\sum_{i\in N}\lambda_i\int_0^t\mathbb{E}\,\|S(t-s)P_1\|^2\|G_i(X(s+\tau))-G_i(Y(s))\|^2\,\mathrm{d}s\\ &\quad\le\int_0^t\|S(t-s)P_1\|^2L_G^2\mathbb{E}\,\|X(s+\tau)-Y(s)\|^2\,\mathrm{d}s.\end{aligned}$$

Further,

$$\begin{aligned}&\|\int_0^t S(t-s)P_1[F(X(s+\tau))-F(Y(s))]\|^2\,\mathrm{d}s\\ &\quad\le\Big(\int_0^t\|S(t-s)P_1\|^{1/2}\|S(t-s)P_1\|^{1/2}\|F(X(s+\tau))-F(Y(s))\|\,\mathrm{d}s\Big)^2\\ &\quad\le\int_0^t\|S(s)P_1\|\,\mathrm{d}s\int_0^t\|S(t-s)P_1\|\|F(X(s+\tau))-F(Y(s))\|^2\,\mathrm{d}s.\end{aligned}$$

Hence

$$\begin{aligned}&\mathbb{E}\,\|P_1[X(t+\tau)-Y(t)]\|^2\le\\ &\quad 3\mathbb{E}\,\|S(t)P_1[X(\tau)-X_0]\|^2\\ &\qquad+3\int_0^t\|S(s)P_1\|\,\mathrm{d}s\int_0^t\|S(t-s)P_1\|L_F^2\mathbb{E}\,\|X(s+\tau)-Y(s)\|^2\,\mathrm{d}s\\ &\qquad+3\int_0^t\|S(t-s)P_1\|^2L_G^2\mathbb{E}\,\|X(s+\tau)-Y(s)\|^2\,\mathrm{d}s\\ &\quad\le 3M^2e^{-2\alpha_1 t}\mathbb{E}\,\|P_1[X(\tau)-X_0]\|^2\\ &\qquad+3M^2\|P_1\|^2(L_F^2/\alpha_1+L_G^2)\int_0^t e^{-\alpha_1(t-s)}\mathbb{E}\,\|X(s+\tau)-Y(s)\|^2\,\mathrm{d}s.\end{aligned}$$

Since $\mathbb{E}\,\|X(s+\tau)-Y(s)\|^2ds\le 2z_1(s)+2z_2(s)$ for all $s\ge 0$, we obtain

$$z_1(t)\le 3M^2e^{-2\alpha_1 t}z_1(0)+2L_1\int_0^t e^{-\alpha_1(t-s)}(z_1(s)+z_2(s))\,\mathrm{d}s.$$

In case (2) we have that P_1 is the identity map and the assertion is established. To prove the second estimate in case (1), let $t_0 \geq 0$ and $t \in [0, t_0]$. Since

$$X(u+v) = S(u)X(v) + \int_v^{u+v} S(u+v-s)F(X(s))\,\mathrm{d}s + \sum_{i\in N} \int_v^{u+v} S(u+v-s)G_i(X(s))\,\mathrm{d}Z_i(s)$$

we find with $u = t_0 - t$ and $v = t + \tau$ that

$$X(t_0+\tau) = S(t_0-t)X(t+\tau) + \int_t^{t_0} S(t_0-s)F(X(s+\tau))\,\mathrm{d}s + \sum_{i\in N} \int_t^{t_0} S(t_0-s)G_i(X(s+\tau))\,\mathrm{d}Z_i^\tau(s).$$

Apply $S(-(t_0-t))P_2$ and rearrange terms. We obtain

$$\begin{aligned} P_2X(t+\tau) = S(-(t_0-t))P_2Y(t_0+\tau) &\\ - \int_t^{t_0} S(-(s-t))P_2F(X(s+\tau))\,\mathrm{d}s &\\ - \sum_{i\in N} \int_t^{t_0} S(-(s-t))P_2G_i(X(s+\tau))\,\mathrm{d}Z_i^\tau(s).& \end{aligned}$$

Similarly,

$$\begin{aligned} P_2Y(t) = S(-(t_0-t))P_2X(t_0) &\\ - \int_t^{t_0} S(-(s-t))P_2F(Y(s))\,\mathrm{d}s &\\ - \sum_{i\in N} \int_t^{t_0} S(-(s-t))P_2G_i(Y(s))\,\mathrm{d}Z_i^\tau(s).& \end{aligned}$$

