EXACT COVERS OF BALANCED SEQUENCES AND FRAENKEL'S CONJECTURE.

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Abstract. A striking conjecture of Fraenkel asserts that the only decomposition of $\mathbf{Z}$ into $m \geq 3$ sequences $\{\lfloor \alpha_i n + \beta_i \rfloor\}_{n \in \mathbf{Z}}$ with α_i and β_i real, $\alpha_i > 1$ and α_i's distinct for $i = 1, ..., m$ is given by

$$\{\alpha_1, ..., \alpha_m\} = \{\frac{2^m - 1}{2^k} : 0 \leq k < m\}.$$

A weaker conjecture states that then α_i/α_j is an integer for some $i \neq j$. The weaker statement has been proved by Simpson if some α_i is at most 2. Fraenkel's conjecture was proved by Simpson if some α_i is at most $3/2$ and by Morikawa if $m = 3$ and, under some condition, if $m = 4$. An unconditional proof for $m = 4$ was given by Altman, Gaujal and Hordijk and for $m = 5, 6$ by the author. Proofs for the more general case of balanced sequences have been given for $m = 3$ by the author and for $m = 4$ by Altman, Gaujal and Hordijk. In the present paper we extend the other results mentioned above to balanced sequences. We start with a survey of the related literature and give self-contained proofs.

Keywords: complementary sequences, eventual covering family,
Fraenkel's conjecture, Beatty sequences, balanced words.

1 Exact congruence covers

For integers a, b with $a > 1$ denote by $S(a, b)$ the arithmetic sequence $\{an + b\}_{n \in \mathbf{Z}}$. A finite set $\{S(a_i, b_i) | 1 \leq i \leq m\}$ is called an *exact congruence cover* if every integer occurs in exactly one $S(a_i, b_i)$. Obviously $\sum_{i=1}^{m} a_i^{-1} = 1$ for such a cover. Fraenkel [F3],[F4], Znám [Zn5] and Beebee [Bee] have given criteria which characterise exact congruence covers. Some criteria involve exponential functions, others Bernoulli polynomials or Euler polynomials.

Let $\{S(a_i, b_i) | 1 \leq i \leq m\}$ be an exact congruence cover with $a_1 \leq a_2 \leq ... \leq a_m$. Mirsky, Newman, Davenport and Radoindependently showed that $a_{m-1} = a_m$ (see Erdös [E], Fraenkel [F3], Lemma 1). Thus there is no exact congruence cover with $a_1 < a_2 < ... < a_m$. Znám [Zn1] showed that $a_{m-2} = a_{m-1} = a_m$ if a_m is odd and that $a_{m-4} = a_{m-3} = ... = a_m$ if a_m is odd and not divisible by 3. Stein [Ste] proved a 'uniqueness theorem' on the way a set of integers can be decomposed as a union of arithmetic progressions with distinct moduli (cf. Znám [Zn4]). His theorem implies the above mentioned result on $a_{m-1} = a_m$. Moreover, it follows that if $a_1 < a_2 < ... < a_{m-1} = a_m$, then $a_i = 2^i$ $(1 \leq i \leq m-1)$ and $a_m = 2^{m-1}$. Let l be such that $a_1 < a_2 < ... < a_{m-l+1} = ... = a_m$. Znám [Zn1] proved that $l = 3$ implies $a_i = 2^i$ $(1 \leq i \leq m-3)$, $a_{m-2} = a_{m-1} = a_m = 3 \times 2^{m-3}$. Porubsky [P1],[P3] characterised the cases $l = 4$ and $l = 5$. Subsequently Berger, Felzenbaum and Fraenkel [BeFF3] characterised the cases with $l \leq 9$. Znám [Zn2] and Newman [Ne] independently proved that l is at least the smallest prime dividing a_m. This result was refined by Berger, Felzenbaum and Fraenkel [BeFF2]. In particular, if a_m is a prime, then the exact congruence cover is the set $\{S(i, a_m) | 0 \leq i < a_m\}$. A conjecture of Znám [Zn1] that if there exist only pairs of equal moduli a_i (no three being equal) then all moduli are composed of primes 2 and 3 was disproved by Burshtein and Schönheim [BuS]. Short proofs of some basic properties mentioned above can be found in [NZ].

Znám [Zn1],[Zn3], Porubský [P2],[P3], Burshtein [Bu] and others generalised the theory to vector covering systems, exact m-covers (sets of arithmetic sequences which cover each integer exactly m times) and related systems. Excellent surveys have been given by Porubský [P4] and Znám [Zn6]. For later developments into this direction which involve extensions of results mentioned above, see [P5], [P6] and [ChP].

Exact congruence covers have been investigated in the theory of optimal routing in queueing networks. See [AGH] where congruences are called 'constant gap series'.

