THE RECTANGLE COMPLEXITY OF FUNCTIONS ON TWO-DIMENSIONAL LATTICES

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Abstract Let X be a non-empty set. Let $f : \mathbb{Z}^2 \to X$. All vectors which occur have integer coefficients, and for $\vec{a} = (a_1, a_2)$, $\vec{b} = (b_1, b_2)$ we write $\vec{a} \le \vec{b}$ or $\vec{a} < \vec{b}$ if $a_j \le b_j$ or $a_j < b_j$ for $j = 1, 2$, respectively. Let $\vec{b} > \vec{0}$. A $\vec{b}$-block is a set of the form $B_{\vec{b}}(\vec{c}) := \{\vec{x} \in \mathbb{Z}^2 \mid \vec{c} \le \vec{x} < \vec{c} + \vec{b}\}$. A $\vec{b}$-pattern is the restriction of f to some $\vec{b}$-block. The total number of distinct $\vec{b}$-patterns is called the $\vec{b}$-complexity of f. A conjecture of the authors implies that f is periodic if there is a $\vec{b} > \vec{0}$ such that the $\vec{b}$-complexity of f does not exceed $b_1 b_2$. In the paper we prove the statement for $\vec{b} = (n, 2)$ where n is any positive integer.

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1 Introduction.

Let X be a non-empty set and $k \in \mathbb{Z}_{>0}$. Let $f : \mathbb{Z}^k \to X$. Let A be a non-empty finite subset of $\mathbb{Z}^k$. We call A a configuration. An A-pattern is a function $f_{\vec{c}} : A \to X$ defined by $f_{\vec{c}}(\vec{a}) = f(\vec{c} + \vec{a})$ for $\vec{a} \in A$. The set of A-patterns of f is denoted by $P_f(A)$ and its cardinality $|P_f(A)|$ is called the A-complexity of f. We call f periodic if there is a $\vec{p} \in \mathbb{Z}^k \setminus \{\vec{0}\}$ such that $f(\vec{x}) = f(\vec{x} + \vec{p})$ for all $\vec{x}$. Throughout the paper all vectors will have integer coordinates.

Let for a moment $X = \{0, 1\}$. Then obviously $|P_f(A)|$ is at most $2^{|A|}$. Morse and Hedlund [3] observed that $f : \mathbb{Z} \to \{0, 1\}$ is periodic if $|P_f(A)| \leq |A|$ for some block $A = \{a, a+1, \ldots, a+n-1\}$ where $n \in \mathbb{Z}_{>0}$. The authors [4] noted that $f : \mathbb{Z} \to \{0, 1\}$ is periodic if $|P_f(A)| \leq |A|$ for any configuration $A \subset \mathbb{Z}$. They further showed that $f : \mathbb{Z}^k \to \{0, 1\}$ is periodic if $|P_f(A)| \leq |A|$ for some $A \subset \mathbb{Z}^k$ with $|A| \leq 3$ and that the corresponding result does not hold for $k \geq 2$ and $|A| \geq 4$. The authors formulated a conjecture (PPC) on periodicity of functions $f : \mathbb{Z}^2 \to \{0, 1\}$ which satisfy the inequality $|P_f(A)| \leq |A|$ for some 'convex body 'A.

In the present paper we consider rectangular configurations A. For $\vec{a} = (a_1, a_2)$, $\vec{b} = (b_1, b_2)$ we write $\vec{a} \leq \vec{b}$ or $\vec{a} < \vec{b}$ if $a_j \leq b_j$ or $a_j < b_j$ for $j = 1, 2$, respectively. Let $\vec{b} > \vec{0}$. Put $B_{\vec{b}} = \{\vec{x} \in \mathbb{Z}^2 \mid \vec{0} \leq \vec{x} < \vec{b}\}$. For brevity we write $\vec{b}$-pattern instead of $B_{\vec{b}}$-pattern and complexity function $|P_f(\vec{b})|$ in place of $|P_f(B_{\vec{b}})|$. A special case of the conjecture (PPC) says that $f : \mathbb{Z}^2 \to \{0, 1\}$ is periodic if there exists a $\vec{b} = (b_1, b_2) > \vec{0}$ such that $|P_f(\vec{b})| \leq b_1 b_2$. This part of conjecture (PPC) was independently formulated by Berthé and Vuillon [1]. They showed that non-periodic functions f exist with 'totally irrational density' such that $|P_f((n,m))| \leq nm+n$ for all $n,m \in \mathbb{Z}_{>0}$. Their examples arise when generalizing the concept of a cutting line to a cutting plane. See also Vuillon [6]. The case $\vec{b}=(2,2)$ of the conjecture was claimed in Boender [2, Stelling 8] and proved in [4, Theorem 5]. Here we prove the conjecture for $\vec{b}=(n,2)$ and arbitrary X where n is any positive integer.