Therefore,

$$\begin{aligned} &\mathbb{E}\,\|P_2[X(t+\tau) - Y(t)]\|^2 \\ &\quad \leq 3M^2\|P_2\|^2 e^{-2\alpha_2(t_0-t)}\mathbb{E}\,\|X(t_0+\tau) - Y(t_0)\|^2 \\ &\qquad + 3(M^2\|P_2\|^2L_F^2/\alpha_2)\int_t^{t_0} e^{-\alpha_2(s-t)}\mathbb{E}\,\|X(s+\tau) - Y(s)\|^2\,\mathrm{d}s \\ &\qquad + 3M^2\|P_2\|^2L_G^2 \int_t^{t_0} e^{-\alpha_2(s-t)}\mathbb{E}\,\|Y(s+\tau) - Y(s))\|^2\,\mathrm{d}s. \end{aligned}$$

Now we use that $t \mapsto \mathbb{E}\,\|X(t)\|^2 = \mathbb{E}\,\|Y(t)\|^2$ is bounded and let $t_0 \to \infty$. Then

$$z_2(t) \leq 2L_2 \int_t^\infty e^{-\alpha_2(s-t)}(z_1(s) + z_2(s))\,\mathrm{d}s.$$

□

In the situation of Proposition 5.3(2), application of Gronwall's lemma yields

$$z_1(t) \leq 3M^2 z_1(0)e^{-(\alpha_1-2K_1)t}, \quad t \geq 0,$$

so

$$\mathbb{E}\,\|X(t+\tau) - Y(t)\|^2 \leq 3M^2 e^{-(\alpha_1-2K_1)t}\mathbb{E}\,\|X(\tau) - Y(0)\|^2.$$

If $2K_1 < \alpha_1$ then we can infer that $t \mapsto \mathbb{E}\,\|X(t)\|^2$ is bounded. Indeed, by weak uniqueness of solutions of (4), $X(t)$ and $Y(t)$ have the same distribution for all $t \geq 0$. Hence

$$\begin{aligned}
&\left|(\mathbb{E}\,\|X(t+\tau)\|^2)^{1/2} - (\mathbb{E}\,\|X(t)\|^2)^{1/2}\right| \\
&\quad = \left|(\mathbb{E}\,\|X(t+\tau)\|^2)^{1/2} - (\mathbb{E}\,\|Y(t)\|^2)^{1/2}\right| \\
&\quad \leq (\mathbb{E}\,\|X(t+\tau) - Y(t)\|^2)^{1/2} \text{ for all } t \geq 0.
\end{aligned}$$

If we let

$$c_n := \sup_{n\tau \leq t \leq (n+1)\tau} (\mathbb{E}\,\|X(t)\|^2)^{1/2}, \quad n \in \mathbb{N},$$

then we obtain

$$c_n \leq \Big(3M^2 e^{-(\alpha_1 - 2K_1)n\tau}\mathbb{E}\,\|X(\tau) - Y(0)\|^2\Big)^{1/2} + c_{n-1}, \quad n \geq 1,$$

so for all $n \geq 0$,

$$c_n \leq \sqrt{3}M\Big(e^{(\alpha_1 - 2K_1)\tau/2} - 1\Big)^{-1}\Big(\mathbb{E}\,\|X(\tau) - Y(0)\|^2\Big)^{1/2} + c_0.$$

In particular we obtain that X is bounded and, with $\tau = 1$,

$$\sup_{t \geq 0} \mathbb{E}\,\|X(t)\|^2 \leq \left(\frac{2\sqrt{3}M}{e^{(\alpha_1 - 2K_1)/2} - 1} + 1\right)^2 \sup_{0 \leq t \leq 1} \mathbb{E}\,\|X(t)\|^2.$$

Thus, as $\mathbb{E}\,\|X(\tau) - Y(0)\|^2 \leq 2\mathbb{E}\,\|X(\tau)\|^2 + 2\mathbb{E}\,\|Y(0)\|^2 \leq 4\sup_{t\geq 0}\mathbb{E}\,\|X(t)\|^2$,

$$\mathbb{E}\,\|X(t+\tau) - Y(t)\|^2 \leq De^{-(\alpha_1 - 2K_1)t} \quad \text{for all } t \geq 0 \text{ and } \tau \geq 0,$$

with

$$D = 12M^2\left(\frac{2\sqrt{3}M}{e^{(\alpha_1 - 2K_1)/2} - 1} + 1\right)^2 \sup_{0 \leq t \leq 1} \mathbb{E}\,\|X(t)\|^2.$$

A similar conclusion can be drawn in the case of Proposition 5.3(1), as the next lemma shows.

