Mastermath Functional Analysis

Homework 1

I. Let X be an infinite dimensional normed space.

   (a) Show that there exists a linear functional on X that is not continuous.

   (b) Show that, for each integer $n \geq 1$, X has a dense subspace of codimension n, i.e., a dense subspace L such that $\dim X/L = n$.

II. Let X be a vector space and $f : X \to \mathbb{F}$ a linear functional. Suppose $\mathcal{P}$ is a separating family of seminorms on X, and introduce the locally convex topology τ on X as in Theorem 1.37. Prove that the following are equivalent:

   (a) f is continuous when X has the τ-topology;

   (b) There exist $n \geq 1$, $p_i \in \mathcal{P}$, and $c_i > 0$ ($i = 1, \ldots, n$), such that $|f(x)| \leq \sum_{i=1}^{n} c_i p_i(x)$, for all $x \in X$.

III Let $\Omega \subset \mathbb{R}^n$ be open and non-empty, and introduce the Fréchet topology on $C^\infty(\Omega)$ as in Example 1.46.

   (a) Let α be a fixed multi-index. Show that the differentiation operator $D^\alpha : C^\infty(\Omega) \to C^\infty(\Omega)$, sending $\phi \in C^\infty(\Omega)$ to $D^\alpha \phi$, is continuous.

   (b) Let $f \in C^\infty(\Omega)$ be fixed. Show that the multiplication operator $M_f : C^\infty(\Omega) \to C^\infty(\Omega)$, sending $\phi \in C^\infty(\Omega)$ to $f\phi$, is continuous.

Furthermore, from Chapter 1:

2 (convex hull)

3: b, d, e, f (various properties of operations on sets)

10 (surjective linear maps onto a finite dimensional space)

— *Due: October 8, 2013* —

Teaching assistant: Björn de Rijk, `brijk@math.leidenuniv.nl`
If handed in by email, send the work as 1 (!) PDF, and make sure your name, university and student number are in the file.

Leiden University
Marcel de Jeu
Onno van Gaans