Induced Substitutions

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Induced Substitutions

Sierk W. Rosema

The following chapters of this thesis have appeared or will appear as journal articles (with minor modifications).

Chapter 2:

S. W. Rosema, R. Tijdeman, The Tribonacci substitution,
Integers, Electron. J. Combin. Number Th. **5(3)** (2005), A13.

Chapter 3:

S. W. Rosema, Substitutions on two letters, cutting segments and their projections,
J. Théor. Nombres Bordeaux **19** (2007), 523-545.

Chapter 4 has been accepted by Monatsh. Math.

Chapter 5 has been submitted.

THOMAS STIELTJES INSTITUTE
FOR MATHEMATICS

Contents

Chapter 1

Introduction

This thesis deals with words and substitutions. By a word we mean a sequence of symbols which we call letters, and by a substitution a map defined over the set of words. The first significant work on the subject of combinatorics on words was done by A. Thue in his papers on square-free words at the beginning of the twentieth century, [57], [58]. Besides in pure combinatorics, this subject has been studied by researchers in the field of computer science, probability theory and algebra, among others.

We will start with some definitions and notation. An *alphabet* $\mathcal{A}$ is a finite set of elements that are called *letters*. A *word* is a finite or infinite string of letters. We denote the set of all finite words by $\mathcal{A}^*$. If $w = w_0w_1 \dots w_{k-1} \in \mathcal{A}^*$, then k is called the length of the word w and denoted by $|w|$. We denote by $|w|_a$ the number of occurrences of the letter $a \in \mathcal{A}$ in the word w, and by w_i we mean the letter at position i in the word w. If $u = u_0u_1 \dots u_k$, then we mean by u^2 the word $uu = u_0u_1 \dots u_ku_0u_1 \dots u_k$, the concatenation of the word u with itself. If $v = v_0 \dots v_m$ is a finite word and if $u = u_0u_1 \dots$ is a finite or infinite word, and there exists a k such that $v_l = u_{k+l}$ for $l = 0, \dots, m$, then v is called a *subword* of u. Let $v = v_0 \dots v_n$ and $w = w_0 \dots w_m$ be finite words with $n \leq m$. Then we say v is a *prefix* (*suffix*, respectively) of w when $v_i = w_i$ ($v_i = w_{i+m-n}$, respectively) for $i = 0, \dots, n$. If v is not equal to w we call v a *proper prefix* (*suffix*, respectively). By a *k-dimensional word* we mean a map from a subset of $\mathbb{Z}^k$ to $\mathcal{A}$.

A *substitution* σ is a map from the alphabet $\mathcal{A}$ to the set of finite words on $\mathcal{A}$. We extend this map to a map $\sigma : \mathcal{A}^* \to \mathcal{A}^*$ by letting $\sigma(wa) = \sigma(w)\sigma(a)$ for $w \in \mathcal{A}^*, a \in \mathcal{A}$, where the operation is concatenation of words. By $\sigma^2(a)$ we mean $\sigma(\sigma(a))$, and similarly we define $\sigma^n(a)$ for $n > 2$. For background information on words and substitutions we refer to [34], [35], [27].

In this thesis we always take $\mathcal{A} = \{0, 1, \dots, k\}$ for some fixed $k \geq 1$. We define the incidence matrix M_σ of a substitution σ by

$$M_\sigma = \big(|\sigma(i)|_j\big)_{i=0,\dots,k;j=0,\dots,k}.$$

So the incidence matrix of a substitution σ indicates for each letter i exactly which letters, and how many of each, $\sigma(i)$ contains, but not in which order. Also we will consider the sequence $(u^{(n)})_{n\geq1}$ given by