Lemma 5.4. *Let $z_1, z_2 : [0,\infty) \to [0,\infty)$ be bounded continuous functions such that there exist constants $C, K_1, K_2 \geq 0$ and $\alpha_1 > 0$, $\alpha_2 > 0$ with*

$$\begin{aligned}
z_1(t) &\leq Ce^{-2\alpha_2 t} + K_1\int_0^t e^{-\alpha_1(t-s)}(z_1(s) + z_2(s))\,ds, \\
z_2(t) &\leq K_2\int_t^\infty e^{-\alpha_2(s-t)}(z_1(s) + z_2(s))\,ds
\end{aligned}$$

for all $t \geq 0$. If

$$K_1 < \alpha_1, \quad K_2 < \alpha_2, \text{ and } \alpha_1 K_2 + \alpha_2 K_1 < \alpha_1\alpha_2,$$

then

$$z_1(t) + z_2(t) \leq C\left(1 + \frac{(\alpha_1 - \gamma)\alpha_2(\alpha_2 + \gamma)}{(\alpha_1 - K_1 - \gamma)(\alpha_1 + \alpha_2)\gamma}\right)e^{-\gamma t} \text{ for all } t \geq 0,$$

where γ may be any number in $(0, \gamma_0]$, with

$$\begin{aligned}
\gamma_0 &= \frac{(\alpha_1 - K_1 + \alpha_2)(\alpha_1\alpha_2 - \alpha_1 K_2 - \alpha_2 K_1)}{\alpha_1\alpha_2 - \alpha_1 K_2 - \alpha_2 K_1 + \alpha_2(\alpha_2 - K_2)} \qquad (5) \\
&= \alpha_1 - K_1 - \frac{K_1 K_2 \alpha_2}{\alpha_1\alpha_2 - \alpha_1 K_2 - \alpha_2 K_1 + \alpha_2(\alpha_2 - K_2)}
\end{aligned}$$

A proof of Lemma 5.4 is given in the last part of the proof of Theorem 5.1 in [17], which is independent of the stochastic differential equation under consideration there.

Corollary 5.5. *Under the assumptions and with the notations of Proposition* 5.3(1) *one has*

$$\mathbb{E}\,\|X(t+\tau)-Y(t)\|^2 \le De^{-\gamma t} \text{ for all } t \ge 0,$$

where γ may be any number in $(0,\gamma_0]$, D is given by

$$D = 24M^2\left(1+\frac{(\alpha_1-\gamma)(\alpha_2\alpha_2+\gamma)}{(\alpha_1-2L_1-\gamma)(\alpha_1+\alpha_2)\gamma}\right)\|P_1\|^2\left(\sup_{s\ge 0}\mathbb{E}\,\|X(s)\|^2\right),$$

and γ_0 is given by (5)*.*

The next theorem is contained in Theorem 3.2 of [17].

Theorem 5.6. *Let $1 \le p < \infty$. Let $X : [0,\infty) \to L^2(\Omega; E)$ be a bounded continuous function and let $\mu_{X(t)}$ denote the distribution of $X(t)$ for $t \ge 0$. Assume that there exists for every $\tau > 0$ a bounded continuous function $Y_\tau : [0,\infty) \to L^2(\Omega : E)$ such that $Y_\tau(t)$ has the same dsitribution as $X(t)$ for all $t \ge 0$. If $\mathbb{E}\,\|X(t+\tau)-Y_\tau(t)\| \to 0$ uniformly in $\tau > 0$ as $t \to \infty$, then there exists a Borel probability measure μ on E with $\int_E \|x\|^p\, d\mu(x) < \infty$ such that $\mu_{X(t)} \to \mu$ in the sense of weak convergence of measures as $t \to \infty$.*