2 Exact covers of Beatty sequences

For real numbers $\alpha \geq 1$ and β denote by $S(\alpha, \beta)$ the sequence $\{\lfloor n\alpha + \beta \rfloor \mid n = 1, 2, 3, ...\}$. A finite set $\{S(\alpha_i, \beta_i) | 1 \leq i \leq m\}$ is called an *(eventual) exact cover* if every (sufficiently large) positive integer occurs in exactly one $S(\alpha_i, \beta_i)$. If $\{S(\alpha_i, \beta_i)\}_{i=1}^m$ is an eventual exact cover, then $\sum_{i=1}^m \alpha_i^{-1} = 1$. Obviously the only case with $m = 1$ is when $\alpha_1 = 1$. We call α the *rate* of $S(\alpha, \beta)$ and α^{-1} the *density* and β the *rest*.

The astronomer Bernoulli [Ben] studied sequences $S(\alpha, \frac{1}{2})$ already in 1772. In 1875 Christoffel [Chr] studied sequences $S(\alpha, 0)$ and such sequences are known as *Christoffel sequences*. The sequences $S(\alpha, \beta)$ are known as *Beatty sequences* or *cutting sequences* (cf. Stolarsky [Sto], Boshernitzan and Fraenkel [BoF1], [BoF2], and Series [Se]). If α is irrational, they are often called *Sturmian sequences* (cf. Berstel [Bes]). Such sequences have close links with continued fractions. They play a significant role in the theories of dynamical systems and ergodic theory. They occur further in quasicrystallography, formal language theory, computer vision, network scheduling and molecular biology.

In 1926 Beatty [Bea] studied the case $\beta = 0$ and published the following result as a problem: $S(\alpha_1, 0)$ and $S(\alpha_2, 0)$ form an exact cover if and only if α_1 is irrational and $\alpha_1^{-1} + \alpha_2^{-1} = 1$. An independent proof of Beatty's result was given by Skolem [Sk1] and Bang [Ba]. Another proof and further references to papers on Beatty's result can be found in Fraenkel, Levitt and Shimshoni [FLS]. A full treatment of the case $m = 2$ including rational α_i's was given by Niven [Ni, pp. 34-45]. In 1927 Uspensky [U] proved by an application of Kronecker's approximation theorem that there is no eventual exact cover of $m \geq 3$ sequences $S(\alpha_i, 0)$ $(1 \leq i \leq m)$. A simpler proof of this fact was given by Graham [Gr1].

The work of Skolem also implied a criterion for exact covers of two Beatty sequences. Skolem [Sk2] proved that if α_1 is irrational, then $\{S(\alpha_i, \beta_i)\}_{i=1}^2$ is an eventual exact cover if and only if $\frac{1}{\alpha_1} + \frac{1}{\alpha_2} = 1$ and $\frac{\beta_1}{\alpha_1} + \frac{\beta_2}{\alpha_2} \in \mathbf{Z}$. Fraenkel [F1] proved the same result in case α_1 is rational. He further gave similar, but more complicated, necessary and sufficient conditions for $\{S(\alpha_i, \beta_i)\}_{i=1}^2$ to be an exact cover.

For $m = 3$ there exists an exact cover of Beatty sequences with distinct rates, namely $\{S(\frac{7}{4}, 0), S(\frac{7}{2}, -1), S(7, -3)\}$. Fraenkel [F2] noticed that there is an example for every m, namely

$$(1) \qquad \left\{S\left(\frac{2^m - 1}{2^{i-1}}, 1 - 2^{m-i}\right)\right\}_{i=1}^m.$$

He further made the following conjecture [F2, p.10].

Fraenkel's integer conjecture. *If $\{S(\alpha_i, \beta_i)\}_{i=1}^m$ is an eventual exact cover with distinct rates $\alpha_1, ..., \alpha_m$ and $m \geq 3$, then there are two different rates with integral ratio.*

He proved the conjecture for $m = 3$; for $m = 4$; if $m = 5$ and all α_i's but one are integers; if $m \geq 5$, two of the α_i's are integers, at least one of them prime; and if $m \geq 5$ and $\alpha_i \leq 2^i$ for $i = 1, 2, ..., m - 4$. Erdös and Graham [EG, p.19] attributed a much more striking conjecture to Fraenkel:

Fraenkel's conjecture. *If $\{S(\alpha_i, \beta_i)\}_{i=1}^m$ is an eventual exact cover with distinct rates $\alpha_1, ..., \alpha_m$ and $m \geq 3$, then*

$$\{\alpha_1, ..., \alpha_m\} = \{\frac{2^m - 1}{2^k} : 0 \leq k < m\}.$$

.

Graham [Gr2] used Kronecker's approximation theorem and Skolem's work on Beatty sequences to show that if all the rates of an eventual exact cover of $m \geq 3$ sequences $S(\alpha_i, \beta_i)$ are distinct, then all the rates α_i are rational. Actually Graham showed that if a finite union of Beatty sequences with at least one irrational rate forms an eventual exact cover, then the union can be split into two sets which both correspond to an eventual exact cover. If α is rational, then the differences of consecutive elements of $S(\alpha, \beta)$ are periodic. It follows from Graham's result and this observation that no distinction needs to be made between eventual exact covers, exact covers and the corresponding notion of exact covers of $\mathbf{Z}$, when studying exact covers of Beatty sequences with distinct rates.

