I. (a) (Exercise 17 of Chapter 13 of Rudin’s book Functional Analysis): Show that the spectrum of an operator $T : \mathcal{D}(T) \to H$ on a complex Hilbert space is a closed subset of $\mathbb{C}$. Hint: if $ST \subset TS = I$ for a bounded operator $S$ on $H$, and $|\lambda|$ is small, then $S(I - \lambda S)^{-1}$ (why is this well defined?) is a bounded inverse of $T - \lambda$.

(b) Let $\mathcal{D}(T) \subseteq H$ be a proper dense subspace of the complex Hilbert space $H$. Let $\lambda_0 \in \mathbb{C}$ be given, and define $T : \mathcal{D}(T) \to \mathbb{C}$ by $Tx = \lambda_0 x$, for $x \in \mathcal{D}(T)$. Determine the spectrum of $T$.

II. Let $T : \mathcal{D}(T) \to H$ be a closed operator on the complex Hilbert space $H$.

(a) Let $S : H \to H$ be bounded. Show that $T + S$ is a closed operator on $H$.

(b) The resolvent set of $T$ consists of those $\lambda \in \mathbb{C}$ such that there exists a bounded $R : H \to H$ such that $R(T - \lambda) \subset (T - \lambda)R = I$. In this case, the requirement that $R$ should be bounded is redundant: this is automatic, as a consequence of the fact that $T$ is closed. Why?

III. Let $\{e_n\}_{n=1}^\infty$ be an orthonormal basis of the complex Hilbert space $\ell^2$. Fix complex numbers $\lambda_1, \lambda_2, \lambda_3, \ldots$, let

$$\mathcal{D}(T) = \left\{ \sum_{n=1}^\infty x_n e_n \in \ell^2 : \sum_{n=0}^\infty |\lambda_n x_n|^2 < \infty \right\},$$

and define $T : \mathcal{D}(T) \to H$ as

$$T \left( \sum_{n=1}^\infty x_n e_n \right) = \sum_{n=1}^\infty \lambda_n x_n e_n,$$

for $\sum_{n=1}^\infty x_n e_n \in \mathcal{D}(T)$.

(a) Determine the spectrum of $T$. You may find it helpful to use that the spectrum of an arbitrary operator is always closed.

(b) $T$ is densely defined. Why?

(c) Determine the adjoint $T^*$ of $T$. Finite dimensional subspaces can be helpful for a precise argumentation that the domain of $T^*$ is what you think it is.

See reverse side
IV. Let \((H, (\cdot, \cdot))\) be a complex Hilbert space and let \(A: D(A) \to H\) be a densely defined closed operator on \(H\). Suppose that
\[
(Ax, x) \leq 0 \text{ for all } x \in D(A) \text{ and }
\lambda I - A \text{ is surjective for every } \lambda > 0.
\]
Such an operator \(A\) is called \textit{m-dissipative}. Prove that \(A\) is the infinitesimal generator of a \(C_0\)-semigroup in \(H\), that is, a semigroup \(\{Q(t)\}\) satisfying the conditions of Definition 13.34(a)–(c) of Rudin’s book.

Hint: show first that for each \(\lambda > 0\) the operator \(\lambda I - A\) has a bounded inverse with norm \(\leq \frac{1}{\lambda}\).

— Due: January 21, 2014 —

— One point subtraction deadline: January 28, 2014 —

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If handed in by email, send the work as 1 (!) PDF, and make sure your name, university and student number are in the file.
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