It remains to show that the limit measure is invariant. Let for each $X_0 \in L^2(\Omega; E)$ that is $\mathcal{F}_0$-measurable $t \mapsto X(t, X_0)$ denote the mild solution of (4) with $X(0, X_0) = X_0$, and let $\mu_{X(t,X_0)}$ denote the distribution of $X(t, X_0)$, $t \ge 0$. Let

$$C_b(E) := \{f : E \to \mathbb{R} : f \text{ is continuous and bounded}\}.$$

The operator $T(t)$, $t \ge 0$, defined by

$$(T(t)f)(x) := \mathbb{E}\, f(X(t,x)), \quad x \in H,\ f \in C_b(E),$$

are called the *transition operators* of (4). For a bounded linear operator $T : C_b(E) \to C_b(E)$ we denote its adjoint by T^*. It is well known (and admits almost verbatim the proofs of [17, Section 7]) that

(1) $T(t)$ is a bounded linear operator from $C_b(E)$ to $C_b(E)$ for every $t \ge 0$;

(2) $\mu_{X(t,X_0)} = T(t)^*\mu_{X_0}$ for every $t \ge 0$;

(3) $(T(t))_{t\ge 0}$ is a semigroup on $C_b(E)$.

Definition 5.7. Let $(T(t))_{t\ge 0}$ be the transition operators of (4). A Borel probability measure μ on E such that $T(t)^*\mu = \mu$ for all $t \ge 0$ is called an *invariant measure* of (4). If an $\mathcal{F}_0$-measurable $X_0 \in L^2(\Omega; E)$ is such that $\mu_{X(t,X_0)} = \mu_{X_0}$ for all $t \ge 0$, then μ_{X_0} is called a *stationary distribution* of (4) and $t \mapsto X(t, X_0)$ is then called a *stationary solution*. With these definitions, an invariant measure μ is a stationary distribution of (4) if and only if

$$\int_E \|x\|^2\, \mathrm{d}\mu(x) < \infty,$$

if $\mathcal{F}_0$ is assumed to be rich enough to allow existence of an $\mathcal{F}_0$-measurable random variable with distribution μ.

As in [17, Proposition 7.2] we have the following.

Proposition 5.8. *Let $X_0 \in L^2(\Omega; E)$ be $\mathcal{F}_0$-measurable. If $\mu_{X(t,X_0)}$ converges weakly to some Borel probability measure μ on E, then μ is an invariant measure of (4). If $t \mapsto \mathbb{E}\,\|X(t,X_0)\|^2$ is bounded, then $\int_E \|x\|^2\, d\mu(x) < \infty$.*

By combining the above results we conclude the next theorem on invariant measures.

Theorem 5.9. *Assume that $(S(t)_{t\geq 0}$ is hyperbolic and let E_1, E_2, α_1, α_2 be as in Definition 5.2. Let L_1, L_2, and K_1 be as in Proposition 5.3.*

(1) *If L_F and L_G are so small that*

$$2L_1 < \alpha_1, \quad 2L_2 < \alpha_2, \text{ and } 2\alpha_1 L_2 + 2\alpha_2 L_1 < \alpha_1\alpha_2,$$

and if there exists a mild solution X of (4) with $\sup_{t\geq 0} \mathbb{E}\,\|X(t)\|^2 < \infty$, then there exists an invariant measure μ of (4) with with $\int_E \|x\|^2\, d\mu(x) < \infty$.

(2) *If $S(t))_{t\geq 0}$ is exponentially stable and L_F, L_G are so small that $2K_1 < \alpha_1$, then there exists a unique invariant measure μ of (4) with $\int_E \|x\|^2\, d\mu(x) < \infty$.*

Proof. (1) and the existence in (2) follow from Corollary 5.5, Theorem 5.6, and Proposition 5.8. To show the uniqueness in (2), suppose that μ and ν are both invariant measures of (4) with finite second moments. Let X_0 and Y_0 be $\mathcal{F}_0$-measurable random vaiables on $(\Omega, \mathcal{F}, \mathbb{P})$ with distributions μ and ν, respectively. Let X be the solution of (4) with $X(0) = X_0$ and Y the solution of (4) with $Y(0) = Y_0$. If we paraphrase the first steps of the proof of Proposition 5.3, we obtain that

$$\begin{aligned} \mathbb{E}\,\|X(t) - Y(t)\|^2 \quad &\leq \quad 3M^2 e^{-2\alpha_1 t}\mathbb{E}\,\|X_0 - Y_0\|^2 \\ &\quad + K_1 \int_0^t e^{-\alpha_1(t-s)}\mathbb{E}\,\|X(s) - Y(s)|^2\,\mathrm{d}s, \end{aligned}$$

for all $t \geq 0$. Multiplication by $e^{\alpha_1 t}$ and application of Gronwall's lemma yields

$$\mathbb{E}\,\|X(t) - Y(t)|^2 \leq 3M^2\mathbb{E}\,\|X_0 - Y_0\|^2 e^{-(\alpha_1 - K-1)t}, \text{ for } t \geq 0.$$