$$u^{(n)} = \sigma^n(0) \qquad (n = 1, 2, \dots).$$

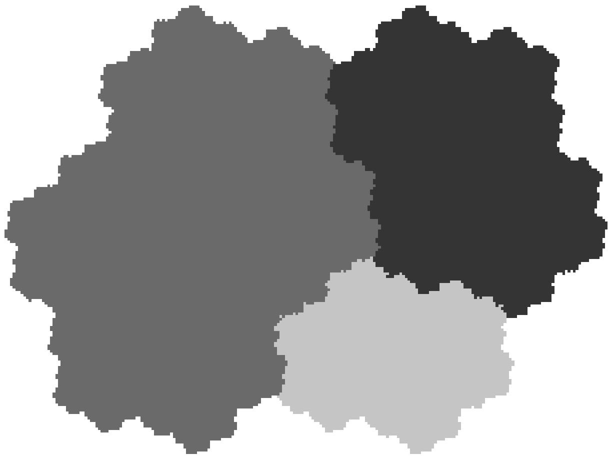
We will assume that the first letter of $\sigma(0)$ is 0, and $|\sigma(0)| > 1$. Then there exists a unique infinite word U such that each word $u^{(n)}$ is a prefix of U. In order to be able to say something about the frequencies of the different letters that appear in U, we assume that the incidence matrix M_σ has a positive dominant eigenvalue larger than 1, which we will call β. By dominant we mean that it is larger in absolute value than all other eigenvalues. We put $\tau = 1/\beta$. If this assumption is satisfied, then

$$\vec{b} := \lim_{n\to\infty} (|u^{(n)}|_0, \ldots, |u^{(n)}|_k)/|u^{(n)}|$$

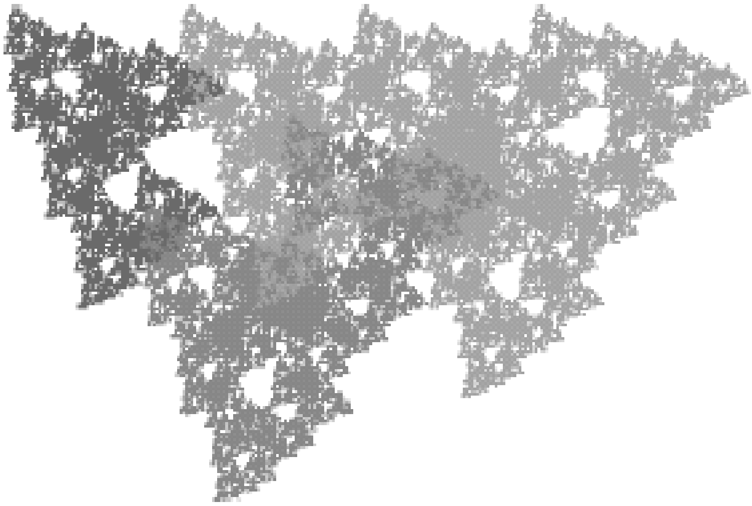
is well-defined, see [29] Lemma 2.2 and the remarks following it. The limit word U generates a broken halfline in $\mathbb{R}^{k+1}$ in the following way: we start in $\vec{0}$ and for $n = 0, 1, \ldots$ go 1 in the direction of the x_i-axis when $U_n = i$. The broken halfline approximates the halfline $\mathbb{R}_{\geq 0}\vec{b}$. For $i = 0, 1, \ldots, k$ let $\vec{e}_i^{(k+1)}$ be the vector of length $k+1$ that has only zeros, except for the $i+1$th entry which is 1. Then the integer points P_m for $m \in \mathbb{N}$ on the broken halfline are given by