If the rate of a sequence $S(\alpha, \beta)$ is rational, we may assume without loss of generality that β is a rational number with the same denominator as α. Suppose $\{S(\frac{a_i}{c_i}, \frac{b_i}{c_i})\}_{i=1}^m$ is an exact cover with rates $\frac{a_i}{c_i}$ where $(a_i, c_i) = 1$ for $i = 1, ..., m$ and $m \geq 3$. Berger, Felzenbaum and Fraenkel [BeFF1] proved that if $a_1 \leq a_2 \leq ... \leq a_m$, then $a_{m-1} = a_m$ and moreover if $b_{m-1} \neq b_m$, then $a_{m-2} = a_{m-1} = a_m$. It follows from this result that Fraenkel's integer conjecture is true if at most two of the rates are nonintegral.

A thorough study of exact covers of Beatty sequences was made by Morikawa [M*] in a series of five papers. In [M1] he gave two methods to construct exact covers. He further classified all exact covers with $m = 3$. The only ones with distinct rates are the shifts of (1). In [M2] he proved a surprising result on the rests, earlier conjectured by himself, namely that if $\{S(\frac{a_i}{c}, \frac{b_i}{c})\}_{i=1}^m$ is an exact cover with $(a_i, c) = 1$ for all i, then

$$b_1 + b_2 + ... + b_m \equiv -\frac{m}{2} (\text{mod } c).$$

In [M3] he studied exact covers which are composed of sequences $S(\frac{a}{c}, \frac{b}{c})$ where $\frac{a}{c}$ can assume only two values. In [M4] he classified all exact covers of the form $S(\frac{a_i}{c}, \frac{b_i}{c})\}_{i=1}^4$ with $(a_i, c) = 1$ for $i = 1, 2, 3, 4,$. Finally in [M5] he investigated the sets of residues of exact covers for general m. (This is an incomplete description of the results in the five papers.)

In 1991 Simpson [Si] obtained some remarkable results. Through an analysis of the arithmetic properties of residues he proved Fraenkel's integer conjecture if some α_i is at most 2. He further showed that Fraenkel's conjecture is true if $\min_i \alpha_i \leq \frac{3}{2}$.

Quite recently Altman, Gaujal and Hordijk [AGH] established Fraenkel's conjecture in case $m = 4$. Since they do so in the more general framework of balanced words, I shall deal with their paper in the next section.

3 Exact covers of balanced sequences

Let Q be a finite set, a so-called *alphabet*. We consider functions $W : \mathbf{Z} \rightarrow \mathbf{Q}$, so-called *bi-infinite words* or **Z**-*words* or just *words*. A *block* is a finite set of consecutive integers. A block of N integers is called a block of *length* N or an N-*block*. A sequence S of integers is called a *balanced sequence* if the numbers of terms of S in any two blocks of equal lengths differ by at most 1. The word $W = \{W(s)\}_{s=-\infty}^{\infty}$ is called a *balanced word* if for every $q \in Q$ the sequence $\{s \in \mathbf{Z} \mid \mathbf{W}(\mathbf{s}) = \mathbf{q}\}$ is balanced. Thus W is balanced if the inverse image of each letter in W is balanced. It is obvious that a $\mathbf{Z}$-word on Q represents an exact cover of $\mathbf{Z}$.

The concept of balanced sequences occurs in Morse and Hedlund [MoH1], [MoH2] in the context of topological dynamics. For the context of the use of balanced words in formal language theory and optimal routing in queueing networks, see [L] and [AGH], respectively. Characterisations of balanced sequences can be found in [MoH2] as well as in [GH], [CoH], [DG] and [T3]. In fact, S is balanced if and only if one of the following conditions holds:

(a) (periodic case) there exist rational numbers α, β with $\alpha > 1$ such that $S = \{\lfloor n\alpha + \beta \rfloor\}_{n \in \mathbf{Z}}$.
(b) (irrational case) there exist real numbers α, β with $\alpha \notin \mathbf{Q}$ and $\alpha > 1$ such that $S = \{\lfloor n\alpha + \beta \rfloor\}_{n \in \mathbf{Z}}$ or $S = \{\lceil n\alpha + \beta \rceil\}_{n \in \mathbf{Z}}$.
(c) (skew case) there exist integers a, b, c with $\gcd(a, c) = 1$ and $a > c > 0$ such that

$$S = \left\{\left\lfloor \frac{na}{c} + b \right\rfloor\right\}_{n<c} \cup \left\{\left\lfloor \frac{na-1}{c} + b \right\rfloor\right\}_{n>0}$$

or

$$S = \left\{\left\lfloor \frac{na-1}{c} + b \right\rfloor\right\}_{n<c} \cup \left\{\left\lfloor \frac{na}{c} + b \right\rfloor\right\}_{n>0}$$

(d) (trivial case) S is either $\emptyset$ or a single integer $\{b\}$ or $\mathbf{Z}\backslash\{\mathbf{b}\}$ or $\mathbf{Z}$.