For every $f : E \mapsto \mathbb{R}$ that is Lipschitz continuous with Lipschitz constant L_f, we obtain, by invariance of the measures μ and ν, that

$$\begin{aligned} \left|\int f\,\mathrm{d}\mu - \int f\,\mathrm{d}\nu\right| \quad &= \quad \left|\int f\,\mathrm{d}\mu_{X(t)} - \int f\,\mathrm{d}\mu_{Y(t)}\right| \\ &= \quad |\mathbb{E}\,f(X(t)) - \mathbb{E}\,f(Y(t))| \\ &\leq \quad L_f\mathbb{E}\,\|X(t) - X(t)|^2 \to 0 \text{ as } t \to \infty. \end{aligned}$$

Thus $\mu = \nu$. □

Now Theorem 1.1 and Theorem 1.2 follow easily.

Proof. of Theorem 1.1 Let Q be the covariance operator of $(S(t))_{t\geq 0}$ and let $(h_i)_{i\in N}$ $(N \subset \mathbb{N})$ be a basis of H consisting of eigenvectors of Q and let $(\lambda_i)_{i\in N}$ be the corresponding eigenvalues. Let

$$\begin{aligned} &m := \mathbb{E}\,Z(1), \qquad \tilde{Z}(t) := Z(t) - mt, \\ &\tilde{Z}_i(t) := \langle \tilde{Z}(t), h_i \rangle, \quad t \geq 0, \ i \in N. \end{aligned}$$

Let

$$\begin{aligned} &\tilde{F}(x) := F(x) + m, \\ &G_i(x) := G(x)h_i, \quad x \in E, \ i \in N. \end{aligned}$$

Then

$$\begin{aligned}
&\|\tilde F(x)-\tilde F(y)\| \le L_F\|x-y\|,\\
&\sum_{i\in N}\lambda_i\|G_i(x)-G_i(y)\|^2 \le (\text{trace } Q)L_G^2\|x-y\|,\\
&\sum_{i\in N}\lambda_i\|G_i(0)\|^2 \le (\text{trace } Q)|||G(0)||| < \infty,
\end{aligned}$$

for all $x, y \in E$. Every mild solution of (1) is a mild solution of

$$dX(t) = [AX(t) + \tilde F(X(t))]dt + \sum_{i\in N} G_i(X(t))d\tilde Z_i(t), \quad t \ge 0,$$

by definition of the stochastic integral. Recall that trace $Q = \mathbb{E}\,\|\tilde Z(1)\|^2$. Application of Theorem 5.9(1) completes the proof. □

Proof. of Theorem 1.2(2) The assertion follows from Theorem 5.9(2) in the same way as Theorem 1.1 is concluded from Theorem 5.9(1). □

6 Examples

Example 6.1. (Heat equation) Let $\Lambda \subset \mathbb{R}^d$ be a bounded open set with a C^2 boundary $\partial\Lambda$. Consider $E = L^2(\Lambda)$ and let $F : E \to E$ and $G : E \to \mathcal{L}(H, E)$ be Lipschitz maps with Lipschitz constants L_F and L_G, respectively. Here H is a separable real Hilbert space on which we consider a stationary independent increments process $Z = (Z(t))_{t\ge 0}$, which is defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $(\mathcal{F}_t)_{t\ge 0}$. Assume $Z(0) = 0$ and

$$\sup_{0\le t\le T} \mathbb{E}\,\|Z(t)\|^2 < \infty \text{ for all } T > 0. \tag{6}$$

