Mastermath Functional Analysis

Homework 2

From Chapter 2:

1. A Hamel basis is, of course, simply a basis for a vector space in the purely algebraic sense. Since "basis" in functional analysis often refers, for short, to a so-called Schauder basis of a Banach space (which in infinite dimension is *not* a basis in the algebraic sense), such a terminology is necessary.

4. Make sure you prove all statements under (a), (b) and (c), and state clearly which results in the text of Rudin's book (that may not all have been mentioned explicitly during the lectures) imply that L^2 is of the first category in L^1. You need only cover the case of L^1 and L^2. Fatou's Lemma may be handy for (a).

Not in the book:

I. (Characterisation of weak*-convergent sequences in c_0^*.) Let $\{x_n\}_{n=1}^\infty \subset \ell_1$ be a sequence in ℓ_1, with $x_n = (x_n(1), x_n(2), x_n(3), \ldots)$.

   (a) Show that
   $$\lim_{n\to\infty} \sum_{j=1}^{\infty} x_n(j)y(j) = 0$$
   for all $y \in c_0$ if and only if $\sup_n \|x_n\|_1 < \infty$ and $\lim_{n\to\infty} x_n(j) = 0$ for all $j = 1, 2, \ldots$. You can use that c_0^* can be identified isometrically with ℓ_1 via de canonical pairing between c_0 and ℓ_1.

   (b) If $x \in \ell_1$ and $\{x_n\}_{n=1}^\infty \subset \ell_1$, use part (a) to formulate a necessary and sufficient condition for $x = \mathrm{w}^* - \lim_{n\to\infty} x_n$ to hold, where the weak*-convergence is that in $c_0^* \simeq \ell_1$.

II. Let X be a Banach space. Show that there exists a compact Hausdorff space K such that X is isometrically isomorphic to a closed subspace of $(C(K), \|\,.\,\|_\infty)$. (Hint: there exists a canonical embedding of X into X^{**}. If you have not seen this already in an introductory course, have a look at Section 4.5 in Rudin's book).

III. (Generalisation of Chapter 3 Exercise 17). Let K be a compact Hausdorff space. For both real valued and complex valued functions, the extreme points of the unit ball $\{f \in C(K) : \|f\|_\infty \leq 1\}$ in $(C(K), \|\,.\,\|_\infty)$ are precisely the unimodular continuous functions, i.e., those $f \in C(K)$ such that $|f(k)| = 1$ for all $k \in K$. Prove this.

— Due: October 22, 2013 —

Please note that there is no lecture on October 22, so you may need to send your work in time by ordinary mail or email!!

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If handed in by email, send the work as 1 (!) PDF, and make sure your name, university and student number are in the file.

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