Every balanced sequence has a density, namely α^{-1} in cases (a) and (b), c/a in case (c), 0 in case $S = \emptyset$ and $\{b\}$, and 1 otherwise. The *rates* are defined as the inverses of the densities where $1/0 = \infty$. The number of terms of S in the image of an N-block is equal to $\lfloor N\alpha \rfloor$ or $\lceil N\alpha \rceil$ if α is the density of S and $N\alpha \notin \mathbf{Z}$. If $N\alpha \in \mathbf{Z}$, it is $N\alpha$ in the periodic case and if $S = \emptyset$ or $\mathbf{Z}$. Otherwise it is either in the set $\{N\alpha, N\alpha + 1\}$ or in the set $\{N\alpha - 1, N\alpha\}$. The proofs of these results of Morse and Hedlund [MH2, p.17] are given in Lemma 2.

Since every Beatty sequence coincides with some balanced sequence, the theory of exact covers of balanced sequences is more general than the theory of exact covers of Beatty sequences. A disadvantage of the generalisation may be that there is not an easy equivalent for balanced sequences of the 'rest' of a Beatty sequence. On the other hand, it seems that results on exact covers of balanced sequences are as general as the corresponding results for exact covers of Beatty sequences and that the proofs are sometimes simpler. For example, it is non-trivial that the complement of a Beatty sequence is a Beatty sequence. Almost by definition the complement of a balanced sequence is a balanced sequence. So every balanced sequence defines a balanced word on two letters (take its characteristic function), and every balanced sequence is characterised by a balanced word (the inverse image of one letter).

Altman, Gaujal and Hordijk ([AGH, Lemma 2.26]) observed that every balanced word on $m \geq 3$ letters with distinct rates is periodic. This is an important restriction of the possibilities. The proof of this fact is based on the decomposition property introduced by Graham to exclude irrational rates in case of exact covers of Beatty sequences and derived by Hubert [H] for balanced sequences.

The balanced words on three letters were characterised by the author [T1]: a balanced word on the letters a, b, c originates from any balanced sequence $S = \{s_i\}_{i \in \mathbf{Z}}$ with $s_i < s_{i+1}$ for all i by putting letters a on places s_{2i}, letters b on places s_{2i+1} and letters c on the remaining places, or corresponds with the periodic word with period cycle $(abacaba)$. (Of course, the choice a, b, c is arbitrary.) Note that in the former case the letters a and b have the same rates and that the exceptional case is in agreement with Fraenkel's conjecture.

Altman, Gaujal and Hordijk [AGH] gave a simplified proof of the latter result and generalised it to balanced words on four letters. To be precise, they proved that the only balanced word on four letters with distinct rates is the periodic word with cycle $(abacabadabacaba)$ and rates $\frac{15}{8}, \frac{15}{4}, \frac{15}{2}, 15$. This established Fraenkel's conjecture in case $m = 4$, without the restriction imposed by Morikawa. (On the other hand, Morikawa's paper [Mo4] also deals with exact covers of four Beatty sequences where the rates are not distinct.)

In the present paper we show that the method can be extended to cover balanced words on five letters:

Theorem 1. *Every balanced word on five letters with distinct rates is isomorphic to the periodic word with period cycle* $(abacabadabacabaeabacabadabacaba)$ *and has* *rates* $\frac{31}{16}, \frac{31}{8}, \frac{31}{4}, \frac{31}{2}, 31$.

Furthermore, we prove an extension of an observation of Simpson.

Theorem 2. *If a balanced word on at least three letters has distinct rates, then all rates are greater than* $3/2$.

Theorems 1 and 2 have the following consequence.

Corollary. *Fraenkel's conjecture is true if* $m \leq 5$ *and also if one of the rates is at most* $\frac{3}{2}$.

In [T4] I shall showed that Fraenkel's conjecture is true for $m = 6$. It is easy to combine the arguments given there with the arguments in the present paper in order to derive a proof for Fraenkel's conjecture in case $m = 6$ for balanced words. I doubt whether the presented methods will suffice to prove the general Fraenkel's conjecture.

We shall further prove a refinement of another result of Simpson. In fact Proposition 1 and its proof provide more information than Theorem 3.

Theorem 3. *If a balanced word on at least three letters has some letter with rate at most 2, then this rate divides the rate of another letter.*

The proofs of Theorems 1-3 are self-contained. We prove in Section 4 that there does not exist an exact congruence cover with distinct moduli. Section 5 contains some fundamental properties of balanced words. Lemma 2 is the result due to Morse and Hedlund on the distribution of letters in a balanced word. Lemmas 3 and 5 give information on the relative behaviour of two letters in a balanced word. Lemma 6 states that if a word is balanced on letters with distinct rates and the letter a has rate ≤ 3, then the word remains balanced if all a's are omitted. This is an extension of a result of Altman, Gaujal and Hordijk [AGH, Lemma 2.27] who proved the result when a has rate ≤ 2. Lemma 7, already contained in [T2], states that if a word on $m \geq 3$ letters with distinct rates is balanced, then the integer parts of the rates are distinct. Theorem 2 is proved in Section 6, Theorem 3 in Section 7 and Theorem 1 in Section 8.

4 A basic result for exact congruence covers

The following lemma shows that there are no exact congruence covers with distinct moduli.