$$P_0 = \vec{0}, \quad P_{m+1} = P_m + \vec{e}_{U_m}^{(k+1)}.$$

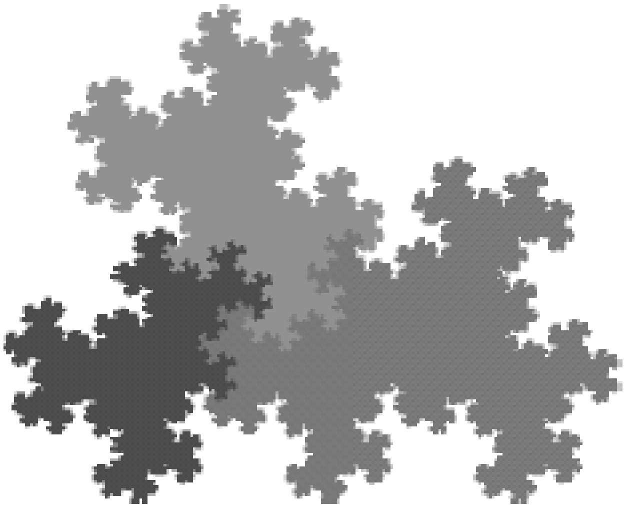
In 1982 G. Rauzy published an article in which he considers the integer points that lie on the broken halfline that corresponds to the so-called Tribonacci substitution $\sigma(0) = 01, \sigma(1) = 02, \sigma(2) = 0$, see [41]. He projects these points parallel to the halfline $\mathbb{R}_{\geq 0}\vec{b}$ onto a two-dimensional plane through the origin. The projected points define a fractal, which is now known as the Rauzy fractal. It can be seen as a tiling with which one can tile the whole plane. By coloring the projected points with three different shades, depending on whether the next step from the original point on the broken halfline is in the x, y or z direction, we get the picture below.



Note that this fractal contains three scaled copies of itself. It has been studied intensively by others, see for example [36], [37], [53], [55]. Since the article by Rauzy, many researchers have looked for similar results for other types of substitutions, see [3], [4], [6], [20], [25], [32], [48], [51], [54], [59]. The next picture comes from the flipped Tribonacci substitution given by $\sigma(0) = 01, \sigma(1) = 20, \sigma(2) = 0$.



The fractal below corresponds to the substitution $\sigma(0) = 01, \sigma(1) = 2, \sigma(2) = 0$, which can also be used to tile the plane.



In this thesis, we examine the behavior of the projections of the points on that part of the broken halfline that corresponds to $u^{(n)}$, for $n = 1, 2, 3, \ldots$. This part is given by the first $|u^{(n)}| + 1$ integer points on the broken halfline, connected by line segments of lengths 1. We call this the *cutting path* corresponding to $u^{(n)}$. We project these integer points parallel to the line between the origin and the endpoint of the cutting path, onto the plane through the origin that is perpendicular to the x_0-axis. We call this projection the *canonical projection* Φ_n^*. If we assume that M_σ is unimodular Φ_n^* is injective on its domain. We define a function $w^{(n)} : D_n \to \mathbb{Z}_{\geq 0}$, where $D_n \subset \mathbb{Z}^k$, by setting its value at position $\Phi_n^*(P_i)$ equal to i for $i = 0, \ldots, |u^{(n)}| - 1$. By abuse of notation we also call $w^{(n)}$ a k-dimensional word over the infinite alphabet $\{0, 1, 2, \ldots\}$. In Figure 1.1 we can see that if $k = 1$ and we take the Fibonacci substitution given by $\sigma(0) = 01$, $\sigma(1) = 0$ and consider $u^{(3)} = 01001$, then we find $w^{(3)} = 41302$.