THEOREM. *Let X be a non-empty set, $f : \mathbb{Z}^2 \to X$ and $n \in \mathbb{Z}_{>0}$. Put $\vec{b}=(n,2)$. If $|P_f(\vec{b})| \leq 2n$, then f is periodic.*

It follows from the condition of the theorem that $f(\mathbb{Z}^2)$ is finite. Example 5 in [4] implies that the corresponding result for rectangular blocks in dimensions ≥ 3 cannot be true. It is further obvious that there exist non-periodic functions f with $|P_f(\vec{b})| = b_1 b_2 + 1$ for all $\vec{b} > \vec{0}$, viz. $f : \mathbb{Z}^2 \to \{0,1\}$ with $f(\vec{0}) = 1$ and $f(\vec{x}) = 0$ if $\vec{x} \neq \vec{0}$. In [5] the authors describe all functions f with $|P_f(A)| \leq |A| + 1$ for all finite A.

The proof of the theorem is by induction on n. In the rest of the paper X, f and n have the above meaning.

2 The case $n = 1$.

Let $n = 1$. Hence $\vec{b} = (1,2)$ and we have patterns $(f(\vec{c}),\ f(\vec{c}+(0,1)))$. Put $\vec{e}_1 = (1,0)$ and $\vec{e}_2 = (0,1)$. The condition $|P_f(\vec{b})| \leq 2$ says that the number of distinct vectors $(f(\vec{c}),\ f(\vec{c}+\vec{e}_2))$ for $\vec{c} \in \mathbb{Z}^2$ is at most 2. We call the vector an equal pair if $f(\vec{c}) = f(\vec{c} + \vec{e}_2)$ and an unequal pair otherwise. If the pairs $(f(\vec{c}),\ f(\vec{c} + \vec{e}_2))$ are equal, then f has period vector $\vec{e}_2$. If the pairs are unequal, then f has period vector $2\vec{e}_2$. It cannot occur that there is one equal and one unequal pair, since the unequal pair can be continued into only one direction. This proves the case $n = 1$ with period vector $2\vec{e}_2$.

3 Unique continuation to the right implies periodicity.

Let $m \in \mathbb{Z}_{>0}$. In this section we prove that if each (n, m)-pattern of f has only one right extension to an $(n + 1, m)$-pattern of f, then f is periodic. By right, left, upward, downward, we mean into the direction of $\vec{e}_1$, $-\vec{e}_1$, $\vec{e}_2$, $-\vec{e}_2$, respectively. Similarly, we speak of horizontal and vertical direction.

Let $\ell \in \mathbb{Z}_{\geq 0}$. Define $f_{\vec{c}}^{(\ell)} : B_{(n+\ell,m)} \to X$ for $\vec{c} \in \mathbb{Z}^2$ by $f_{\vec{c}}^{(\ell)}(\vec{x}) = f(\vec{c} + \vec{x})$ for $\vec{x} \in B_{(n+\ell,m)}$. Suppose that for every $\vec{c}$ the function $f_{\vec{c}}^{(1)}$ is completely determined by $f_{\vec{c}}^{(0)}$. Then, by iteration, $f_{\vec{c}}^{(\ell)}$ is determined by $f_{\vec{c}}^{(0)}$ for every $\ell \geq 1$. Since $f(\mathbb{Z}^2)$ is finite, $N := |P_f((n+1, m))| \leq |f(\mathbb{Z}^2)|^{m(n+1)}$. Among the $(n + 1, m)$-patterns $f_{\vec{c}+j\vec{e}_1}^{(1)}$ $(j = 0, 1, \ldots, N)$ at least two are equal. Suppose $f_{\vec{c}+i\vec{e}_1}^{(1)} = f_{\vec{c}+j\vec{e}_1}^{(1)}$ for some i, j with $0 \leq i < j \leq N$. Then, by unique continuation, $f_{\vec{c}+(i+1)\vec{e}_1}^{(1)} = f_{\vec{c}+(j+1)\vec{e}_1}^{(1)}$, $f_{\vec{c}+(i+2)\vec{e}_1}^{(1)} = f_{\vec{c}+(j+2)\vec{e}_1}^{(1)}, \ldots$, so that $\{f_{\vec{c}+h\vec{e}_1}^{(1)}\}_{h=i}^{\infty}$ is periodic with period $j - i$. Since $j - i \leq N$, the sequence $\{f_{\vec{c}+h\vec{e}_1}^{(1)}\}_{h=N}^{\infty}$ is periodic with period dividing $N!$. Applying this result to $\vec{c} := \vec{x} - N\vec{e}_1$ we obtain that $f_{\vec{x}}^{(1)} = f_{\vec{x}+N!\vec{e}_1}^{(1)}$ for every $\vec{x}$. In particular, $f(\vec{x}) = f(\vec{x} + N!\vec{e}_1)$ for every $\vec{x}$. Thus f is periodic with period vector $N!\vec{e}_1$.

