Mastermath Functional Analysis

Homework 6

I. (Exercise 13 of Chapter 12 of Rudin's book Functional Analysis): Let $\mathcal{H}$ be a Hilbert space over $\mathbb{C}$ and let $T \in \mathcal{B}(\mathcal{H})$ be a normal operator. Show that there exists a unitary operator $U \in \mathcal{B}(\mathcal{H})$ such that $T^* = UT$. When is U unique? Find a necessary and sufficient condition on T formulated in terms of the norm of T, the range of T, the kernel of T, or something similar.

II. Exercise 18 of Chapter 12 of Rudin's book Functional Analysis, on point spectrum, continuous spectrum and residual spectrum. Recall for (c) that the Hilbert space ℓ^2 is defined by

$$\ell^2 = \{x\colon \mathbb{Z}_+ \to \mathbb{C}\colon \sum_{n=0}^{\infty} |x(n)|^2 < \infty\}$$

with the inner product given by $(x, y) = \sum_{n=0}^{\infty} x(n)\overline{y(n)}$. The left shift and right shift on ℓ^2 are the operators $S_L\colon \ell^2 \to \ell^2$ and $S_R\colon \ell^2 \to \ell^2$ given by

$$(S_L x)(n) = x(n+1), \quad n \in \mathbb{Z}_+,$$

and

$$(S_R x)(n) = \begin{cases} 0, & n = 0, \\ x(n-1), & n \in \mathbb{Z}_+,\ n \geq 1, \end{cases}$$

for $x \in \ell^2$. Here $\mathbb{Z}_+ = \{n \in \mathbb{Z}\colon n \geq 0\}$.

III. Let H be a complex Hilbert space and let T be a densely defined operator on H with domain $\mathcal{D}(T)$.

(a) Suppose S is a densely defined operator on H such that $S \subset T$. Prove, with a precise argumentation, that $S^* \supset T^*$.

(b) Suppose that, moreover, T is closed, and that S is another densely defined closed operator. Then the following are equivalent:

(i) $S \subset T$;

(ii) $S^* \supset T^*$.

Prove this equivalence; you may use the result in part (a) and results in the text of Rudin's book if needed.

(c) Suppose S is an operator on H with domain $\mathcal{D}(S)$. Then the following are equivalent:

(i) $(Tx, y) = (x, Sy)$ for all $x \in \mathcal{D}(T)$ and all $y \in \mathcal{D}(S)$;

(ii) $T^* \supset S$.

Prove this equivalence, with a precise argumentation.

See reverse side

IV. Exercise 3 of Chapter 13 in Rudin's book Functional Analysis: show that the suggestion in the exercise *does* provide an example of a densely defined operator on an infinite dimensional separable Hilbert space, such that its adjoint is the zero operator on the zero subspace.
Once you have the correct idea and also realize that $\lim_{n\to\infty}(x, e_n) = 0$ for all $x \in H$, the solution is almost immediate.

— Due: January 7, 2014 –
– One point subtraction deadline: January 14, 2014 –

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