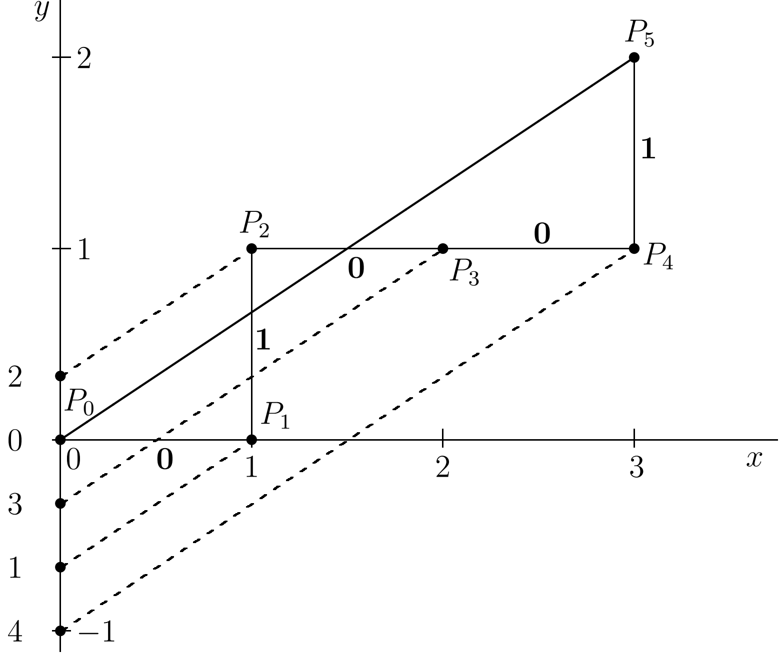
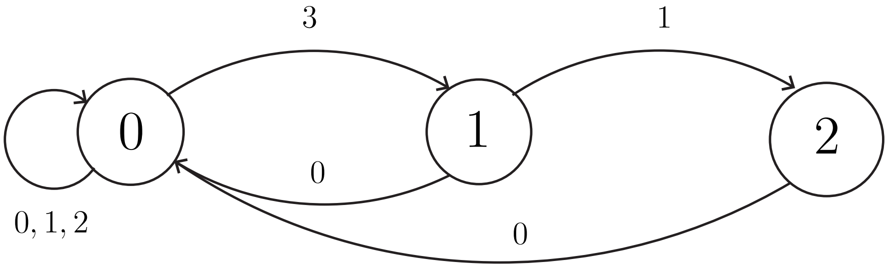


Figure 1.1: Projecting the word $u^{(3)} = 01001$ leads to the word $w^{(3)} = 41302$.

In general $w^{(n)}$ defines a roundwalk on its domain. By this we mean that there exist vectors $a_0, a_1, \dots, a_k \in \mathbb{Z}^k$ such that if $w^{(n)}$ has the letter m at position $x \in \mathbb{Z}^k$, then there exists an $i \in \{0, 1, \dots, k\}$ such that $w^{(n)}$ has the letter $m+1 \pmod{|u^{(n)}|}$ at position $x + a_i$. For more information on roundwalks see [15], [61].

For each substitution σ over the alphabet $\mathcal{A} = \{0, 1, \dots, k\}$ we define the associated *automaton* as follows. The set of states of the automaton equals $\mathcal{A}$, and if there is an i such that $\sigma(i) = a_0 a_1 \dots a_j$ for some j, then there exist arrows from state i to the states $a_0, a_1, \dots, a_j$ that are labelled $0, 1, \dots, j$, respectively. For example if $k = 2$ and $\sigma(0) = 0001$, $\sigma(1) = 02$, $\sigma(2) = 0$, the automaton is given by



A *path* in the automaton is a sequence of transitions from state to state, following the arrows, and where we start in state 0. Each path can be uniquely identified by a string of labels, corresponding to the respective transitions from state to state. The *language* $\mathcal{L}$ of an automaton is the set of all such strings, where we discard strings that start with 0. We define an ordering on $\mathcal{L}$ as follows. Let $u, v \in \mathcal{L}$. Then $u < v$ if and only if $|u| < |v|$ or $|u| = |v|$ and u is smaller than v in the lexicographical ordering (where we assume that the set of labels has the canonical ordering). We define an *abstract number system* by giving the nth string in the ordered set $\mathcal{L}$ the value n. In our example the first few values of the number system are $1, 2, 3, 10, 11, 12, 13, 20, 21, \dots$.

