THE TRIBONACCI SUBSTITUTION

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Abstract

We study the discretised segments generated by the iterated Tribonacci substitution and the projections of the integer points on them to some plane. After suitable transformations we get a sequence of finite two-dimensional words which tends to a doubly rotational word on $\mathbb{Z}^2$. (Without scaling we would get the Rauzy fractal.) As an introduction we start with the corresponding case of the Fibonacci substitution.

1. The Fibonacci Word

If there would exist Miss Word elections, the Fibonacci word would be an excellent candidate to win. In this section we give an overview of the properties of the Fibonacci word. For background information for this and other sections we refer to [L] and [B].

The Fibonacci substitution is the substitution ϕ over the 2-letter alphabet $\mathcal{A} := \{0, 1\}$ defined by $\phi(0) = 01$, $\phi(1) = 0$. If we start with 0 and repeatedly apply ϕ we get successively

$$
\begin{aligned}
&u_0 = 0\\
&u_1 = 01\\
&u_2 = 010\\
&u_3 = 01001\\
&u_4 = 01001010\\
&u_5 = 0100101001001\\
&u_6 = 010010100100101001010\\
&\dots
\end{aligned}
$$

Note that $u_n = u_{n-1}u_{n-2}$ for every integer $n > 1$. This sequence of words converges to the so-called *Fibonacci word* $f = (f_m)_{m=1}^{\infty}$. If we define $F_0 = 0, F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$

for any integer $n > 1$, then the number of symbols of u_n, denoted by $|u_n|$, equals F_{n+2} and the number of 0's and 1's in u_n, denoted by $|u_n|_0$ and $|u_n|_1$, equals F_{n+1} and F_n, respectively. The Fibonacci word has the following properties:

- The *frequency* of 1 equals $\lim_{n\to\infty} \frac{F_n}{F_{n+2}} = \frac{3}{2} - \frac{1}{2}\sqrt{5} = \gamma^2$ where $\gamma := \frac{1}{2}\sqrt{5} - \frac{1}{2}$, hence $1 = \gamma + \gamma^2$, [L] sect.2.1.1. Since the frequency is irrational, the Fibonacci sequence is non-periodic. The frequency of 0 equals γ.

- The Fibonacci word is *balanced*, which means that for all subwords u, v of f of equal lengths we have $||u|_1 - |v|_1| \leq 1$, [L] sect.2.1.1. Note that $||u|_0 - |v|_0| \leq 1$ is an equivalent requirement.

- The Fibonacci word is *sturmian*, that is, $P(n) = n + 1$ for every n, where $P(n)$ equals the number of different subwords of f of length n, [L] sect.2.1.1. Because a word is (ultimately) periodic if there exists an n for which $P(n) \leq n$ [MH], a sturmian word is in this sense the most regular non-periodic word.

- The Fibonacci word is a *rotation word*, [L] sect.2.1.2. In fact
$$\forall m \geq 1 : \quad f_m = \begin{cases} 0 \text{ if } \{(\mathrm{m}+1)\gamma\} \in (0, \gamma] \\ 1 \text{ if } \{(\mathrm{m}+1)\gamma\} \in \{0\} \cup (\gamma, 1) \end{cases}$$
where $\{\cdot\}$ denotes the fractional part.

- The Fibonacci word is a *Beatty sequence*, [L] sect.2.1.2. In fact
$$\forall m \geq 1 : \quad f_m = \llcorner (m+1)\gamma^2 \lrcorner - \llcorner m\gamma^2 \lrcorner.$$

- The Fibonacci word is a *cutting sequence*, [S]. In fact, in the x-y-plane the *broken Fibonacci halfline* that you get by starting in $(0, 0)$ and going 1 in the direction of the x-axis when $f_m = 0$ and 1 in the direction of the y-axis when $f_m = 1$ is an ideal discrete approximation to the halfline given by $y = \gamma x,\ x \geq 0$. See Figure 1.

Up to now we have considered one-sided words and halflines. The definition of the broken Fibonacci word as a Beatty sequence (or as a rotation sequence) allows a straightforward extension to a biinfinite word $(f_m)_{m\in\mathbb{Z}}$. Actually there is another natural extension, viz.
$$\forall m \in \mathbb{Z} : \quad f^*_m = \ulcorner (m+1)\gamma^2 \urcorner - \ulcorner m\gamma^2 \urcorner \;=\; \begin{cases} 0 \text{ if } \{(m+1)\gamma\} \in [0, \gamma) \\ 1 \text{ if } \{(m+1)\gamma\} \in [\gamma, 1). \end{cases}$$

The sequences $(f_m)_{m\in\mathbb{Z}}$ and $(f^*_m)_{m\in\mathbb{Z}}$ coincide except that $f^*_{-1} = f_0 = 0, f_{-1} = f^*_0 = 1$. Furthermore $f_m = f_{-m-1}$ for all $m > 0$ (cf. Theorem 2.4). Starting at the origin and going in both directions according to $(f_m)_{m\in\mathbb{Z}}$ we obtain an ideal discretisation, called the *broken Fibonacci line*, of the line $y = \gamma x$. It is given by (the 10 of $f_{-1}f_0$ is put in between vertical bars to indicate the positions -1 and 0)

$$\ldots 1001001010010|10|0100101001001 \ldots .$$

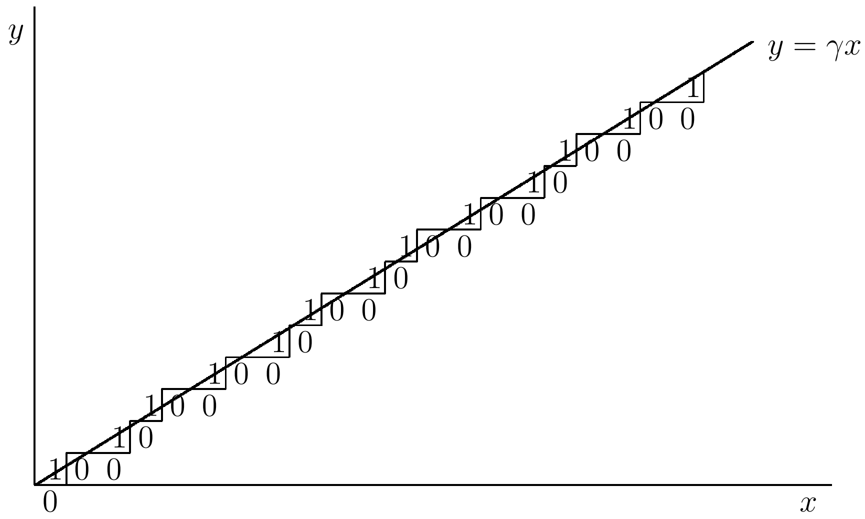

Figure 1: The Fibonacci sequence is a cutting sequence.

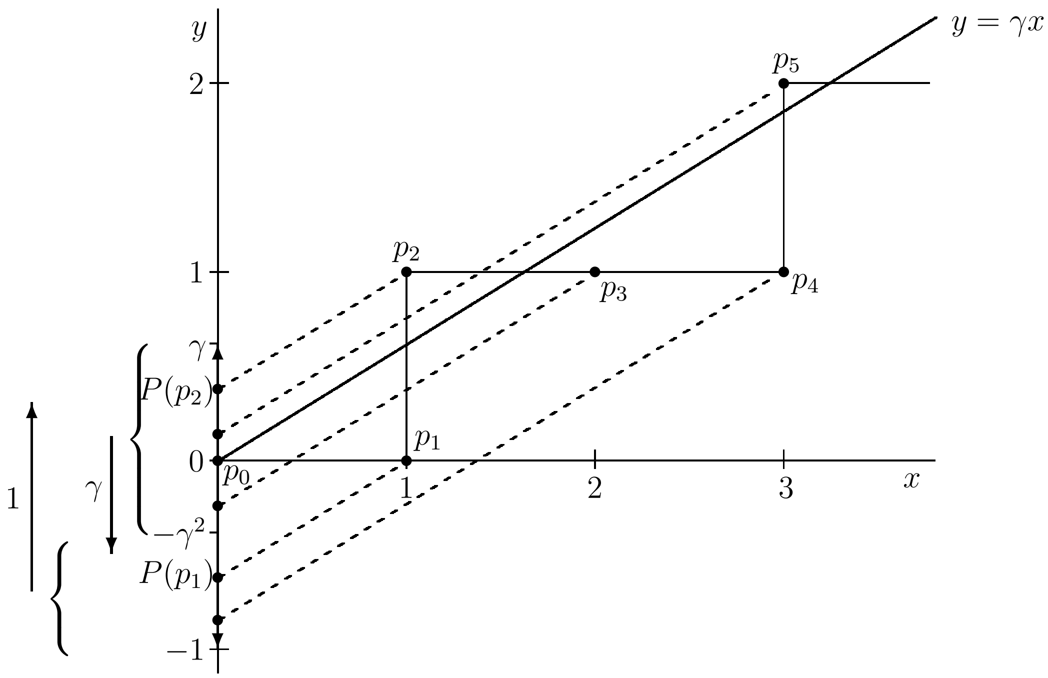

Figure 2: Projecting the broken Fibonacci halfline gives an exchange of intervals.