Consider the nonlinear stochastic heat equation

$$\frac{\partial u}{\partial t}(t, x) = \Delta_x u(t, x) + F(u(t, \cdot))(x) + (G(u(t, \cdot))dZ(t))(x), \tag{7}$$

for a.e. $x \in \Lambda$ and all $t \ge 0$, with initial and boundary conditions

$$\begin{aligned}
u(0, x) &= u_0(x), && x \in \Lambda,\\
u(t, x) &= 0, && x \in \partial\Lambda,
\end{aligned}$$

where u_0 is an E-valued $\mathcal{F}_0$-measurable random variable. The operator $A := \Delta_x$ defined on $D(A) = \{u \in H^2(\Lambda) : u = 0 \text{ on } \partial\Lambda\} \subset E$ generates a strongly continuous semigroup $(S(t))_{t\ge 0}$ on E such that

$$\|S(t)\| \le \exp(-\alpha t) \text{ for all } t \ge 0,$$

where $-\alpha < 0$ is the greatest eigenvalue of A. Good estimates of α are given in [12, Theorem 12.3, p. 304]. According to Theorem 1.2 there exists a unique invariant measure μ for (7) on E with $\int_E \|x\|^2\, d\mu < \infty$ whenever

$$K := 6(L_F^2/\alpha + L_G^2\sigma) < \alpha,$$

where $\sigma = (\mathbb{E}\,\|Z(1) - \mathbb{E}\,Z(1)\|^2)^{1/2}$. Remark that $L_G = 0$ in the case of additive noise. Further, if F is given by

$$F(u) = x \mapsto f(u(x)), \quad u \in L^2(\Lambda),$$

for some Lipschitz continuous function $f : \mathbb{R} \to \mathbb{R}$, then $L_F \le L_f$, where L_f denotes the Lipschitz constant of f. In the linear case, that is, $F = 0$ and $L_G = 0$, the condition (6) on the noise process Z can be relaxed (see [3]).

Example 6.2. (Functional differential equation) Let $r > 0$, $d \in \mathbb{N}$, $d \geq 1$, and denote for a function $x : [-r, \infty) \to \mathbb{R}^d$,

$$x_t(\theta) := x(t+\theta), \quad \theta \in [-r, 0],\ t \geq 0.$$

Consider the functional differential equation

$$\begin{cases} \frac{dx}{dt}(t) = L(x_t), & t \geq 0, \\ x_0 = \varphi, \end{cases} \tag{8}$$

where $\varphi : [-r, 0] \to \mathbb{R}^d$, $L : W^{1,2} \to \mathbb{R}^d$ is a bounded linear functional, and where $W^{1,2}$ is the Sobolev space

$$\begin{aligned} W^{1,2} \quad = \quad & W^{1,2}([-r,0];\mathbb{R}^d) = \{f \in L^2([-r,0];\mathbb{R}^d) : \\ & \exists g \in L^2([-r,0];\mathbb{R}^d) \text{ such that } f(t) = f(-r) + \int_{-r}^t g(s)\,\mathrm{d}s \\ & \text{for all } t \in [-r, 0]\}. \end{aligned}$$

For $f \in W^{1,2}$ we denote the corresponding derivative g by f', and

$$\|f\|_{1,2} = \left(\|f\|_2^2 + \|f'\|_2^2\right)^{1/2},$$

where $\|f\|_2 = \left(\int_{-r}^0 |f(\theta)|^2\,\mathrm{d}\theta\right)^{1/2}$. Here $|\cdot|$ denotes the norm of $\mathbb{R}$. It is well known (see e.g. [1, Theorem 4.4, p.138]) that $W^{1,2}([-r,0];\mathbb{R}^d) \subset C([-r,0];\mathbb{R}^d)$ and

$$\|f\|_\infty = \sup_{-r \leq t \leq 0} |f(\theta)| \leq \left(\frac{r^2+1}{r}\right)^{1/2} \|f\|_{1,2}, \quad f \in W^{1,2}.$$

Let M^2 denote the Delfour-Mitter space

$$M^2 = \mathbb{R}^d \oplus L^2([-r,0];\mathbb{R}^d)$$

endowed with the norm

$$\|(\alpha, y)\|_{M^2} = \left(|\alpha|^2 + \|y\|_2^2\right)^{1/2}.$$

Define

$$\begin{aligned} &D(A) := \{(\alpha, y) \in M^2 : y \in W^{1,2},\ \alpha = y(0)\}, \\ &A(\alpha, y) := (Ly, y'), \quad (\alpha, y) \in D(A). \end{aligned}$$