Recall the function $w^{(n)} : D_n \to \mathbb{Z}_{\geq 0}$, that we found by projecting the cutting path corresponding to $u^{(n)} = \sigma^n(0)$. We define $\hat{w}^{(n)}(\vec{x}) := w^{(n)}(\vec{x})/|u^{(n)}|$. It is easy to show that when $n' > n$, then $D_n \subset D_{n'}$. The sequence $\left(\hat{w}^{(n)}\right)_{n\in\mathbb{N}}$ converges to some k-dimensional limit word $\widehat{W}$, that we calculate by taking the limit on each $\vec{x} \in \mathbb{Z}^k$ for which there exists an m such that $\vec{x} \in D_m$, see [29] Theorem 3.5. If we write out each value of $\widehat{W}$ in the number system defined above, and only consider the digits right of the decimal point, we get the following amazing result, which was discovered by C. Fuchs and R. Tijdeman, [29] Theorem 5.3. If $\vec{x} \in \mathbb{Z}^k$ is in the domain of $w^{(n)}$, then $\widehat{W}(\vec{x})$ has a finite expansion of length at most n. Conversely, for all elements $\vec{x}$ in the domain of $w^{(n)}$ the expansion of $\widehat{W}(\vec{x})$ is different and all words in the language $\mathcal{L}$ of length at most n appear. Fuchs and Tijdeman use this result to state under which conditions a substitution over an alphabet of k letters leads to a sequence of words $\left(w^{(n)}\right)_{n\geq 1}$ that is defined on $\mathbb{Z}^k$ (or a subspace of $\mathbb{Z}^k$).

Using the abstract number system we define for each n integers $b_0, b_1, \ldots, b_{k+1}$ such that $b_0 = 0 < b_1 < \cdots < b_k < b_{k+1} = |u^{(n)}|$. We determine for each value p in $w^{(n)}$ the letter $i \in \mathcal{A}$ such that $b_i \leq p < b_{i+1}$, and replace p by i, to form the k-dimensional word $v^{(n)}$ over the alphabet $\mathcal{A}$. These words turn out to be k-dimensional rotational words. They can be seen as discretisations of a k-dimensional hyperplane in a $(k+1)$-dimensional space, see [5], [17]. In this thesis we examine the properties of these words $v^{(n)}$ for various substitutions σ.

In Chapter 2 we first consider the Fibonacci substitution mentioned above, and show that the sequence of words $\left(v^{(n)}\right)_{n\geq 1}$ converges to the two-sided Fibonacci word when n tends to infinity. Next we consider the Tribonacci substitution and using the Tribonacci number system we show that the two-dimensional limit word $V := \lim_{n\to\infty} v^{(n)}$ is defined on $\mathbb{Z}^2$, and given by

$$V_{i,j} = \begin{cases} 0 \text{ if } \{i\tau^2 + j\tau\} \in [0, \tau) \\ 1 \text{ if } \{i\tau^2 + j\tau\} \in [\tau, \tau + \tau^2) \\ 2 \text{ if } \{i\tau^2 + j\tau\} \in [\tau + \tau^2, 1). \end{cases}$$

In Chapters 3 and 4 we take $\mathcal{A} = \{0, 1\}$. We say that a word u is *balanced* if $||v|_0 - |w|_0| < 2$ for all subwords v, w of u of equal length. An infinite word is called *Sturmian* if it is balanced and not ultimately periodic. The history of Sturmian words goes back to J. Bernoulli in 1772 and E. B. Christoffel in 1875, [23]. The first in-depth study of Sturmian words was made by M. Morse and G. A. Hedlund in 1940, [38]. We call a finite word u a *Lyndon word* if $u = vw$, with u or v not the empty word, implies $u < wv$ in the lexicographical order and a *Christoffel word* if it is a non-constant balanced Lyndon word. We call a substitution σ over two letters *Sturmian* if $\sigma(u)$ is a Sturmian word for every Sturmian word u. A *Christoffel substitution* σ is a Sturmian substitution that has an incidence matrix with positive entries and determinant 1, and for which $\sigma(0)$ and $\sigma(1)$ are Christoffel words. In Chapter 3 we show that Sturmian substitutions are exactly those substitutions for which the sequence of projected words $\left(v^{(n)}\right)_{n\in\mathbb{N}}$ is space-filling. Furthermore, we show that when σ is a Christoffel substitution, there exists a Christoffel substitution τ such that $v^{(n+1)} = \tau\left(v^{(n)}\right)$ for each n. In Chapter 4 we show that when σ is a Sturmian substitution that has an incidence matrix with determinant 1, there exists a Sturmian substitution τ such that $v^{(n+1)} = \tau\left(v^{(n)}\right)$ for each n.