2. Discretisation of the line

In this section we study projections of the broken Fibonacci halfline to the y-axis more closely. The integer points p_m for $m \in \mathbb{Z}_{\geq 0}$ on this halfline are fixed by $p_0 = (0,0)$, $p_m = p_{m-1} + \vec{e}_{f_m}$ where $\vec{e}_0, \vec{e}_1$ denote the unit vectors in the directory of the x-axis and y-axis, respectively. Let $\tilde{u}_m$ denote the word $f_1 f_2 \ldots f_m$. Hence $u_n = \tilde{u}_{F_{n+2}}, |\tilde{u}_m|_1 = \sum_{j=1}^{m} f_m, |\tilde{u}_m|_0 = m - |\tilde{u}_m|_1$ and $p_m = (|\tilde{u}_m|_0, |\tilde{u}_m|_1)$. Now we project each integer point parallel to the line $y = \gamma x,\ x \geq 0$ to the y-axis. We denote the projection of p_m by $P(p_m)$. See Figure 2. Note that a 0 means going one step to the right on the broken Fibonacci halfline, and for the projection this corresponds to going down γ along the y-axis. Similarly a 1 corresponds to going up 1 along the y-axis. We have

$$\begin{aligned} P(p_m) = |\tilde{u}_m|_1 - \gamma|\tilde{u}_m|_0 \ &= \ \lfloor (m+1)\gamma^2 \rfloor - \lfloor \gamma^2 \rfloor - \gamma(m - \lfloor (m+1)\gamma^2 \rfloor + \lfloor \gamma^2 \rfloor) \\ &= (1+\gamma)(-\{(m+1)\gamma^2\} + \{\gamma^2\}) = \gamma - (1+\gamma)\{(m+1)\gamma^2\} \in (-1, \gamma]. \end{aligned}$$

The projected points are all in the interval $(-1, \gamma]$ on the y-axis. Thus there is an exchange of intervals, and every point in $(-1, -\gamma^2]$ is moved 1 up and every point in $(-\gamma^2, \gamma]$ is moved γ down.

2.1 Local behaviour

We consider the broken line segments in the x-y-plane corresponding with the words u_n where we start from the origin and a 0 means going 1 in the direction of the x-axis and a 1 corresponds to going 1 in the direction of the y-axis. We project parallel to the line through the origin and the end point $p_{|u_n|}$ of the broken line segment, and we project to the y-axis. We denote the projection of the point p_m by $P_n(p_m)$. See Figure 3.

In Figure 3 we see that $u_2 = 010$ leads to $w'_2 = 102$ and that $u_3 = 01001$ leads to $w'_3 = 41302$, where w'_n is given by the increasing order of the projected points $P_n(p_m)$ on the y-axis. Note that $|w'_n| = |u_n| = F_{n+2}$ for every non-negative integer n.

$$
\begin{array}{ccl c ccccccc|cc|cccc}
n & |u_n| & u_n & & & & & & & & & \multicolumn{2}{c}{w'_n} & & & & \\
0 & 1 & 0 & \to & & & & & & & & & 0_0 & & & & \\
1 & 2 & 01 & \to & & & & & & & & 1_1 & 0_0 & & & & \\
2 & 3 & 010 & \to & & & & & & & & 1_1 & 0_0 & 2_0 & & & \\
3 & 5 & 01001 & \to & & & & & & 4_1 & 1_1 & 3_0 & 0_0 & 2_0 & & & \\
4 & 8 & 01001010 & \to & & & & & 4_1 & 1_1 & 6_1 & 3_0 & 0_0 & 5_0 & 2_0 & 7_0 & \\
5 & 13 & 0100101001001 & \to & 12_1 & 4_1 & 9_1 & 1_1 & 6_1 & 11_0 & 3_0 & 8_0 & 0_0 & 5_0 & 10_0 & 2_0 & 7_0
\end{array}
$$

$$\ldots\ldots$$

The subscripts refer to the corresponding values in u_n. For even n we reflect the w'_n in the origin and interchange the 0's and 1's in the subscripts. We call the resulting words w_n.

$$
\begin{array}{ccl c ccccccc|cc|cccc}
n & |u_n| & u_n & & & & & & & & & \multicolumn{2}{c}{w_n} & & & & \\
0 & 1 & 0 & \to & & & & & & & & & 0_0 & & & & \\
1 & 2 & 01 & \to & & & & & & & & 1_1 & 0_0 & & & & \\
2 & 3 & 010 & \to & & & & & & & & 2_1 & 0_1 & 1_0 & & & \\
3 & 5 & 01001 & \to & & & & & & 4_1 & 1_1 & 3_0 & 0_0 & 2_0 & & & \\
4 & 8 & 01001010 & \to & & & & & & 7_1 & 2_1 & 5_1 & 0_1 & 3_1 & 6_0 & 1_0 & 4_0 \\
5 & 13 & 0100101001001 & \to & 12_1 & 4_1 & 9_1 & 1_1 & 6_1 & 11_0 & 3_0 & 8_0 & 0_0 & 5_0 & 10_0 & 2_0 & 7_0
\end{array}
$$

$$\ldots\ldots$$

Observe that the numbers form a complete system of representatives mod $|u_n|$ and that the number i is placed in a position congruent to $iF_n (\bmod F_{n+2})$ for $i = 0, 1, \ldots, |u_n| - 1$. Moreover, the subscript is 0 if the next jump is to the left and it is 1 if the next jump is to the right. Subsequently we replace each number in w_n which is less than F_{n+1} with 0 and each number at least F_{n+1} with 1. We call the resulting word v_n. If we underline a word, it means that the letters are not only placed in order but also in the right position.

$$
\begin{array}{cc ccccccc|cc|cccc}
\underline{v_0} & = & & & & & & & & & 0 & & & & \\
\underline{v_1} & = & & & & & & & & 1 & 0 & & & & \\
\underline{v_2} & = & & & & & & & & 1 & 0 & 0 & & & \\
\underline{v_3} & = & & & & & & 1 & 0 & 1 & 0 & 0 & & & \\
\underline{v_4} & = & & & & & & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
\underline{v_5} & = & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0
\end{array}
$$

$$\ldots\ldots$$

We shall show that the sequence of words $(\underline{v_n})_{n=0}^{\infty}$ converges to the two-sided Fibonacci word when n tends to infinity. It means that the generated broken line is the broken Fibonacci line.

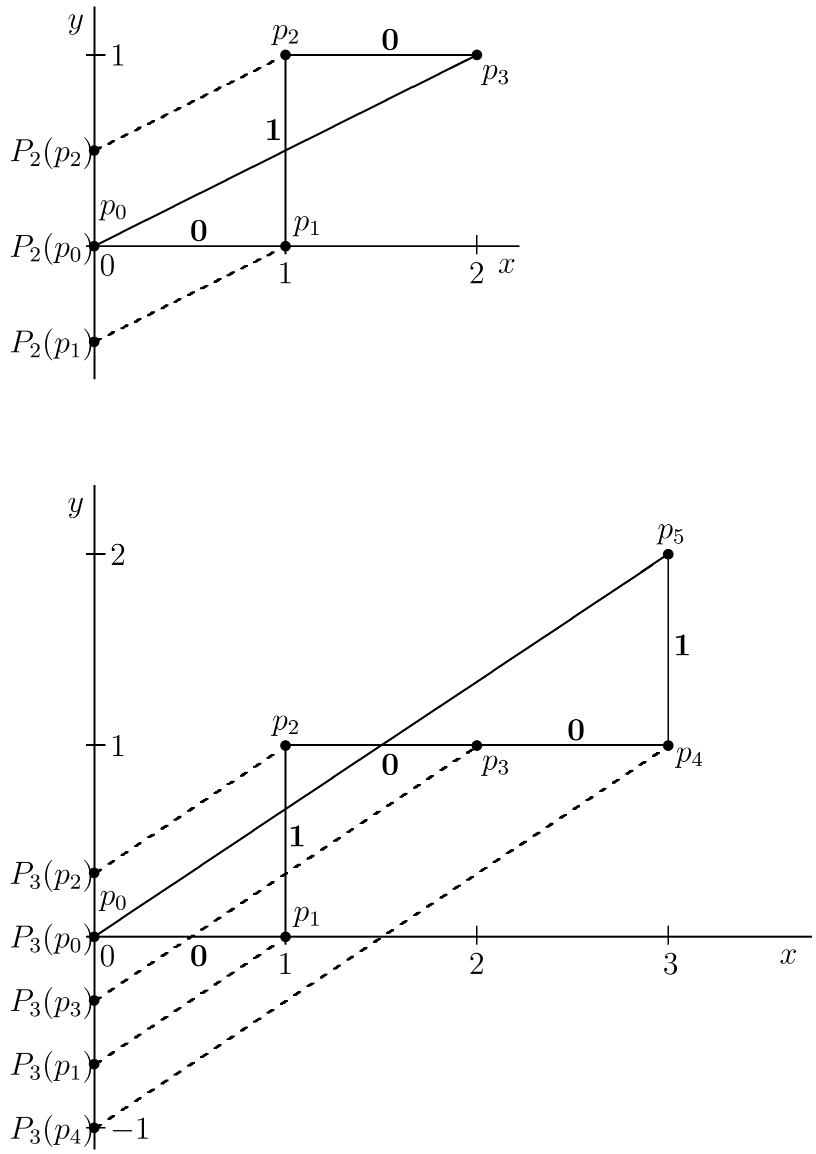

Figure 3: Projection of $u_2 = 010$ and $u_3 = 01001$.