According to [7], there exists a strongly continuous semigroup $(S(t))_{t\geq 0}$ on M^2 with generator $(A, D(A))$. Moreover, for each $\varphi = (\alpha, y) \in M^2$ one has that a function $x : [-r, \infty) \to \mathbb{R}^d$ with a continuous restriction to $[0, \infty)$ satisfies

$$\begin{cases} x(t) = \alpha + L(\int_0^t x_s\,\mathrm{d}s), & t \geq 0, \\ x(t) = y(t), & -r \leq t < 0, \end{cases} \tag{9}$$

if and only if $(x(t), x_t) = S(t)(\alpha, y)$ for all $t \geq 0$. To emphasize the initial condition, we denote this solution by $x(\cdot, \varphi)$. If in addition $y \in W^{1,2}$ and $\alpha = y(0)$, then this function x satisfies $x_t \in W^{1,2}$ for all $t \geq 0$ and equation (8) holds.

Because $(S(t))_{t\geq 0}$ is strongly continuous, there exist constants $C, \nu \in \mathbb{R}$ such that

$$\|S(t)\varphi\|_{M^2} \leq Ce^{\nu t}\|\varphi\|_{M^2} \text{ for all } t \geq 0,\ \varphi \in M^2. \tag{10}$$

Hence

$$|x(t,\varphi)| = |(S(t)\varphi)_1| \leq Ce^{\nu t}|\varphi\|_{M^2}, \quad t \geq 0.$$

In some cases the possible values of the exponent ν in (10) can be determined. Suppose that L is continuous with respect to the supremum norm $\|\cdot\|_\infty$ on $W^{1,2}$. Then for $\varphi \in M^2$ and $t \geq \tau \geq 0$,

$$\begin{aligned}|x(t,\varphi) - x(\tau,\varphi)| &= \left| L\left(\int_\tau^t x_s(\varphi)\,\mathrm{d}s\right)\right| \\ &\leq \ell \sup_{-r\leq\theta\leq 0}\left|\int_\tau^t x_s(\varphi)(\theta)\,\mathrm{d}s\right| \\ &\leq \ell(t-\tau)Ce^{\nu t}\|\varphi\|_{M^2},\end{aligned}$$

where ℓ denotes the norm of L with respect to $\|\cdot\|_\infty$. Hence for any $T \geq 0$, the set

$$\{x(\cdot,\varphi) : [0,T] \to \mathbb{R} : \ \varphi \in M^2, \ \|\varphi\|_{M^2} \leq 1\}$$

is equicontinuous and by the Arzela-Ascoli theorem relatively compact with respect to $\|\cdot\|_\infty$. It follows that $\{S(t)\varphi : \varphi \in M^2, \ \|\varphi\|_{M^2} \leq 1\}$ is relatively compact in M^2 for each $t \geq r$. Hence the semigroup S is eventually compact. Therefore the growth bound of S equals the spectral radius of A (see [9, Corollary IV.3.12, p.281]). That means that for every

$$\nu > s(A) := \sup\{\Re\lambda : \lambda \in \sigma(A)\}$$

there exists a constant $C \geq 1$ such that (10) holds. It is fairly often possible to estimate or even compute the spectrum of A and thus the possible values of ν (see [8, 9]). It seems that much less is known about the corresponding constant C. Some upper bounds for C can be derived from [7].

Next, let $m \in \mathbb{N}$, and let $f : M^2 \to \mathbb{R}^d$ and $g : M^2 \to \mathbb{R}^{d\times m}$ be Lipschitz continuous functions. Let $(Z(t))_{t\geq 0}$ be a stationary independent increments process in $\mathbb{R}^m$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $(\mathcal{F}_t)_{t\geq 0}$ and such that $Z(0) = 0$ and $\sup_{0\leq t\leq T} \mathbb{E}\,|Z(t)|^2 < \infty$ for all $T \geq 0$. Consider the nonlinear stochastic functional differential equation

$$dx(t) = \Big(L(x_t) + f(x(t), x_t)\Big)dt + g(x(t), x_t)dZ(t), \quad t \geq 0. \tag{11}$$

We can rewrite this equation in our framework as

$$dX(t) = \Big(AX(t) + F(X(t))\Big)dt + G(X(t))dZ(t), \quad t \geq 0, \tag{12}$$

where $X(t) = (x(t), x_t)$, $t \geq 0$, is a process in M^2, A is the generator of $(S(t))_t$ as defined above, and

$$F(\alpha, y) = (f(\alpha, y), 0), \qquad G(\alpha, y) = (g(\alpha, y), 0), \quad (\alpha, y) \in M^2.$$