4 Directed graphs.

We assume henceforth that the statement of the theorem is true for n and will derive its truth for $n+1$. We therefore assume $|P_f((n,2))| > 2n$ and $|P_f((n+1,2))| \leq 2n+2$ and that f is not periodic. From now on we write $P(x,y)$ in place of $P_f((x,y))$.

Suppose first $|P(n,2)| = |P(n+1,2)|$. Since every pattern has a right extension, every $(n,2)$-pattern has a unique right extension to an $(n+1,2)$-pattern. By the result of Section 3 f is periodic. Thus for the further proof we may restrict our attention to the case

$$|P(n,2)| = 2n+1, \qquad |P(n+1,2)| = 2n+2. \tag{1}$$

We make a directed graph with the $(n,2)$-patterns

$$\begin{matrix} x_1 & x_2 & \dots & x_n \\ y_1 & y_2 & \dots & y_n \end{matrix} \tag{2}$$

of f as vertices and a directed edge from one pattern to another if there exists an $(n+1,2)$-pattern of f such that the edge leaves the vertex which is the left $(n,2)$-subpattern and goes to the vertex which is the right $(n,2)$-subpattern. By (1) there are $2n+1$ vertices, of which $2n$ have only one outgoing edge and one has two outgoing edges and similarly for ingoing edges. The graph may consist of more than one component, but the component which contains the vertex with two outgoing edges contains also the vertex with two ingoing edges, since the graph is finite and each vertex has at least one outgoing vertex. We call it the exceptional component. Consider a path starting from the vertex with two outgoing edges until a vertex is reached which occurred earlier along the path. Since there are no other vertices with two outgoing edges, the exceptional component is the union of the two subgraphs formed by the possible paths.

By a 2-row of f we mean $f|_{\mathbb{Z}\times\{x,x+1\}}$ for some $x \in \mathbb{Z}$. Every 2-row corresponds with an orbit in one of the components. The non-exceptional components yield periodic 2-rows with period at most $2n$. If the exceptional component would also only generate 2-rows with period at most $2n$, then f would have a horizontal period at most $(2n)!$. So we may assume that the exceptional component generates a 2-row which does not have a period at most $2n$.

If the vertices (2) of the exceptional component contain only vertically equal pairs and there is another component, then we take a 2-row which has not a period $\leq 2n$ and move vertically until we meet a 2-row generated by another component. This 2-row cannot have a period $\leq 2n$ either which is a contradiction. We conclude that the exceptional component contains a vertex with an unequal vertical pair.

Consider the unique vertex from which two edges leave, $\begin{matrix} x_0 & x_1 & \dots & x_{n-1} \\ y_0 & y_1 & \dots & y_{n-1} \end{matrix}$ say, and let

$$(3) \qquad \begin{matrix} x_1 & x_2 & \dots & x_{n-1} & x_n \\ y_1 & y_2 & \dots & y_{n-1} & y_n \end{matrix} \qquad \text{and} \qquad \begin{matrix} x_1 & x_2 & \dots & x_{n-1} & x_n' \\ y_1 & y_2 & \dots & y_{n-1} & y_n' \end{matrix}$$

be the two corresponding edges with $(x_n, y_n) \neq (x_n', y_n')$. In Section 5 we deal with the situation $x_n = y_n$, $x_n' = y_n'$. In Section 6 we assume $x_n \neq x_n'$, $y_n \neq y_n'$ and use the result of Section 5 to derive a contradiction. In Section 7 we treat the remaining case $x_n = x_n'$ or $y_n = y_n'$.