Lemma 2.1. *The positions of w_n run from $-F_{n+1}$ up to $F_n - 1$ if n is odd, and from $-F_n$ up to $F_{n+1} - 1$ if n is even.*

Proof. For each projected point we have $P_n(p_m) = |\tilde{u}_m|_1 - \frac{F_n}{F_{n+1}}|\tilde{u}_m|_0$ with $0 \leq |\tilde{u}_m|_1 \leq F_n, 0 \leq |\tilde{u}_m|_0 \leq F_{n+1}$. Since $\gcd(F_n, F_{n+1}) = 1$, it follows that if $m < m'$ we get $P_n(p_m) = P_n(p_{m'}) \Rightarrow m = 0, m' = |u_n| = F_{n+2}$. In other words all F_{n+2} projected points $P_n(p_0), \ldots, P_n(p_{|u_n|-1})$ are distinct and of the form $\frac{x}{F_{n+1}}, x \in \mathbb{Z}$. Note that to get from one projected point to the next, we either add $\frac{F_{n+1}}{F_{n+1}}$ or subtract $\frac{F_n}{F_{n+1}}$ which is the same modulo $\frac{F_{n+2}}{F_{n+1}}$. Hence x passes through all cosets modulo F_{n+2}.
Now we construct w_n by placing index m in position a in w_n if $P_n(p_m) = (0, a)$, and for even n reflecting w_n in the origin. The lemma implies that w_n contains no gaps if it is constructed in this way.
We will prove it by induction. The lemma is true for $n = 2, 3$. Assume the lemma is true for $n-1$ and assume n is odd. To go from u_{n-1} to u_n in u_{n-1} every 0 is replaced with 01 and every 1 is replaced with 0. It follows that to go from w_{n-1} to w_n every jump to the right of length F_{n-2} is replaced by a jump to the left of length F_{n-1} followed by a jump to the right of length F_n and every jump to the left of length F_{n-1} in w_{n-1} leads to a jump to the left of the same length in w_n. Thus w_n consists of F_{n+1} numbers without gaps placed directly under w_{n-1} and F_n numbers on the left-hand side of them, possibly containing gaps. But because these F_n numbers on the left are in different cosets modulo F_{n+2} and are only one jump to the left of length F_n away from the F_{n+1} numbers directly under w_{n-1}, they must occupy exactly the first F_n positions left of w_{n-1}, and it follows that w_n has no gaps either. Because w_{n-1} runs from position $-F_{n-1}$ up to $F_n - 1$, it follows that w_n runs from position $-F_{n+1}$ up to $F_n - 1$. The situation for even n is similar. □

Lemma 2.2. $\underline{v_n} = \begin{cases} \underline{v_{n-1}}v_{n-2} & \textit{if } n \textit{ is even}, \\ v_{n-2}\underline{v_{n-1}} & \textit{if } n \textit{ is odd}. \end{cases}$

Proof. Note that $p_{F_{n+1}} = (|u_{n-1}|_0, |u_{n-1}|_1) = (F_n, F_{n-1})$, so $P_n(p_{F_{n+1}}) = F_{n-1} - F_n\frac{F_n}{F_{n+1}} = \frac{(-1)^2}{F_{n+1}}$. Because for n is even w_n is reflected in the origin, w_n has F_{n+1} in position -1 for every n. Assume n is odd. Because F_{n+1} in w_n is placed directly below F_n in w_{n-1}, below every number smaller than F_n in w_{n-1} a number is placed in w_n that is smaller than F_{n+1}, and below every number larger than F_n in w_{n-1} a number is placed that is larger than F_{n+1}. It follows that the part of $\underline{v_n}$ placed directly below $\underline{v_{n-1}}$ is equal to $\underline{v_{n-1}}$. Because of the way w_n is constructed from w_{n-1} in the proof of Lemma 2.1, the left F_n numbers of $\underline{v_n}$ are an exact copy of the left F_n numbers of v_{n-1}, which by induction are an exact copy of the F_n numbers of v_{n-2}. This proves the lemma for n is odd. The case n is even is similar. □

We still need another lemma. By overlining we indicate that the word should be reversed. The lemma states that $01u_n$ for even n and $10u_n$ for odd n are palindromes.

Lemma 2.3.
$$\begin{aligned} \overline{0\,1\,u_n} &= 0\,1\,u_n \quad \textit{if } n \textit{ is even}, \\ \overline{1\,0\,u_n} &= 1\,0\,u_n \quad \textit{if } n \textit{ is odd}. \end{aligned}$$

Proof. The equalities hold for $n = 0$ and $n = 1$. Let n be even. Then, by induction on n,

$$\begin{aligned} 01u_n &= 01u_{n-1}u_{n-2} = 01u_{n-2}u_{n-3}u_{n-2} = \overline{u_{n-2}}10u_{n-3}u_{n-2} = \overline{u_{n-2}u_{n-3}}01u_{n-2} \\ &= \overline{u_{n-2}u_{n-3}u_{n-2}}10 = \overline{01u_{n-2}u_{n-3}u_{n-2}} = \overline{01u_{n-1}u_{n-2}} = \overline{01u_n}. \end{aligned}$$

For odd n the proof is similar. □

Now we can prove the theorem.

Theorem 2.4.

$$\lim_{n\to\infty} \underline{v_n} = \overline{f} \mid 1\ 0 \mid f$$

Proof. It suffices to show by induction on n that

$$0\underline{v_n}1 = \begin{cases} \overline{u_{n-2}}|10|u_{n-1} & \text{if } n \text{ is even,} \\ \overline{u_{n-1}}|10|u_{n-2} & \text{if } n \text{ is odd.} \end{cases}$$

We have $0\underline{v_2}1 = 0|10|01 = \overline{u_0}|10|u_1$ and $0\underline{v_3}1 = 010|10|01 = \overline{u_2}|10|u_1$ indeed. Suppose the statement is true for all integers less than n. Then, for even n, indicating by subscript p that the last digit is missing and by subscript s that the first digit is missing, from Lemma 2.2 and the induction hypothesis we get

$$0\underline{v_n}1 = 0\underline{v_{n-1}}v_{n-2}1 = \overline{u_{n-2}}|10|(u_{n-3})_p(\overline{u_{n-4}})_s10u_{n-3}.$$

Hence, by Lemma 2.3 and the facts that u_{n-4} starts with a 0 and u_{n-3} ends with a 1,

$$\begin{aligned} 0\underline{v_n}1 &= \overline{u_{n-2}}|10|(u_{n-3})_p(\overline{01u_{n-4}})_su_{n-3} = \overline{u_{n-2}}|10|(u_{n-3})_p1u_{n-4}u_{n-3} = \\ &\overline{u_{n-2}}|10|u_{n-3}u_{n-4}u_{n-3} = \overline{u_{n-2}}|10|u_{n-2}u_{n-3} = \overline{u_{n-2}}|10|u_{n-1}. \end{aligned}$$

The proof in case n is odd is similar. □

2.2 Incidence vectors and matrices

Let u be a finite word over a k-letter alphabet $\mathcal{A} := \{0, 1, \ldots, k-1\}$. Then we call $\vec{u} := (|u|_0, |u|_1, \ldots, |u|_{k-1})$ its *incidence vector*. If α is a substitution over $\mathcal{A}$, then the *incidence matrix* M_α belonging to α has $|\alpha(i-1)|_{j-1}$ as entry (i, j). Incidence vectors and matrices contain the global information (the numbers of each letter), but not the local information (precise order).