Lemma 1. (Mirsky, Newman, Davenport, Rado, cf. [E].)
Let $\{S(a_i, b_i) | 1 \leq i \leq m\}$ *be an exact congruence cover with* $a_1 \leq a_2 \leq ... \leq a_m$*. Then* $a_{m-1} = a_m$*.*

Proof. Let $S(a_i, b_i) = \{a_i n + b_i\}_{n \in \mathbf{Z}}$ with $0 \leq b_i < a_i$ for $i = 1, ..., m$. Consider the generating function $f_i(z) = \sum_{n=0}^{\infty} z^{a_i n + b_i}$. Then

$$f_i(z) = \frac{z^{b_i}}{1 - z^{a_i}}.$$

Since we have an exact cover,

$$\sum_{i=1}^{m} \frac{z^{b_i}}{1 - z^{a_i}} = \sum_{i=1}^{m} f_i(z) = \sum_{n=0}^{\infty} z^n = \frac{1}{1 - z}.$$

The rational function on the right side has only a simple pole at $z = 1$. Hence the rational function on the left side has no poles at the a_m-th primitive roots of 1. Thus $a_{m-1} = a_m$. □

5 Some basic results for balanced words

In this section we derive some fundamental properties of balanced sequences and balanced words which are used in the proofs of the theorems. We denote the number of letters of the finite word V and the number of times that the letter x occurs in V by $|V|$ and $c_x(V)$, respectively. We further denote the rate of the letter a (in a given word W) by r_a. Observe that r_a is the average distance between pairs of consecutive a's, where we define the distance between consecutive letters to be 1.

Lemma 2. (cf.[MH2, p.17], [T3, p.116].) *Let $S = \{s_i\}_{i \in \mathbf{Z}}$ be an infinite balanced sequence of rate r. Let k be a positive integer. If $kr \notin \mathbf{Z}$, then $s_{i+k} - s_i$ equals $\lfloor kr \rfloor$ or $\lceil kr \rceil$ for all i. If $kr \in \mathbf{Z}$, then either $s_{i+k} - s_i \in \{kr, kr+1\}$ for all i or $s_{i+k} - s_i \in \{kr-1, kr\}$ for all i and it can occur for only one i that $s_{i+1} - s_i - r$ is non-zero.*

Proof. Suppose $s_{i+k} - s_i = v$ and $s_{j+k} - s_j = w$ and $v - w \geq 2$. Then the blocks $[s_i + 1, s_{i+k} - 1]$ and $[s_j, s_j + s_{i+k} - s_i - 2]$ have equal lengths, but the former block contains at most $k-1$ elements of S and the latter block at least $k+1$ such elements because of $s_j + s_{i+k} - s_i - 2 = s_j + v - 2 \geq s_j + w = s_{j+k}$. This contradiction shows that there is some u with $s_{i+k} - s_i \in \{u, u+1\}$ for every i. Since r is the average distance between consecutive elements of S, both a value $\geq kr$ and a value $\leq kr$ should be attained. Thus $u \leq kr \leq u+1$. Only the last assertion of Lemma 2 remains to be proved. Suppose it is false. Then there is an integer $h > 1$ such that $|s_{i+1} - s_i - r| \geq 1$ and $|s_{i+h} - s_{i+(h-1)} - r| \geq 1$. By applying the already found result to the intermediate differences $s_{i+2} - s_{i+1} - r, ..., s_{i+h-1} - s_{i+h-2} - r$, we obtain $|s_{i+h} - s_i - hr| \geq 2$, a contradiction. □

Lemma 2 implies that for every k we have $\lfloor kr \rfloor \leq s_{i+k} - s_i \leq \lceil kr \rceil$ with only finitely many exceptions i.

Lemma 3. *If W is a balanced word on letters $\{a, b, ...\}$, then there is a w such that the number of a's between consecutive b's is w or $w+1$.*

Proof. Suppose between one pair of consecutive b's there are k letters a and between another pair more than $k+1$. By Lemma 2 there is some u such that the distance between two consecutive b's is u or $u+1$. If $b...b$ contains k letters a and no further b's, then in W the distance between the a to the left of $b...b$ and the a to the right of it is at least $u+2$, whence $(k+1)r_a > u+1$ by Lemma 2. If the block $b...b$ contains more than $k+1$ letters a and no further b's, then the distance between the first a and the last a in $b...b$ is at most $u-1$, whence $(k+1)r_a < u$ again by Lemma 2. We have found a contradiction. □

Lemma 4. *If W is a balanced word and V_1 and V_2 are finite subwords with $c_x(V_1) < c_x(V_2)$, then $|V_1| < |V_2| + r_x$.*

Proof. Suppose $|V_1| > |V_2|$ and $c_x(V_1) < c_x(V_2)$. Since W is balanced, $k := c_x(V_1) = c_x(V_2) - 1$. Let $\{s_k\}$ be the balanced sequence representing the places where x occurs in W. Let $s_{i+1}, ..., s_{i+k}$ be the locations of x in V_1 and $s_j, s_{j+1}, ..., s_{j+k}$ in V_2. Then $s_{i+k+1} - s_i \geq |V_1| + 1$ and $s_{j+k} - s_j \leq |V_2| - 1$. It follows from Lemma 2 that $s_{i+k+1} - s_i \leq (k+1)r_x + 1$ and $s_{j+k} - s_j \geq kr_x - 1$, where we cannot have equalities in both cases. On combining the inequalities we obtain $|V_1| < |V_2| + r_x$. $\square$