5 Two equal pairs.

Assume that the edges starting from the same vertex are the ones in (3) with $x_n \neq x'_n$. We know from Section 3 that there exists an $(n,3)$-pattern which cannot be continued to the right in a unique way. If one of the $(n,2)$-subpatterns has a unique right extension, then there is only one way to extend the $(n,3)$-pattern to the right. Therefore both left $(n,2)$-subpatterns are equal to $\begin{smallmatrix} x_0 & x_1 & \dots & x_{n-1} \\ y_0 & y_1 & \dots & y_{n-1} \end{smallmatrix}$ which implies $x_j = y_j$ for $j = 0, 1, \dots, n-1$. Hence the $(n,3)$-pattern is of the form

$$\begin{matrix} x_0 & x_1 & \dots & x_{n-1} \\ x_0 & x_1 & \dots & x_{n-1} \\ x_0 & x_1 & \dots & x_{n-1} \end{matrix}. \tag{4}$$

We have seen in Section 4 that the exceptional component is the union of the subgraphs formed by the two paths starting with the edges (3) and that it contains a vertex with an unequal vertical pair. Without loss of generality we may assume that the path starting with $\begin{smallmatrix} x_0 & x_1 & \dots & x_n \\ x_0 & x_1 & \dots & x_n \end{smallmatrix}$ and ending at the first vertex which occurred before, contains a vertex with an unequal vertical pair. Since the unequal pairs are uncompatible in the middle row, it is not possible that the $(n,3)$-pattern is continued by $(x_n, x_n, x_n)^{\mathrm{T}}$. Hence there is a unique extension by $(x'_n, x'_n, x'_n)^{\mathrm{T}}$ which yields a contradiction.

6 Pairs with no common coordinate.

Assume that f is not periodic and that $x_n \neq x'_n$ and $y_n \neq y'_n$ in (3). We know by Section 3 that there exists an $(n,3)$-pattern which has two distinct extensions to the right to an $(n+1,3)$-pattern. If one of the two involved $(n,2)$-subpatterns of the $(n,3)$-pattern has a unique right extension, then necessarily $x_n = x'_n$ or $y_n = y'_n$ which is excluded. Thus both $(n,2)$-subpatterns are equal to $\begin{smallmatrix} x_0 & x_1 & \dots & x_{n-1} \\ y_0 & y_1 & \dots & y_{n-1} \end{smallmatrix}$. It follows that the $(n,3)$-pattern with two distinct right extensions is of the form (4). Because of Section 5 we may assume $x_n \neq y_n$. If every time that (4) occurs the upper 2-row of (4) is continued to the right by the same pair, then (4) has a unique right extension in view of the continuation of the lower 2-row of (4) and $x_n \neq x'_n$. Thus the upper 2-row of (4) is sometimes continued by $(x_n, y_n)^{\mathrm{T}}$ and sometimes by $(x'_n, y'_n)^{\mathrm{T}}$. Similarly we deduce that the lower 2-row of (4) is sometimes continued by $(x_n, y_n)^{\mathrm{T}}$ and sometimes by $(x'_n, y'_n)^{\mathrm{T}}$, because of $y_n \neq y'_n$. Since $x_n \neq y_n$, we conclude that $y_n = x'_n$, whence $x_n = y'_n$. It follows that the $2n+1$ vertices are given by

$$\begin{array}{cccc} x_0 & x_1 & \dots & x_{n-1} \\ x_0 & x_1 & \dots & x_{n-1} \end{array} \quad \text{and} \quad \begin{array}{cccccc} x_j & x_{j+1} & \dots & x_{n-1} & x_n & \dots \\ x_j & x_{j+1} & \dots & x_{n-1} & y_n & \dots \end{array}$$
$$\text{and} \quad \begin{array}{cccccc} x_j & x_{j+1} & \dots & x_{n-1} & y_n & \dots \\ x_j & x_{j+1} & \dots & x_{n-1} & x_n & \dots \end{array} \tag{5}$$

for $j = 1, \dots, n$. If we extend the $(n,3)$-pattern (4) to the left to an $(n+1,3)$-pattern, the new left $(n,2)$-subpatterns have to be vertices from (5) with all vertical pairs equal except possibly for the most left column. Hence the new column of the $(n+1,3)$-pattern is either $(x_n, y_n, x_n)^{\mathrm{T}}$ or $(y_n, x_n, y_n)^{\mathrm{T}}$ or $(x_0, x_0, x_0)^{\mathrm{T}}$. In the latter case $x_0 = x_1 = \dots = x_{n-1}$ and the vertex $\begin{smallmatrix} x_0 & x_1 & \dots & x_{n-1} \\ x_0 & x_1 & \dots & x_{n-1} \end{smallmatrix}$ would have three distinct right extensions which is a contradiction. In the former cases $\begin{smallmatrix} x_0 & x_1 & \dots & x_{n-1} \\ x_0 & x_1 & \dots & x_{n-1} \end{smallmatrix}$ is the unique vertex with double left and right extension $(x_n, y_n)^{\mathrm{T}}$ and $(y_n, x_n)^{\mathrm{T}}$. Hence each of the other vertices has a unique left extension. By applying the box principle it follows that the $2n+1$ vertices are given by