When applying substitution ϕ defined above to a finite word, the new incidence vector is obtained by multiplying the old one on the right by $M_\phi = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. Starting with the incidence vector $(1, 0)$ of u_0 and repeatedly multiplying with M_ϕ yields successively $(1, 1)$, $(2, 1)$, $(3, 2)$, $(5, 3)$, $(8, 5)$, $(13, 8)$, …. This agrees with the sequence (p_n, q_n) where p_n/q_n are the convergents of the continued fraction expansion of $\gamma^{-1} = \frac{1}{2} + \frac{1}{2}\sqrt{5}$.

3. The Tribonacci Word

We can ask ourselves what happens in higher dimensions. Does a substitution over a 3-letter alphabet generate a discretisation of a line or of a plane? Rauzy [R] considered as a generalisation of the Fibonacci substitution the substitution $\sigma : \begin{cases} 0 & \rightarrow & 01 \\ 1 & \rightarrow & 02 \\ 2 & \rightarrow & 0 \end{cases}$ over the alphabet $\{0, 1, 2\}$. Starting with 0 and repeatedly applying σ we get successively

$$\begin{array}{ll}
u_0 = 0 & \vec{u}_0 = (1, 0, 0) \\
u_1 = 01 & \vec{u}_1 = (1, 1, 0) \\
u_2 = 0102 & \vec{u}_2 = (2, 1, 1) \\
u_3 = 0102010 & \vec{u}_3 = (4, 2, 1) \\
u_4 = 0102010010201 & \vec{u}_4 = (7, 4, 2) \\
u_5 = 010201001020101020100102 & \vec{u}_5 = (13, 7, 4) \\
u_6 = 0102010010201010201001020102 01\ldots & \vec{u}_6 = (24, 13, 7) \\
\ldots &
\end{array}$$

On the right-hand side the incidence vectors are given. Note that $u_n = u_{n-1}u_{n-2}u_{n-3}$ for every integer $n > 2$. The limit word is called the *Tribonacci word* $t = (t_m)_{m=1}^{\infty}$. Note that if we define $T_0 = 0, T_1 = T_2 = 1, T_n = T_{n-1} + T_{n-2} + T_{n-3}$ for $n > 2$, then the number of symbols of u_n equals T_{n+2}, $|u_n|_0 = T_{n+1}$, $|u_n|_1 = T_n$, $|u_n|_2 = T_{n-1}$.

We check which properties of the Fibonacci word mentioned in Section 1 have analogues for the Tribonacci word.

- The frequency of 0 equals $\lim_{n\to\infty} \frac{T_{n+1}}{T_{n+2}} = \tau$, the frequency of 1 equals $\lim_{n\to\infty} \frac{T_n}{T_{n+2}} = \tau^2$, the frequency of 2 equals $\lim_{n\to\infty} \frac{T_{n-1}}{T_{n+2}} = \tau^3$, where $\tau + \tau^2 + \tau^3 = 1$. Since the frequencies are irrational, the Tribonacci sequence is non-periodic.
- The Tribonacci word is not balanced with respect to each letter. The word u_5 contains subwords of length 3 without and a subword of length 3 with two 1's. The word u_6 contains subwords of length 5 without 2's and equally long subwords with two 2's. By applying σ to the subword 201020 of u_6 we get 0010201001, and it follows that u_7 contains subwords of length 9 with four 0's and with six 0's.
- The Tribonacci word is *episturmian* on three letters which implies that $P(n) = 2n+1$ for every n [AR]. Because a word on three letters is periodic if the frequencies of its letters are independent over $\mathbb{Q}$ and there exists an n for which $P(n) \leq 2n$ [T1], an episturmian word is in this sense a most regular non-periodic word on three letters with independent frequencies.
- If we generalise the notion of rotation word to three letters by splitting the unit interval in three instead of two intervals and giving a letter to each interval, then

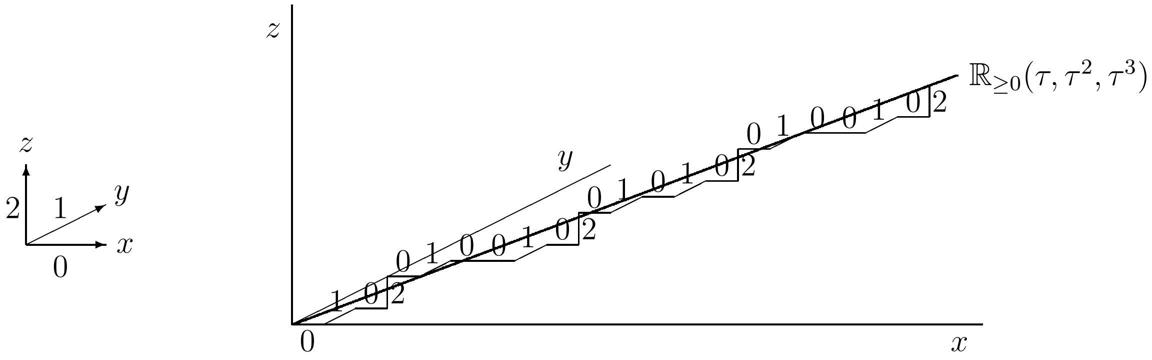

Figure 4: The Tribonacci sequence is a cutting sequence.

the resulting 3-letter word is balanced with respect to each letter. Therefore in this sense the Tribonacci word is not rotational.

- There is not a natural extension of the concept of Beatty sequence for 3-letter words.

- The Tribonacci word can be described as a cutting sequence. In fact, in the x-y-z-space the *broken Tribonacci halfline* that we get by starting in $(0, 0, 0)$ and going 1 in the direction of the x-axis when $t_m = 0$, 1 in the direction of the y-axis if $t_m = 1$ and 1 in the direction of the z-axis if $t_m = 2$ provides an excellent discrete approximation of the halfline $\mathbb{R}_{\geq 0}(\tau, \tau^2, \tau^3)$, see Figure 4.

4. Discretisation of the plane

In this section we study projections of the broken Tribonacci halfline to the y-z-plane. The integer points p_m for $m \in \mathbb{Z}_{\geq 0}$ on this line are given by $p_0 = (0, 0, 0)$, $p_m = p_{m-1} + \vec{e}_{t_m}$ $(m > 0)$ where $\vec{e}_0, \vec{e}_1, \vec{e}_2$ denote the unit vectors in the direction of the x-axis, y-axis, z-axis, respectively. Now we project each integer point parallel to the halfline $\mathbb{R}_{\geq 0}(\tau, \tau^2, \tau^3)$ to the y-z-plane. Note that for the projection to the y-z-plane, a 0 means moving over $(-\tau, -\tau^2)$, a 1 corresponds to going 1 to the right and a 2 to going 1 upwards.

4.1 Local behaviour

For $n = 1, 2, \ldots$ we consider the broken line segment corresponding with the word u_n in the x-y-z-space which is obtained by taking the segment of the broken Tribonacci halfline from p_0 to p_k where $k = |u_n|$. We define σ_n as the projection of the integer points on this line segment parallel to the line through the origin and the end point of the broken line segment to the y-z-plane. Hence a 1 means for the projection going 1 in the direction of the y-axis and a 2 means for the projection going 1 in the direction of the z-axis. Since there are T_{n+1} 0's, T_n 1's and T_{n-1} 2's and the end point is projected to the origin, we see that for the projection a 0 means a translation over $(-\frac{T_n}{T_{n+1}}, -\frac{T_{n-1}}{T_{n+1}})$. Observe that projecting to any other generic plane would change the projection by some linear transformation only. We number the projections $P_n(p_m)$ of the points p_m according to the index m. See Figure 5.

We see that, after suitable linear transformations, $u_2 = 0102$, $u_3 = 0102010$ and $u_4 = 0102010010201$ lead to the configurations

$$
\begin{array}{ccc}
w_2 & w_3 & w_4 \\
\begin{array}{cc} & \\ & \\ 2_0 & 3_2 \\ 0_0 & 1_1 \\ & \end{array}
&
\begin{array}{ccc} & & \\ & & \\ & 4_0 & 6_0 \\ 5_1 & 0_0 & 2_0 \\ 1_1 & 3_2 & \end{array}
&
\begin{array}{ccc} & 8_1 & 12_1 \\ 10_2 & 1_1 & 5_1 \\ 3_2 & 7_0 & 11_0 \\ 9_0 & 0_0 & 4_0 \\ 2_0 & 6_0 & \end{array}
\end{array}
$$

respectively. The subscripts refer to the corresponding values in u_n. Observe that the subscript indicates which jump has to made to reach the next number.