Lemma 5. *If W is a balanced word on letters $\{a, b, ...\}$ and $r_a \leq 2$, then the number of letters a between two b's in W is determined by the total number of letters in between the b's.*

Proof. Suppose there is some pair of b's in W with k letters in between among which l letters a and there is some other pair of b's with k letters in between among which more than l letters a. Let V_1 be the subword of all letters between the former two b's including the b's and V_2 the subword of all letters between the latter two b's excluding the two b's. The, by Lemma 4, $|V_1| < |V_2| + 2$, whence $k < (k-2) + 2$. $\square$

Remark 1. In Lemma 5 the bound 2 is the best possible.

Lemma 6. *Let W be a balanced word on a finite alphabet $\{a, b, ...\}$. Suppose the rate of a in W is at most 3. Then the word W' obtained from W by omitting all a's is balanced too.*

The corresponding result if a has rate at most 2 is due to Altman, Gaujal and Hordijk [AGH, Lemma 2.27]. The statement is obvious if $m \leq 2$.

Proof. Choose two subwords V_1', V_2' of length n in W'. Let V_1 and V_2 be subwords of W such that their first and last letters are not a's and that the words reduce to V_1', V_2', respectively, if the a's are omitted. Let $k := |V_1| - n$ and $l := |V_2| - n$ denote the number of appearances of the letter a in V_1 and V_2, respectively. We may assume $l \geq k$.

Case 1: $l = k$. Since the number of occurrences of any letter in V_1 and V_2 differs by at most 1, the same is true for V_1' and V_2'. Hence W' is balanced with respect to V_1' and V_2'.

Case 2: $l > k + 1$. Note that V_1 has $n + k$ letters among which exactly k letters a. Let $\hat{V}_2$ be the word obtained from V_2 by truncating the first letter and the last $l - k - 1$ letters. Then $\hat{V}_2$ has also $n + k$ letters, but the number of a's in $\hat{V}_2$ is at least $l - (l - k - 2) = k + 2$. The comparison of the subwords V_1 and $\hat{V}_2$ of W reveals that W is not balanced. So Case 2 cannot occur.

Case 3: $l = k+1$. Suppose W' is not balanced with respect to V_1' and V_2', that means there is an x such that $|c_x(V_1') - c_x(V_2')| > 1$. Then $x \neq a$ and $|c_x(V_1) - c_x(V_2)| > 1$. Since V_1 and V_2 are subwords of the balanced word W with $|V_2| = |V_1| + 1$, we obtain that $c_x(V_2) = c_x(V_1) + 2$ and that $V_2 = xVx$ for some subword V of W. Since W is balanced, we further deduce that it contains xV_1x as a subword. Hence between the last a preceding xV_1x and the first a succeeding it there are at least $|V_1| + 2 = n + k + 2$ letters. On the other hand, between the first and the last a in V_2 there are at most $|V| - 2 = n + k - 3$ letters. The fact that xV_1x contains k letters a and V contains $k + 1$ letters a, implies by Lemma 4 that $|V_1| + 2 < |V| + r_x$. Hence $r_x > 3$. □

Remark 2. If in Lemma 6 the rates of the letters in W are distinct, then the rates of the letters in W' are distinct and we have in fact for the rate r_b' of $b \neq a$ in W' that $r_b' = r_b(1 - r_a^{-1})$. This follows from the observations that the ratios of the rates of letters $\neq a$ are the same in W and W' and that the sum over all letters of the inverses of the rates equals 1.

Remark 3. The bound 3 in Lemma 6 cannot be peplaced by 4. For example, the word with period cycle (*bacdabecadf*) is balanced with $r_a = \frac{11}{3}$, but the word with period cycle (*bcdbecdf*) is not balanced.

Lemma 7. *Let W be a balanced word on a finite alphabet $\{a, b, ...\}$. Let k and l be integers such that $\lfloor kr_a \rfloor = \lfloor lr_b \rfloor$. Then $kr_a = lr_b$.*

Proof. Suppose $kr_a < lr_b$. Then the situation occurs infinitely often that there are two a's with exactly $k - 1$ letters a and fewer than l letters b in between. By Lemma 2 it occurs infinitely often that there are at least $\lfloor kr_a \rfloor - 1$ letters between the two a's, whence there are at least $\lfloor kr_a \rfloor + 1$ letters between the last b preceding the first a and the first b succeeding the last a. Thus $\lceil lr_b \rceil \geq \lfloor kr_a \rfloor + 2 = \lfloor lr_b \rfloor + 2$ which is a contradiction. □

Corollary. *If $\lfloor r_a \rfloor = \lfloor r_b \rfloor$, then $r_a = r_b$. If the rates of a balanced word are distinct, then each letter is characterised by the integer part of its rate.*

Remark 4. In the proof of Theorem 1 we shall often denote a letter by the integer part of its rate. By the Corollary this characterises the letter uniquely.