In Chapter 5 we consider substitutions over an alphabet containing arbitrarily many letters. We limit ourselves to a specific type of substitution, the so-called β-substitution. A β-substitution over the alphabet $\mathcal{A} = \{0, 1, \ldots, k\}$ is a substitution of the form

$$\sigma(0) = 0^{n_0}1, \quad \sigma(1) = 0^{n_1}2, \quad \ldots, \quad \sigma(k-1) = 0^{n_{k-1}}(k), \quad \sigma(k) = 0^{n_k},$$

where $n_0, n_k \geq 1$ and $n_i \geq 0$ for $i = 1, \ldots, k-1$. For more information about β-expansions and β-substitutions we refer the reader to [1], [2], [13], [28] and [56]. Because we want the incidence matrix of M_σ to be unimodular, we assume that $n_k = 1$. Furthermore we define the word $N := n_0 n_1 \dots n_{k-1} 1$ and make the following assumption:

each proper suffix of N is smaller than N.

Using the theory developed in [29] we show that if σ is a β-substitution under the stated conditions, then there exists a function ϕ so that $v^{(n+1)} = \phi\left(v^{(n)}\right)$ for every n.

Chapter 2

The Tribonacci substitution

2.1 The Fibonacci word

If there would exist Miss Word elections, the Fibonacci word would be an excellent candidate to win. In this section we give an overview of the properties of the Fibonacci word. For background information for this and other sections we refer to [35] and [12].

The Fibonacci substitution is the substitution ϕ over the 2-letter alphabet $\mathcal{A} := \{0, 1\}$ defined by $\phi(0) = 01$, $\phi(1) = 0$. If we start with 0 and repeatedly apply ϕ we get successively

$$\begin{aligned} &u_0 = 0 \\ &u_1 = 01 \\ &u_2 = 010 \\ &u_3 = 01001 \\ &u_4 = 01001010 \\ &u_5 = 0100101001001 \\ &u_6 = 010010100100101001010 \\ &\dots \end{aligned}$$

Note that $u_n = u_{n-1}u_{n-2}$ for every integer $n > 1$. This sequence of words converges to the so-called *Fibonacci word* $f = (f_m)_{m=1}^{\infty}$. If we define $F_0 = 0, F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for any integer $n > 1$, then the number of symbols of u_n, denoted by $|u_n|$, equals F_{n+2} and the number of 0's and 1's in u_n, denoted by $|u_n|_0$ and $|u_n|_1$, equals F_{n+1} and F_n, respectively. The Fibonacci word has the following properties:

- The frequency of 0 equals $\lim_{n\to\infty} \frac{F_{n+1}}{F_{n+2}} = -\frac{1}{2} + \frac{1}{2}\sqrt{5} =: \gamma$, the *frequency* of 1 equals $\lim_{n\to\infty} \frac{F_n}{F_{n+2}} = \frac{3}{2} - \frac{1}{2}\sqrt{5} = \gamma^2$, hence $\gamma + \gamma^2 = 1$, [35] sect.2.1.1. Since the frequencies are irrational, the Fibonacci sequence is non-periodic.