$$\begin{array}{cccc} x_0 & x_1 & \dots & x_{n-1} \\ x_0 & x_1 & \dots & x_{n-1} \end{array} \quad \text{and} \quad \begin{array}{cccccccc} x_j & x_{j+1} & \dots & x_{n-1} & x_n & x_0 & \dots & x_{j-1} \\ x_j & x_{j+1} & \dots & x_{n-1} & y_n & x_0 & \dots & x_{j-1} \end{array}$$

$$\text{and} \quad \begin{array}{cccccccc} x_j & x_{j+1} & \dots & x_{n-1} & y_n & x_0 & \dots & x_{j-1} \\ x_j & x_{j+1} & \dots & x_{n-1} & x_n & x_0 & \dots & x_{j-1} \end{array}$$

for $j = 1, \dots, n$. Thus the graph has only one component and every row of f is of the form

$$\dots * x_0 x_1 \dots x_{n-1} * x_0 x_1 \dots x_{n-1} * x_0 x_1 \dots x_{n-1} * \dots$$

where $*$ can be x_n or y_n.

If the $*$'s in two adjacent rows are not strictly above each other, then all $*$'s are uniquely determined and there is a horizontal period $n+1$. Thus the image of f consists of cycles of vertical columns of $x_0, x_1, \dots, x_{n-1}, *$, respectively. Since two $*$'s above each other are unequal, there is a vertical period 2, a contradiction.

7 Pairs with a common coordinate.

We assume that again f is not periodic and that $x_n = x'_n$ or $y_n = y'_n$, but $(x_n, y_n) \neq (x'_n, y'_n)$ in (3). Without loss of generality we consider the case $x_n = x'_n$, $y_n \neq y'_n$. The two edges starting from the unique vertex with double right extension then are

(6)

$$\begin{array}{ccccc} x_0 & x_1 & \dots & x_{n-1} & x_n \\ y_0 & y_1 & \dots & y_{n-1} & y_n \end{array} \quad \text{and} \quad \begin{array}{ccccc} x_0 & x_1 & \dots & x_{n-1} & x_n \\ y_0 & y_1 & \dots & y_{n-1} & y'_n \end{array} \quad \text{with } y_n \neq y'_n.$$

Since the other vertices have a unique right extension, every vertex has a unique right extension at the upper row. It follows that the function $f_1(x, y) := f(x + y, y)$ satisfies the conditions $|P_{f_1}((n, 2))| \leq 2n + 1$ and $|P_{f_1}((n + 1, 2))| \leq 2n + 2$ too. Moreover f is periodic if and only if f_1 is periodic and the directed graph induced by f_1 is uniquely determined by the directed graph induced by f. By our assumption f_1 is not periodic. Because of the preceding sections, in particular (1), we may assume

$$|P_{f_1}((n, 2))| = 2n + 1, \qquad |P_{f_1}((n + 1, 2))| = 2n + 2 \tag{7}$$

and the two edges starting from the same vertex in the graph of f_1 are of the form (6). We can repeat the procedure as often as we like, so defining $f_m(x, y) = (x + my, y)$ for $m = 2, 3, \dots$. Each time there has to be a common coordinate, since otherwise the previous sections yield a contradiction. Moreover, the common coordinate has to be in the upper row, since otherwise y_n or y'_n is excluded and only one of the extensions (6) can occur. Thus the complete right upper row starting with some pattern $\begin{array}{cccc} x_0 & x_1 & \dots & x_{n-1} \\ y_0 & y_1 & \dots & y_{n-1} \end{array}$ is determined by the directed graph induced by f.

Consider the directed graphs induced by $f, f_1, f_2, \dots$. Since at most finitely many values of X are in the range of f, by (7), the number of possible graphs for $f, f_1, f_2, \dots$ is bounded in terms of n. Therefore there exist j and $m > 0$, both bounded in terms of n, so that the graph of f_j equals the graph of

f_{j+m}. It follows that the graph of f_{j+1} equals the graph of f_{j+m+1}, and so on. Hence, starting from an arbitrary $(n+1,2)$-pattern, the upper right half row is periodic with period m. Thus every row is periodic with period m, that is, f has horizontal period m. This contradiction completes the proof of the theorem.

REMARK. It follows from the given proof that under the conditions of the Theorem f has a period vector with second coordinate 0, 1 or 2. The bound for the first coordinate is effectively computable, but quite high. We conjecture that there is a period vector (p,q) with $|p| \leq n$ and $|q| \leq 2$.

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