6 Fraenkel's conjecture with one rate at most $3/2$

Proof of Theorem 2. Let W be a balanced word on at least three letters $\{a, ...\}$ with distinct rates and $r_a \leq 3/2$. Denote by W' the word obtained from W by omitting all a's. By Lemma 6 we know that W' is a balanced word. Obviously the rates of its letters are distinct. By Lemma 1 we know that there is a letter b whose index set with respect to W' is not a congruence class. This means that there exist two finite subwords V_1' and V_2' of W' of equal lengths n such that V_1' starts and ends with b and that V_2' starts with b, but ends with a letter $\neq a, b$. Moreover, we may assume that V_1' and V_2' contain no other b's. Let V_1 and V_2 be subwords of W both not beginning or ending with a so that V_1' and V_2' are obtained from V_1 and V_2, respectively, by omitting all a's. Hence V_1 starts and ends with b and V_2 starts with b, but ends with a letter $\neq a, b$. Since $2r_a \leq 3$ in W, the last letter of V_2 is followed in W by at least two a's. Since W is balanced with respect to b and V_1 contains two b's and V_2aa ends with $|V_2| + 1$ letters $\neq b$, we obtain $|V_1| > |V_2| + 1$. Compare V_2 with the word V_3 defined as the subword of W consisting of the first $|V_2|$ letters of V_1 after its first letter. The number of non-a's in V_2 is equal to the number of non-a's in V_1, hence at least two larger than the number of non-a's in V_3. Therefore the number of a's in V_2 is at least two smaller than the number of a's in V_3. Since V_2 and V_3 are subwords of W and have equal length, this contradicts the balancedness of W. $\square$

7 Fraenkel's integer conjecture with one rate at most 2

We shall prove the following refinement of Theorem 3.

Proposition 1. *Let W be a balanced word on $m \geq 3$ letters with distinct rates such that letter a has rate $r_a \leq 2$. Then*
(i) *there exists a letter $b \neq a$ with $r_a | r_b$.*
(ii) *the rate of each letter is either divisible by r_a or by the rate $(1 - r_a^{-1})^{-1}$ of non-a's.*

Proof of Proposition 1.
(ii) Let b be some letter $\neq a$. By Lemma 2 there is a v such that the distance in W between consecutive b's is v or $v+1$. If the distance between consecutive b's is v, then, by Lemma 5, the number of a's in between is constant, w say. If the distance is $v+1$, then the number of a's is also constant, w' say. By a reasoning similar to the proof of Lemma 5, we have $w \leq w'$. By Lemma 3 we have $w' \leq w+1$. If $w = w'$, then the number of a's between consecutive b's is constantly w whence $r_b = wr_a$. Otherwise $w' = w+1$. By Lemma 6 we can construct a word W' with distinct rates by omitting all a's in W. The distance between consecutive b's in W' is constantly $w - v = w' - (v+1)$. Hence the density of b in W is $(w-v)^{-1}(1 - r_a^{-1})$. Thus $r_b = (w-v)(1 - r_a^{-1})^{-1}$.
(i) If there is no letter $b \neq a$ with $r_a | r_b$, then, by the proof of (ii), the number of non-a's between any pair of consecutive b's is constant, $w - v$. By Lemma 6, the word W', obtained from W by omitting all a's, is balanced and has distinct rates. It means that W' is an exact congruence cover with distinct rates. This is a contradiction with Lemma 1. □

8 Fraenkel's conjecture in cases $m = 3, 4$ and 5

In this section we assume that W is a balanced word on a finite alphabet Q so that the rates of the letters are distinct. We shall show that W contains at least 6 different letters. By the Corollary of Lemma 7 the letters are characterised by the integer parts of their rates. We shall denote a letter by the integer part of its rate if we know the value. An $*$ will denote some unknown letter which may be equal to one of the already used letters. By r_x we denote the rate of x. We recall from Lemma 2 that for every x from Q there is at most one i such that the distance between two consecutive x's is not $\lfloor r_x \rfloor$ or $\lceil r_x \rceil$. Hence there is some place in W so that to the right of it such exceptions do not occur. Call this subword, with its first ten letters removed, W^+. Thus if a letter is denoted by 3, then the distance between consecutive 3's in W^+ is 3 or 4.

Lemma 8. *If each of the letters* $3, 4$ *and* 5 *occur in* W, *then* $|Q| \geq 6$.

Proof. Suppose the letters $3, 4$ and 5 and no more than two other letters appear in W. Since $r_5 > r_4$, the pattern $5\ 4\ *\ *\ *\ 4\ 5$ will occur. The middle $*$ has to be 3, since 3 can have only jumps 3 and 4. Thus we have the pattern $*\ 3\ 5\ 4\ *\ 3\ *\ 4\ 5\ 3\ *$ where $*$ denotes letters different from $3, 4$ and 5. It follows from Lemma 7 that the rates of the other letters are at least 6. We therefore have $a\ 3\ 5\ 4\ b\ 3\ a\ 4\ 5\ 3\ b$ where $3, 4, 5, a, b$ are distinct. It follows that $a \leq 6$ and $b \leq 6$, which is impossible by Lemma 7. □

Lemma 9. *If* $3, 4, 6$ *and* 7 *occur in* W, *then* $|Q| \geq 6$.