Subsequently we replace in w_n every number less than T_{n+1} with 0, every number at least T_{n+1} but less than $T_{n+1} + T_n$ with 1 and every number at least $T_{n+1} + T_n$ but less than T_{n+2} with 2 to obtain v_n. We underline the 0 which is at the origin. This yields

$$
\begin{array}{ccc}
v_2 & v_3 & v_4 \\
\begin{array}{cc} & \\ & \\ 1 & 2 \\ \underline{0} & 0 \\ & \end{array}
&
\begin{array}{ccc} & & \\ & & \\ & 1 & 2 \\ 1 & \underline{0} & 0 \\ 0 & 0 & \end{array}
&
\begin{array}{ccc} & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 2 \\ 1 & \underline{0} & 0 \\ 0 & 0 & \end{array}
\end{array}
$$

respectively. Observe that v_n is an extension of v_{n-1} for $n = 3, 4$ and that the new part of v_3 is obtained by translating v_2 over $(-1, -1)$ and the new part of v_4 by translating v_3 over $(0, 2)$. We shall study the behaviour of the sequence $(v_n)_{n=1}^{\infty}$ and show that there are many similarities with the one-dimensional case of the broken Fibonacci line.

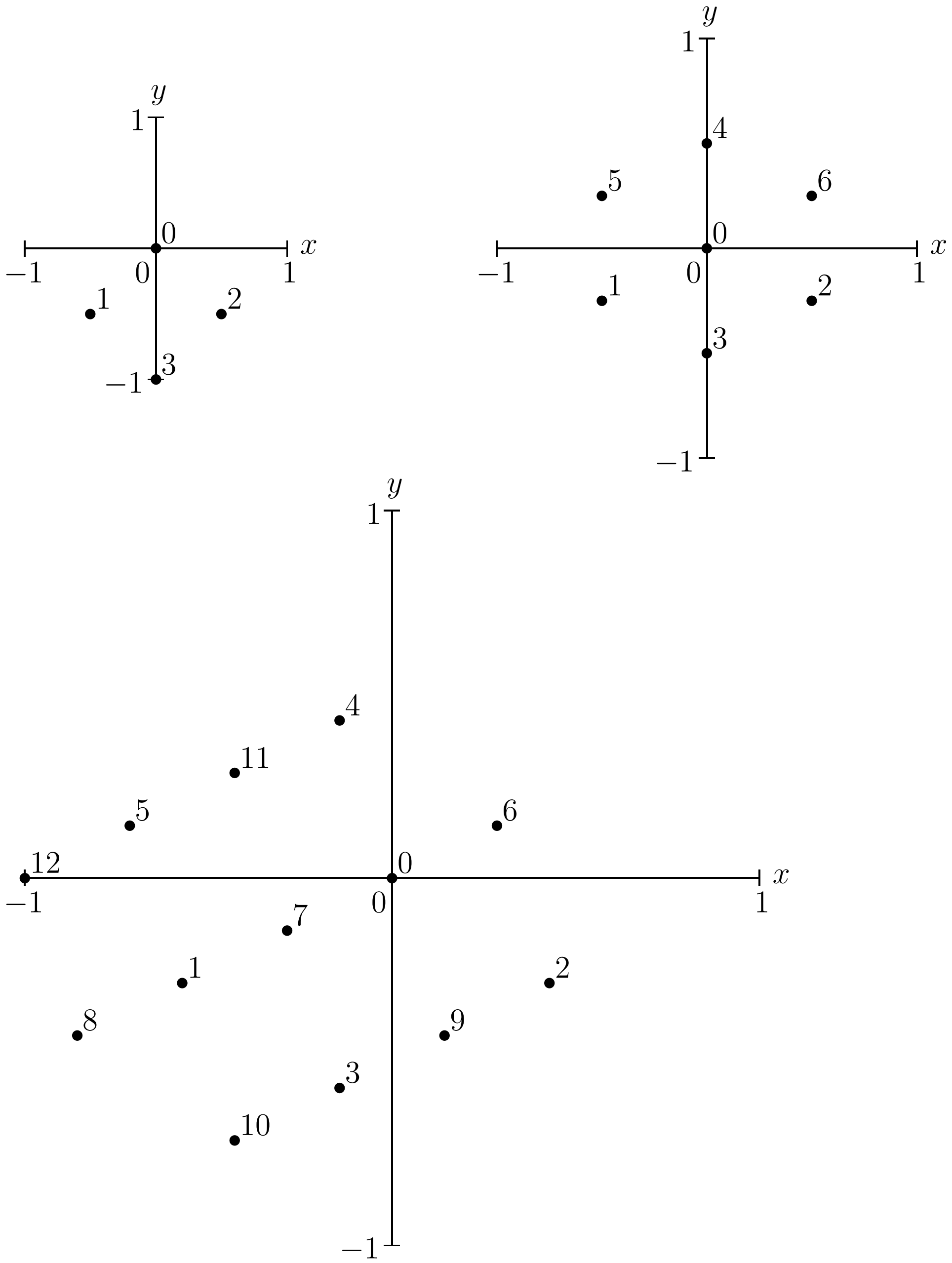

Figure 5: Projection w_2' of $u_2 = 0102$, projection w_3' of $u_3 = 0102010$ and projection w_4' of $u_4 = 0102010010201$. We number the projections $P_n(p_m)$ of the points p_m according to m.

4.2. Incidence vectors and matrices

Incidence vectors and matrices have been introduced in Sec. 2.2. The incidence vector of u_2 is $\vec{u}_2 = (2,1,1)$. The incidence matrix of the substitution σ is given by $M_\sigma = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$. Hence $\sigma(u)$ has incidence vector $\vec{u}M_\sigma$. In particular, the incidence vector of the word u_n is given by $(1,0,0)M_\sigma^n$ for $n = 0, 1, \dots$. By induction we find that

$$M_\sigma^n = \begin{pmatrix} T_{n+1} & T_n & T_{n-1} \\ T_n + T_{n-1} & T_{n-1} + T_{n-2} & T_{n-2} + T_{n-3} \\ T_n & T_{n-1} & T_{n-2} \end{pmatrix}.$$

It follows that for $n \geq 2$ the word $u_n = \sigma^{n-2}(u_2)$ originates from the word $u_2 = 0102$ where in u_n the first T_n letters come from the first 0 in u_2, the next $T_{n-1} + T_{n-2}$ letters come from 1, the next T_n letters have their origin in the second 0 and the last T_{n-1} letters are generated by the 2. The total number is T_{n+2} indeed.

Suppose that the first m letters of u_{n-1} have incidence vector (a,b,c), so $p_m = (a,b,c)$. The letter in position m is mapped by the substitution σ to one or two letters in u_n the last of which has incidence matrix $(a+b+c, a, b) = (a,b,c)M_\sigma$.

Lemma 4.1 *For $n = 2, 3, \dots$ we can choose the linear transformation such that in w_n the number at the origin is 0, the number in position $(1,0)$ is T_n, the number in position $(0,1)$ is T_{n+1} and every number m with $0 \leq m < T_{n+2}$ is in the lattice $\mathbb{Z}^2$.*

Proof. Without loss of generality we can place 0 in w_n at the origin for $n = 2, 3, \dots$. We use induction on n. The lemma is true for $n = 2$. Suppose it is true for $n - 1 \geq 2$. Then the number in position $(1,0)$ in w_{n-1} is the first number which originates from the 1 in $u_2 = 0102$, and the number in position $(0,1)$ in w_{n-1} is the first number in u_{n-1} which comes from the second 0 in $u_2 = 0102$. Moreover, every number in w_{n-1} is in a position in the lattice $\mathbb{Z}^2$. If the position of a letter in w_{n-1} is (i,j), then $(i,j) = i * P_{n-1}((T_{n-1}, T_{n-2}, T_{n-3})) + j * P_{n-1}((T_n, T_{n-1}, T_{n-2}))$. When we apply σ to the word u_{n-1} to obtain u_n, we apply M_σ to the endpoint of broken line segment corresponding to u_{n-1} to obtain the endpoint of the broken line segment corresponding to u_n and therefore $P_n M_\sigma$ to get their projections. If (a,b,c) is the integer point on the broken line segment corresponding to u_{n-1} which is projected to (i,j), then

$$(a,b,c) \in i * (T_{n-1}, T_{n-2}, T_{n-3}) + j * (T_n, T_{n-1}, T_{n-2}) + \mathbb{R}(T_{n+1}, T_n, T_{n-1}).$$

Applying M_σ on the right we obtain

$$(a,b,c)M_\sigma \in i * (T_n, T_{n-1}, T_{n-2}) + j * (T_{n+1}, T_n, T_{n-1}) + \mathbb{R}(T_{n+2}, T_{n+1}, T_n).$$

If we now apply P_n we get that the image of (i,j) equals

$$i * P_n((T_n, T_{n-1}, T_{n-2})) + j * P_n((T_{n+1}, T_n, T_{n-1})).$$

Thus the number in position (i,j) in w_{n-1} is kept in w_n in the same position (i,j). In case the corresponding letter in u_{n-1} is a 2, then the image is the letter 0 and the next letter in u_n corresponds also with a number at a lattice point in w_n. It follows that a step 0 along the broken line segment corresponding to u_n is projected by P_n to the difference of two lattice points. Since all new points are obtained by translating a known lattice point by the projection of a step 0, all the new lattice points are also in the lattice and can be found by translating the corresponding old lattice points by the vector $P_n((1,0,0))$. This completes the proof of the lemma. $\square$

4.3 Continued lattices

Note that in w_4 the jump to reach the next number is $(0,2),(-1,-3),(2,-1)$, if the subscript is 0,1,2, respectively. For w_3 the translation vectors are $(-1,-1),(2,1),(0,2)$, respectively. We shall study the relation between the translation vectors of consecutive w_n's.

