Mastermath Functional Analysis

Homework 4

I. Let A be a Banach algebra with a unit e.

(a) If $a \in A$ is nilpotent (i.e. there exists $n \in \mathbb{N}$ such that $a^n = 0$), then the spectral radius of a satisfies $\rho(a) = 0$. Verify this.

(b) Show by a counterexample that the converse of the statement in (a) is not true. A good candidate for a not nilpotent a with $\rho(a) = 0$ is the right shift operator on the space of bounded complex functions on $\mathbb{N}$ endowed with a suitably weighted variant of the supremum norm.

II. Let A be a Banach algebra with unit e and let $b, c \in A$ be such that

$$\sigma(b^2 - c) \subseteq \{\alpha \in \mathbb{R} \colon \ \alpha > 0\}.$$

Show that there exists $x \in A$ such that

$$x^2 + bx + xb + c = 0.$$

Does there exist such an x with $\sigma(x + b) \subseteq \{\alpha \in \mathbb{R} \colon \ \alpha > 0\}$?

III. Let A be a Banach algebra with unit e and let $a \in A$.

(a) Show that

$$\exp((\lambda + \mu)a) = \exp(\lambda a) \exp(\mu a) \quad \text{for every } \lambda, \mu \in \mathbb{C}.$$

(b) Show that $\lambda \mapsto \exp(\lambda a)$ is a continuous map from $\mathbb{C}$ into A.

(c) Suppose that $\sigma(a) \subseteq \{\lambda \in \mathbb{C} \colon \ \operatorname{Re} \lambda < 0\}$. Show that there exist $M, \omega \in \mathbb{R}$ with $\omega > 0$ such that

$$\| \exp(ta) \| \leq M e^{-\omega t} \quad \text{for every } t \in [0, \infty).$$

IV. Let X be a compact Hausdorff space and let $C(X)$ be the commutative Banach algebra of all complex-valued continuous functions on X endowed with the maximum-norm. Let J be an ideal of $C(X)$. For a closed subset $Y \subseteq X$ define

$$J_Y := \{f \in C(X) \colon\ f(y) = 0 \text{ for all } y \in Y\}.$$

(a) If $Y \subseteq X$ is closed, then J_Y is a closed ideal of $C(X)$.

(b) If $g \in J$ and $0 < \varepsilon < 1$, then there exists $u \in J$ with $u(x) = 1$ for all $x \in X$ with $|g(x)| \geq 1$, $u(x) = 0$ for all $x \in X$ with $|g(x)| \leq \varepsilon$, and $0 \leq u(x) \leq 1$ for all $x \in X$.

(c) If J is a closed ideal of $C(X)$, then there exists a closed $Y \subseteq X$ such that $J = J_Y$.

Prove that every closed ideal J of $C(X)$ is of the form J_Y for some closed $Y \subset X$, by proving the statements (a), (b) and (c).

When dealing with spaces of continuous functions, you may want to have Urysohn's lemma on your mind: if X is a compact Hausdorff space and A and B are two disjoint closed subsets of X then there exists a continuous function $u \colon X \to [0, 1]$ such that $u(x) = 1$ for all $x \in A$ and $u(x) = 0$ for all $x \in B$.

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If you hand it in by email, send the work as *one* PDF and make sure your name, university and student numer are in the file.

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