Proof. Suppose $3, 4, 6$ and 7 occur. Since $r_7 > r_6$, the following pattern is in W: $7\ 6\ *\ *\ *\ *\ *\ 6\ 7$. If there are two 4's between the two 6's, then we have $7\ 6\ 4\ *\ *\ *\ 4\ 6\ 7$ and the 3's cannot be placed. Thus there is only one 4 in between and we have $4\ 7\ 6\ 3\ a\ 4\ a\ 3\ 6\ 7\ 4$ where a denotes a letter different from $3, 4, 6, 7$. The rate of a would be less than 3, but this is impossible. □

We prove the following proposition which implies Fraenkel's conjecture for $m = 3, 4, 5$.

Proposition 2. *The only balanced word* W *on* m *letters* $a_1, ..., a_m$ *with distinct rates* $r_1 < r_2 < ... < r_m$ *and* $3 \leq m \leq 5$ *is the periodic word with period cycle* (F_m) *where* $F_1 = a_1$ *and* $F_m = F_{m-1} a_m F_{m-1}$ *for* $m = 2, 3, ...$. *Moreover,* $r_j = \frac{2^m - 1}{2^{m-j}}$ *for* $j = 1, 2, ..., m$.

Proof. We write a for a_1, b for a_2, c for a_3, ..., with rates $r_a < r_b < r_c < ....$
We follow [AGH, Theorem 2.19] to give a short proof for $m = 3$.
Step 1.The pattern aca must appear in W.
There exists a pair of consecutive a's with no b in between. Thus the pattern aa or aca appears. If aa appears, then $r_a \leq 2$ and c is necessarily surrounded by a's.
Step 2. The pattern $abaaba$ appears in W.
Choose two b's with no c in between. Then the pattern is ba^kb (where a^k means k consecutive a's). Since b is balanced in W and aca appears, we have $k \geq 2$. Every c is surrounded by a^{k-1}'s. Since b is balanced in W and both ba^kb and $a^{k-1}ca^{k-1}$ occur, we must have $k + 2 > 2k - 1$. Thus $k = 2$. Then $r_a \leq 2$ and $abaaba$ appears in W.
Step 3. The letter c appears only in the pattern $abacaba$.
Every b and every c is surrounded by a's, since aa occurs. The pattern aca is surrounded by b's, since $baab$ occurs. By $r_a \leq 2$, the only possible extension is $abacaba$.
Step 4. W is the periodic word with period cycle ($abacaba$).
By Steps 3 the letter c appears only in $abacaba$. By Steps 2 and 3, c has always jumps ≥ 5, b has always jumps ≥ 3, every subword of W of two letters contains one a and every subword of W of five letters contains at most three a's. Hence $abacaba$ is surrounded by a's, then by b's, then by a's, then by c's. By applying Step 3 we see that every pattern $abacaba$ is surrounded by the same pattern. This proves the claim for $m = 3$.

Suppose W is a balanced word on at most five letters a, b, c, d, e with distinct rates. If $r_a \geq 3$ and $r_b \geq 5$, then by Lemma 7,

$$\sum_{x \in Q} r_x^{-1} \leq \frac{1}{3} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} < 1$$

which is impossible. If $r_a \geq 3$ and $r_b < 5$, then by Lemmas 7,8 and 9,

$$\sum_{x \in Q} r_x^{-1} \leq \frac{1}{3} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{9} < 1$$

which is also impossible. Thus $r_a < 3$.

Next we prove the claim for $m = 4$. If we omit all a's in W, we obtain, by Lemma 6, a balanced word W' on the letters b, c, d with distinct rates. This has to be the word with period cycle ($bcbdbcb$) according to the above proved case $m = 3$. We see that the pattern ba^kb has to occur in W^+ for some k. Since $r_a < r_b$, we have $k > 0$. If $k = 1$, then $r_b < 3$. Hence $r_a < 2$, by Lemma 7. So c is surrounded by a's which contradicts $r_b < 3$. If $k > 2$, then $r_a \leq \frac{3}{2}$ which is excluded by Theorem 2. Thus $k = 2$, and $baab$ appears in W. Hence $r_b < 4$ and $r_a < 2$. It follows that between two consecutive b's in W only aa, aca and ada can appear. We now find from $W' = (bcbdbcb)$ that $W = (abacabadabacaba)$.

The proof for case $m = 5$ is similar. Since $r_a < 3$, we can apply Lemma 3 to obtain a balanced word W' on the letters b, c, d, e with distinct rates. This has to be the word with period cycle ($bcbdbcbebcbdbcb$) according to the above proved case $m = 4$. We again find $k = 2$, $r_b < 4$ and $r_a < 2$. It follows that between two consecutive b's in W only aa, aca, ada and aea can appear. Thus W is periodic with period cycle ($abacabadabacabaeabacabadabacaba$). $\square$

Remark 5. The essential difficulty for general m is to prove that $r_a < 3$. Then the proof can be completed by induction on m. It is proved in Section 4 of [T4] that $r_a < 3$ if $m = 6$. This implies the validity of Fraenkel's conjecture for $m = 6$.

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