Lemma 4.2 *Let $n>2$. Let $\vec{a}_0^{(n)},\vec{a}_1^{(n)},\vec{a}_2^{(n)}$ be the translation vectors corresponding to w_n for $n=2,3,\ldots$. Then, for $n=3,4,\ldots$,*

$$\vec{a}_0^{(n)}=\vec{a}_2^{(n-1)},\ \vec{a}_1^{(n)}=\vec{a}_0^{(n-1)}-\vec{a}_2^{(n-1)},\ \vec{a}_2^{(n)}=\vec{a}_1^{(n-1)}-\vec{a}_2^{(n-1)}.$$

Moreover, the domain of w_n is a fundamental domain of the lattice

$$\Lambda_n:=\mathbb{Z}(\vec{a}_1^{(n)}-\vec{a}_0^{(n)})+\mathbb{Z}(\vec{a}_2^{(n)}-\vec{a}_0^{(n)}).$$

If $d_0\vec{a}_0^{(n)}+d_1\vec{a}_1^{(n)}+d_2\vec{a}_2^{(n)}=\vec{0}$ for integers d_0,d_1,d_2 then $d_i=kT_{n-i+1}$ for some $k\in\mathbb{Z}$ and $i=0,1,2$.
The number m in position (i,j) in w_n is congruent to $iT_n+jT_{n+1} \pmod{T_{n+2}}$ and (i,j) is congruent to $m\vec{a}_0^{(n)} \pmod{\Lambda_n}$.

Proof. The statements are correct for $n=3$ with $\vec{a}_0^{(2)}=(1,0),\vec{a}_1^{(2)}=(-1,1),\vec{a}_2^{(2)}=(-1,-1)$ and $\vec{a}_0^{(3)}=(-1,-1),\vec{a}_1^{(3)}=(2,1),\vec{a}_0^{(3)}=(0,2)$. The lattice $\Lambda_3=\mathbb{Z}(3,2)+\mathbb{Z}(1,3)$ has determinant $t_5=7$ and the domain of w_3 is a fundamental domain of Λ_3. The number 6, for example, is in position $(1,1)$ in w_3 and indeed $T_3+T_4\equiv 6 \pmod 7$ and $(1,1)\equiv(-6,-6) \pmod{\Lambda_3}$, since $(7,7)=2(3,2)+(1,3)$.

Suppose the statements are true for values smaller than n. According to the substitution σ each jump 0 in w_{n-1} is replaced with a jump 0 followed by a jump 1 in w_n, each jump 1 in w_{n-1} is replaced with a jump 0 followed by a jump 2 in w_n, and each jump 2 is replaced with a jump 0. Hence $\vec{a}_0^{(n-1)}=\vec{a}_0^{(n)}+\vec{a}_1^{(n)},\vec{a}_1^{(n-1)}=\vec{a}_0^{(n)}+\vec{a}_2^{(n)},\vec{a}_2^{(n-1)}=\vec{a}_0^{(n)}$. This implies the first statement of the lemma.

Suppose $d_0\vec{a}_0^{(n)} + d_1\vec{a}_1^{(n)} + d_2\vec{a}_2^{(n)} = \vec{0}$ for integers d_0, d_1, d_2. Then

$$d_0\vec{a}_2^{(n-1)} + d_1(\vec{a}_0^{(n-1)} - \vec{a}_2^{(n-1)}) + d_2(\vec{a}_0^{(n-1)} - \vec{a}_2^{(n-1)}) = \vec{0}.$$

By the induction hypothesis there is an integer k such that $d_1 = kT_n, d_2 = kT_{n-1}$, $d_0 - d_1 - d_2 = kT_{n-2}$, which implies $d_0 = k(T_n + T_{n-1} + T_{n-2}) = kT_{n+1}$. This proves the third statement.

In w_n the number $m+1$ is reached by a jump $\vec{a}_0^{(n)}, \vec{a}_1^{(n)}$ or $\vec{a}_2^{(n)}$ from the number m. Since each vector is congruent to $\vec{a}_0^{(n)} (\bmod\ \Lambda_n)$ and the number 0 is at the origin, it is immediate that m is in a position congruent to $m\vec{a}_0^{(n)} (\bmod\ \Lambda_n)$. This is the last assertion.

Suppose $P_n(p_{m_1})$ and $P_n(p_{m_2})$ with $0 \le m_1 < m_2 < T_{n+2}$ are in positions congruent modulo Λ_n. Then $(m_2 - m_1)\vec{a}_0^{(n)} \in \mathbb{Z}(\vec{a}_1^{(n)} - \vec{a}_0^{(n)}) + \mathbb{Z}(\vec{a}_2^{(n)} - \vec{a}_0^{(n)})$. Let i and j be integers such that

$$(m_2 - m_1)\vec{a}_0^{(n)} = i(\vec{a}_1^{(n)} - \vec{a}_0^{(n)}) + j(\vec{a}_2^{(n)} - \vec{a}_0^{(n)}).$$

Then there exists an integer k such that $m_2 - m_1 + i + j = kT_{n+1}, -i = kT_n, -j = kT_{n-1}$. Hence $m_2 - m_1 = kT_{n+2}$, which yields a contradiction. Thus the domain of w_n is a fundamental domain of Λ_n. This is the second assertion.

Suppose there is an m in position (i,j) of w_n. Then $(i,j) \equiv m\vec{a}_0^{(n)} (\bmod\ \Lambda_n)$. We know from Lemma 4.1 and the last assertion of Lemma 4.2 that $(1,0) \equiv T_n\vec{a}_0^{(n)} (\bmod\ \Lambda_n)$ and $(0,1) \equiv T_{n+1}\vec{a}_0^{(n)} (\bmod\ \Lambda_n)$. Thus $m\vec{a}_0^{(n)} \equiv (iT_n + jT_{n+1})\vec{a}_0^{(n)} (\bmod\ \Lambda_n)$. As in the preceding paragraph it follows that $T_{n+2} \mid m - iT_n - jT_{n+1}$. This completes the proof. $\square$

Corollary 4.3 *If μ is in v_n in position (i,j), then*

$$\mu = \begin{cases} 0 \text{ if } iT_n + jT_{n+1} \pmod{T_{n+2}} \in [0, T_{n+1}) \\ 1 \text{ if } iT_n + jT_{n+1} \pmod{T_{n+2}} \in [T_{n+1}, T_{n+1} + T_n) \\ 2 \text{ if } iT_n + jT_{n+1} \pmod{T_{n+2}} \in [T_{n+1} + T_n, T_{n+2}). \end{cases}$$

Moreover, the value in position (i,j) will remain μ in v_h for all $h > n$.

Proof. According to Lemma 4.2 the number m in position (i,j) in w_n is congruent to $iT_n + jT_{n+1} \pmod{T_{n+2}}$. By definition of v_n, m is replaced with a 0 if $0 \le m < T_{n+1}$, by a 1 if $T_{n+1} \le m < T_{n+1} + T_n$, by a 2 if $T_{n+1} + T_n \le m < T_{n+2}$. Since $0 \le m < T_{n+2}$ this proves the first assertion.

The value μ in v_n is 0, 1, 2, depending on whether the original point on the broken line segment corresponding to u_n is among the first T_{n+1} integer points, among the next T_n integer points, among the remaining T_{n-1} integer points of the broken line segment, respectively. If we apply σ to the word u_n, hence M_σ to the broken line segment, then the images of the integer points are among the first T_{n+2} integer points, among the next T_{n+1} integer points, among the remaining T_n integer points of the broken line segment corresponding to u_{n+1}, respectively. Therefore the resulting value m in w_{n+1} satisfies $0 \leq m < T_{n+1}, T_{n+1} \leq m < T_{n+1} + T_n, T_{n+1} + T_n \leq m < T_{n+2}$ in the respective cases. Hence, the corresponding values in v_{n+1} are 0, 1, 2, respectively. Thus the value of μ remains unchanged. □

4.4 The Tribonacci number system

The preceding theory implies that $\lim_{n\to\infty} v_n$ exists. In this subsection we study the Tribonacci number system to be able to show that every integer point is contained in the domain of v_n for sufficiently large n. Therefore the limit function will be "space-filling". Lemma 4.4 and its consequences can be derived from more general results on so-called beta-expansions by Frougny and Solomyak [FS] and by Akiyama [A]. Since we want to keep the paper self-contained we present a direct proof.