The Lipschitz constants of F and G equal the Lipschitz constants L_f and L_g of f and g, respectively. Denote $\sigma := (\mathbb{E}\,|Z(1) - \mathbb{E}\,Z(1)|^2)^{1/2}$. According to Theorem 1.2 there exists a unique invariant measure μ of (12) with $\int_{M^2} \|x\|_{M^2}\,\mathrm{d}\mu(x) < \infty$ if

$$6C^2(L_f^2/|\nu| + L_g^2\sigma^2) < |\nu| \quad \text{and} \quad \nu < 0.$$

Of course the same conclusion holds for coefficients F and G of more general form with sufficiently small Lipschitz constants and also for infinte dimensional processes $(Z(t))_{t\geq 0}$ that satisfy the condition of Theorem 1.2.

Finally, notice the difference between the required continuity of the linear functional L and the nonlinearities f and g. The functional L must be continuous on $W^{1,2}$ and therefore continuity with respect to $\|\cdot\|_\infty$ suffices. It may contain point delays. The functionals f and g are supposed to be continuous on M^2 and may not contain other point evaluations than at 0. They are typically of an integral form with possibly Dirac measure at 0.

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References

[1] V.I. Burenkov, *Sobolev spaces on Domains*, Teubner-Texte zur Mathematik, Band 137, B.G. Teubner, Stuttgart, Leipzig, 1998.

[2] A. Chojnowska-Michalik, On processes of Ornstein-Uhlenbeck type in Hilbert space, *Stochastics* **21** (1987), 251–286.

[3] A. Chojnowska-Michalik, Stationary solutions for heat equation perturbed by general additive noise, *J. Appl. Anal.* **3**, no. 1 (1997), 129–136.

[4] J.B. Conway, *A Course in Operator Theory*, Graduate Studies in Mathematics 22, American Mathematical Society, Providence, 2000.

[5] G. Da Prato and J. Zabczyk, *Stochastic Equations in Infinite Dimensions*, Encyclopedia of mathematics and its applications, Vol. 45, Cambridge University Press, Cambridge, 1992.

[6] G. Da Prato and J. Zabczyk, *Ergodicity for Infinite Dimensional Systems*, London Mathematical Society Lecture Notes Series 229, Cambridge University Press, Cambridge, 1996.

[7] M.C. Delfour, The largest class of hereditary systems defining a C_0 semigroup on the product space, *Can. J. Math.* **32**, no. 4 (1980), 969–978.

[8] O. Diekmann, S.A. van Gils, S.M. Verduyn Lunel, and H.-O. Walther, *Delay Equations, Functional-, Complex-, and Nonlinear Analysis*, Applied Mathematical Sciences, Vol. 110, Springer-Verlag, New York, 1995.

[9] K.-J. Engel and R. Nagel, *One-parameter Semigroups for Linear Evolution Equations*, Graduate Texts in Mathematics 194, Springer-Verlag, New York, 2000.

[10] A.A. Guschchin and U. Küchler, On stationary solutions of delay differential equations driven by a Lévy process, *Stochastic Process. Appl.* **88** (2000), 195–211.

[11] J. Jacod, *Calcul stochastique et problèmes de martingales*, Lecture Notes in Mathematics 714, Springer-Verlag, Berlin, 1979.

[12] E.H. Lieb and M. Loss, *Analysis* (2nd ed.), Graduate Studies in Mathematics, Vol. 14, American Mathematical Society, Providence, Rhode Island, 2001.

[13] S.-E.A. Mohammed and M.K.R Scheutzow, Lyapunov exponents of linear stochastic functional differential equations driven by semimartingales, *Ann. Inst. Henri Poincaré* **32**, no. 1 (1996), 69–105.

[14] S. Peszat and J. Zabczyk, Lecture notes on SDE with infinite dimensional Lévy noise, in preparation.

[15] Ph. Protter, *Stochastic Integration and Differential Equations, A New Approach*, Springer-Verlag, Berlin, Heidelberg, 1990.

[16] O. van Gaans, A series approach to stochastic differential equations with infinite dimensional noise, *Integral Equations Operator Theory* **51**, no. 3 (2005), 435–458.

[17] O. van Gaans and S. Verduyn Lunel, Long term behavior of dichotomous stochastic differential equations in Hilbert spaces, *Commun. Contemp. Math.* **6**, No. 3 (2004), 349–376.