The Tribonacci number system is obtained by writing the non-negative integers successively in the 2-letter alphabet $\{0, 1\}$ thereby not allowing three consecutive digits 1. So $0 \to 0, 1 \to 1, 2 \to 10, 3 \to 11, 4 \to 100, 5 \to 101, 6 \to 110, 7 \to 1000, 8 \to 1001, 9 \to 1010, 10 \to 1011, 11 \to 1100, 12 \to 1101, 13 \to 10000, \ldots$. It follows by induction on n that for $n > 0$ the number T_n is expressed by a 1 followed by $n - 2$ 0's.

Let τ be the positive root of the polynomial $x^3 + x^2 + x - 1$. Hence $\tau \approx 0.5437$. We shall apply the following result. The condition $k_n k_{n+1} k_{n+2} = 0$ says that $k_N k_{N-1} \ldots k_1$ is a Tribonacci representation of some non-negative integer.

Lemma 4.4 a) *Every number of the form $\sum_{n=1}^{N} k_n \tau^n$ with $k_n \in \{0, 1\}$ and $k_n k_{n+1} k_{n+2} = 0$ for $n = 1, 2, \ldots, N-2$, can be uniquely expressed as $a + b\tau + c\tau^2$ with $a, b, c \in \mathbb{Z}$ and $0 \leq a + b\tau + c\tau^2 < 1$.*
b) *Every number $a + b\tau + c\tau^2$ with $a, b, c \in \mathbb{Z}$ and $0 < a + b\tau + c\tau^2 < 1$ can be uniquely expressed as a finite sum $\sum_{n=1}^{N} k_n \tau^n$ with $k_n \in \{0, 1\}$ and $k_n k_{n+1} k_{n+2} = 0$ for $n = 1, 2, \ldots$, N-2.*

Proof. a) Let $\sum_{n=0}^{N} k_n \tau^n$ be a number of the specified form. Then $0 \leq \sum_{n=0}^{N} k_n \tau^n < 1$. We can replace $k_N \tau^N$ with $-k_N \tau^{N-1} - k_N \tau^{N-2} - k_N \tau^{N-3}$ to transform it into an expression $\sum_{n=0}^{M} k'_n \tau^n$ with $k'_n \in \mathbb{Z}$ for $n = 0, 1, \ldots, M$ and $M < N$. By iterating this reduction we eventually arrive at the wanted representation $a + b\tau + c\tau^2$. All such values are distinct, since τ is an algebraic number of degree 3.

b) For a number $\sum_{n=0}^{N} k_n \tau^n$ with integer coefficients k_n we define its *weight* as $\sum_{n=1}^{N} |k_n|$. Let $k_i \tau^i + k_{i+1} \tau^{i+1} + k_{i+2} \tau^{i+2}$ be some number with $k_i, k_{i+1}, k_{i+2} \in \mathbb{Z}$ and $0 < k_i \tau^i + k_{i+1} \tau^{i+1} + k_{i+2} \tau^{i+2} < \tau^{i-1}$. We claim that if $k_i \tau^i + k_{i+1} \tau^{i+1} + k_{i+2} \tau^{i+2} < \tau^i$, then we can find integers $k'_{i+1}, k'_{i+2}, k'_{i+3}$ such that $k_i \tau^i + k_{i+1} \tau^{i+1} + k_{i+2} \tau^{i+2} = k'_{i+1} \tau^{i+1} + k'_{i+2} \tau^{i+2} + k'_{i+3} \tau^{i+3}$ and $|k_i| + |k_{i+1}| + |k_{i+2}| \geq |k'_{i+1}| + |k'_{i+2}| + |k'_{i+3}|$ and if $k_i \tau^i + k_{i+1} \tau^{i+1} + k_{i+2} \tau^{i+2} \geq \tau^i$, then we can find integers $k'_{i+1}, k'_{i+2}, k'_{i+3}$ such that $k_i\tau^i + k_{i+1}\tau^{i+1} + k_{i+2}\tau^{i+2} = \tau^i + k'_{i+1}\tau^{i+1} + k'_{i+2}\tau^{i+2} + k'_{i+3}\tau^{i+3}$ and $|k_i| + |k_{i+1}| + |k_{i+2}| > |k'_{i+1}| + |k'_{i+2}| + |k'_{i+3}|$.

To prove our claim, first assume $k_i \leq 0$. Then $k_{i+1} > 0$ or $k_{i+2} > 0$. If $k_{i+1} \leq -k_i$ and $k_{i+2} \leq -k_i$, then $k_i\tau_i + k_{i+1}\tau^{i+1} + k_{i+2}\tau^{i+2} \leq k_i(\tau^i - \tau^{i+1} - \tau^{i+2}) = k_i\tau^{i+3} \leq 0$ which is excluded. If $k_{i+1} > -k_i$ or $k_{i+2} > -k_i$, then it is easy to verify the claim.

Now assume $k_i = 1$. If $k_{i+1}\tau^{i+1} + k_{i+2}\tau^{i+2} \geq 0$ then we are in the second case and the claim is obviously true. Otherwise $k_{i+1} < 0$ or $k_{i+2} < 0$. Since $\tau^i + k_{i+1}\tau^{i+1} + k_{i+2}\tau^{i+2} = (k_{i+1}+1)\tau^{i+1} + (k_{i+2}+1)\tau^{i+2} + \tau^{i+3}$, the last expression has weight $|k_{i+1}+1| + |k_{i+2}+1| + 1 \leq 1 + |k_{i+1}| + |k_{i+2}|$ which is the old weight.

Finally assume $k_i \geq 2$. If $k_i + k_{i+1} \leq 0$ or $k_i + k_{i+2} \leq 0$, then the claim is clearly true. So we assume $k_i + k_{i+1} \geq 1$ and $k_i + k_{i+2} \geq 1$. Hence $k_i\tau^i + k_{i+1}\tau^{i+1} + k_{i+2}\tau^{i+2} \geq \tau^i + (k_i + k_{i+1} - 1)\tau^{i+1} + (k_i + k_{i+2} - 1)\tau^{i+2} + (k_i - 1)\tau^{i+3} \geq \tau^i$. We know that

$$(k_i + k_{i+1} - 1)\tau^{i+1} + (k_i + k_{i+2} - 1)\tau^{i+2} + (k_i - 1)\tau^{i+3} < \tau^{i+1} + \tau^{i+2}$$

and has non-negative coefficients. Hence $(k_i+k_{i+1}-1, k_i+k_{i+2}-1) \in \{(0,0), (1,0), (0,1), (0,2)\}$. This leaves only few possibilities for (k_{i+1}, k_{i+2}) and it is easy to check that the claim holds in each remaining case.

We use the claim to prove the lemma. Suppose a, b, c satisfy the conditions of b). Then we start with weight $|a|+|b|+|c|$. We apply the above procedure iteratively starting with replacing a with $a\tau + a\tau^2 + a\tau^3$. In every step the weight does not increase, but each time that $k_i\tau^i + k_{i+1}\tau^{i+1} + k_{i+2}\tau^{i+2} \geq \tau^i$ the weight decreases. If $\tau^j \leq a + b\tau + c\tau^2 < \tau^{j-1}$, then at step j the weight decreases by at least 1. If $a + b\tau + c\tau^2 = \tau^j$, then we are finished. Otherwise there exists a j' such that $\tau^{j'} \leq a + b\tau + c\tau^2 - \tau^j < \tau^{j'-1}$. After at most $|a|+|b|+|c|$ such values j the tail has weight 0 and we have a representation $\sum_{j=1}^{N} k_j\tau^j$ for $a + b\tau + c\tau^2$ with $k_j \in \{0, 1\}$ for all j and at most $|a| + |b| + |c|$ non-zero coefficients. Since we have used the greedy algorithm and $\tau^j + \tau^{j+1} + \tau^{j+2} = \tau^{j-1}$, we have $k_jk_{j+1}k_{j+2} = 0$ for all j. □

The following statement follows immediately from the proof of Lemma 4.4 b).

Corollary 4.5 *Every number $a + b\tau + c\tau^2$ with $a, b, c \in \mathbb{Z}$ and $0 \leq a + b\tau + c\tau^2 < 1$ can be written in Tribonacci representation $\sum_{j=1}^{N} k_j\tau^j$ with $k_j \in \{0, 1\}$ for $j = 1, \ldots, N$ and $k_jk_{j+1}k_{j+2} = 0$ for $j = 1, \ldots, N-2$ with at most $|a| + |b| + |c|$ non-zero coefficients k_j.*

4.5 Discretisation of the plane

In this section we show that the limit function $\lim_{n\to\infty} v_n$ is bilinear on $\mathbb{Z}^2$. The next lemma characterises the domain of v_n. For $(i, j) \in \mathbb{Z}^2$ there exists a unique integer $h = h_{i,j} = -\lfloor i\tau^2 + j\tau \rfloor$ such that $h + i\tau^2 + j\tau \in [0, 1)$. By Lemma 4.4 b) we can write

τ^n $(n > 0)$ in precisely one way as $a+b\tau+c\tau^2$ with $a, b, c \in \mathbb{Z}$. We denote the ordered pair (c, b) as (i_n, j_n). We first observe that $i_n = i_{n+1} + i_{n+2} + i_{n+3}$ and $j_n = j_{n+1} + j_{n+2} + j_{n+3}$ for $n \geq 1$. Indeed this follows immediately from $\tau^n = \tau^{n+1} + \tau^{n+2} + \tau^{n+3}$ and the fact that τ is an algebraic number of degree 3. Using these identities we infer from the assertion of Lemma 4.2 by induction on n that

$$a_0^{(n)} = (i_n, j_n), \quad a_1^{(n)} = (i_{n-1} - i_n, j_{n-1} - j_n), \quad a_2^{(n)} = (i_{n+1}, j_{n+1}).$$

Lemma 4.6. *The domain of v_n consists of all $(i, j) \in \mathbb{Z}^2$ such that $h_{i,j} + i\tau^2 + j\tau$ can be written as $\sum_{i=1}^{n} k_i\tau^i$ with $k_i \in \{0, 1\}$ for $i = 1, \ldots, n$ and $k_ik_{i+1}k_{i+2} = 0$ for $i = 1, \ldots n - 2$.*

Proof. By induction on n. The statement is true for $n = 1$. Suppose it is true for all values below n. Then the domain of v_{n-1} consists of all pairs $(i, j) \in \mathbb{Z}^2$ such that $h_{i,j} + i\tau^2 + j\tau$ is of the form $\sum_{i=1}^{n-1} k_i\tau^i$ with $k_i \in \{0, 1\}$ for $i = 1, \ldots, n - 1$ and $k_ik_{i+1}k_{i+2} = 0$ for $i = 1, \ldots, n - 3$. According to the proof of Lemma 4.1 the domain of v_n (which is the domain of w_n) is the union of the domain of v_{n-1} and the domain of v_{n-1} translated over $a_2^{(n-1)} = a_0^{(n)}$. By the above formula for $a_0^{(n)}$ we know that $a_0^{(n)} = (i_n, j_n)$. Since $h_{i_n,j_n} + i_n\tau^2 + j_n\tau = \tau^n$, points (i, j) in $v_n \setminus v_{n-1}$ are such that $h_{i,j} + i\tau^2 + j\tau = \tau^n + \sum_{i=1}^{n-1} k_i\tau^i$ with $k_i \in \{0, 1\}$ for $i = 1, \ldots, n - 1$ and $k_ik_{i+1}k_{i+2} = 0$ for $i = 1, \ldots, n - 3$. Put $k_n = 1$. Suppose $k_nk_{n-1}k_{n-2} \neq 0$. Then $k_n = k_{n-1} = k_{n-2} = 1$ and $\sum_{i=n-2}^{n} k_i\tau^i \geq \tau^{n-3}$. This implies that the greedy algorithm has not been applied properly. Thus $k_n = 0$ if $k_{n-1} = k_{n-2} = 1$ and $k_nk_{n-1}k_{n-2} = 0$. □

In fact the construction with continued lattices provides an efficient way to compute the Tribonacci expansions of numbers $i\tau^2 + j\tau \pmod 1$ with $|i|, |j|$ below some bound. In Figure 6 the Tribonacci expansions are given for $|i| \leq 3, |j| \leq 4$. We use that

$$a_0^{(1)} = (0, 1),\ a_0^{(2)} = (1, 0),\ a_0^{(3)} = (-1, -1),\ a_0^{(4)} = (0, 2),\ a_0^{(5)} = (2, -1),$$

$$a_0^{(6)} = (-3, -2),\ a_0^{(7)} = (1, 5),\ a_0^{(8)} = (4, -4),\ a_0^{(9)} = (-8, -3).$$

Now we arrive at the main result of this paper.

Theorem 4.7 *We have* $\lim_{n\to\infty} v_n = V := (V_{i,j})_{(i,j)\in\mathbb{Z}^2}$ *where*

$$V_{i,j} = \begin{cases} 0 \text{ if } \{i\tau^2 + j\tau\} \in [0, \tau) \\ 1 \text{ if } \{i\tau^2 + j\tau\} \in [\tau, \tau + \tau^2) \\ 2 \text{ if } \{i\tau^2 + j\tau\} \in [\tau + \tau^2, 1). \end{cases}$$

$i\backslash j$	-3	-2	-1	0	1	2	3
4	1101100	1100001	1100011	1000100	1000110	1010101	1010000
3	1100101	1100000	1100010	1001	1011	1010100	1010110
2	1100100	1100110	1101	1000	1010	11001	11011
1	101001	101011	1100	1	11	11000	11010
0	101000	101010	101	0	10	10001	10011
-1	100001	100011	100	110	10101	10000	10010
-2	100000	100010	110001	110011	10100	10110	10001101
-3	100110	110101	110000	110010	10101001	10101011	10001100
-4	101000100	110100	110110	10101101	10101000	10101010	10000101

Figure 6: The Tribonacci expansions of $i\tau^2 + j\tau$ (mod 1) for $|i| \leq 3, |j| \leq 4$.

Proof. It follows from Lemma 4.6 that every integer point (i, j) is in the domain of V. Let (i, j) be in the domain of v_n. Then its value at (i, j) is determined in Corollary 4.3. Moreover it remains unchanged if $n \to \infty$. The condition in Corollary 4.3 can be rewritten as

$$i\frac{T_n}{T_{n+2}} + j\frac{T_{n+1}}{T_{n+2}} (\text{mod } 1) \in [0, \frac{T_{n+1}}{T_{n+2}}), [\frac{T_{n+1}}{T_{n+2}}, \frac{T_{n+1}+T_n}{T_{n+2}}), [\frac{T_{n+1}+T_n)}{T_{n+2}}, 1),$$

respectively. By letting n tend to ∞, these intervals tend to $[0, \tau), [\tau, \tau + \tau^2), [\tau + \tau^2, 1)$, respectively. Since τ is an algebraic number of degree 3, the number $i\tau^2 + j\tau$ is a boundary point if and only if (i, j) equals $(0, 0), (0, 1)$ or $(1, 1)$. Other points (i, j) are in a fixed half-open interval from some point on. However, it is easy to check that the numbers in v_n in the positions $(0, 0), (0, 1), (1, 1)$ have the right values 0, 1, 2, respectively. □

We have proved that the limit word V is a doubly rotational sequence, and even a BV-sequence. (Cf. Section 4.6)

4.6 Remarks

Remark 1 A *doubly rotational sequence* is a sequence $f : \mathbb{Z}^2 \to \{0, 1, 2\}$ for which numbers α, β, γ with $0 < \alpha < \alpha + \beta < 1$ exist such that, maybe after permuting the function values 0, 1 and 2 and changing f to $-f$,

$$f_{m,n} = \begin{cases} 0 \text{ if } \{m\alpha + n\beta + \gamma\} \in [0, \alpha) \\ 1 \text{ if } \{m\alpha + n\beta + \gamma\} \in [\alpha, \alpha + \beta) \\ 2 \text{ if } \{m\alpha + n\beta + \gamma\} \in [\alpha + \beta, 1) \end{cases}$$

We call a doubly rotational sequence a *BV-sequence*, if $1, \alpha, \beta$ are linearly independent over the rationals. These sequences were investigated by Berthé and Vuillon [BV1,BV2] in their study of discretisations of a 2-dimensional plane in a three-dimensional space. BV-sequences can be considered as two-dimensional analogues of sturmian sequences.

Berthé and Tijdeman [BT1] showed that BV-sequences are not balanced in the sense that there are rectangles of the same size where the numbers of one letter differ more than 1. Berthé and Vuillon [BV2] showed that BV-sequences have the property that $P(m, n) = mn + m + n$ for all integers m, n. They also considered projections of such BV-words to words on two letters and obtained such (uniform recurrent) words with complexity $P(m, n) = mn + m$. There exists a conjecture by Nivat [N] stating that if there exist m, n such that $P(m, n) \leq mn$ then there is periodicity. Partial results to this conjecture can be found in [SaT], [EKM], [QZ].

Remark 2. The theory of continued lattices has been developed in much more general form than in Subsection 4.3 by Berthé and Tijdeman [BT2],[T2]. Simpson and Tijdeman [SiT] applied it to obtain a multi-dimensional Fine and Wilf theorem.

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