- The Fibonacci word is *balanced*, which means that for all subwords u, v of f of equal lengths we have $||u|_1 - |v|_1| \leq 1$, [35] sect.2.1.1. Note that $||u|_0 - |v|_0| \leq 1$ is an equivalent requirement.

- The Fibonacci word is *Sturmian*, that is, $P(n) = n + 1$ for every n, where $P(n)$ equals the number of different subwords of f of length n, [35] sect.2.1.1. Because a word is (ultimately) periodic if there exists an n for which $P(n) \leq n$, [24] sect.2, a Sturmian word is in this sense the most regular non-periodic word.

- The Fibonacci word is a *rotation word*, [35] sect.2.1.2. In fact
$$\forall m \geq 1: \quad f_m = \begin{cases} 0 \text{ if } \{(m+1)\gamma\} \in (0,\gamma] \\ 1 \text{ if } \{(m+1)\gamma\} \in (\gamma,1) \end{cases}$$
where $\{\cdot\}$ denotes the fractional part.

- The Fibonacci word is a *Beatty sequence*, [35] sect.2.1.2. In fact
$$\forall m \geq 1: \quad f_m = \lfloor (m+1)\gamma^2 \rfloor - \lfloor m\gamma^2 \rfloor.$$

- The Fibonacci word is a *cutting sequence*, [50]. In fact, in the x-y-plane the *broken Fibonacci halfline* that you get by starting in $(0,0)$ and going 1 in the direction of the x-axis when $f_m = 0$ and 1 in the direction of the y-axis when $f_m = 1$ is the best possible discrete approximation to the halfline given by $y = \gamma x,\ x \geq 0$. See Figure 2.1.

Up to now we have considered one-sided words and halflines. The definition of the broken Fibonacci word as a Beatty sequence (or as a rotation sequence) allows a straightforward extension to a biinfinite word $(f_m)_{m\in\mathbb{Z}}$. Actually there is another natural extension, viz.
$$\forall m \in \mathbb{Z}: \quad f^*_m = \lceil (m+1)\gamma^2 \rceil - \lceil m\gamma^2 \rceil \ = \ \begin{cases} 0 \text{ if } \{(m+1)\gamma\} \in [0,\gamma) \\ 1 \text{ if } \{(m+1)\gamma\} \in [\gamma,1). \end{cases}$$
The sequences $(f_m)_{m\in\mathbb{Z}}$ and $(f^*_m)_{m\in\mathbb{Z}}$ coincide except that $f^*_{-1} = f_0 = 0, f_{-1} = f^*_0 = 1$. Furthermore $f_m = f_{-m-1}$ for all $m > 0$ (cf. Theorem 2.2.4). Starting at the origin and going in both directions according to $(f_m)_{m\in\mathbb{Z}}$ we obtain an ideal discretisation, called the *broken Fibonacci line*, of the line $y = \gamma x$. It is given by (the 10 of $f_{-1}f_0$ is put in between vertical bars to indicate the positions -1 and 0)
$$\ldots 1001001010010|10|0100101001001\ldots\ .$$

2.2 Discretisation of the line

In this section we study projections of the broken Fibonacci halfline to the y-axis more closely. The integer points p_m for $m \in \mathbb{Z}_{\geq 0}$ on this halfline are fixed by $p_0 = (0,0)$, $p_m = p_{m-1} + \vec{e}_{f_m}$ where $\vec{e}_0, \vec{e}_1$ denote the unit vectors in the direction of the x-axis and y-axis, respectively. Let $\tilde{u}_m$ denote the word $f_1 f_2 \ldots f_m$. Hence $u_n = \tilde{u}_{F_{n+2}}$, $|\tilde{u}_m|_1 = \sum_{j=1}^m f_m$, $|\tilde{u}_m|_0 = m - |\tilde{u}_m|_1$ and $p_m = (|\tilde{u}_m|_0, |\tilde{u}_m|_1)$. Now we project each integer point parallel to the line $y = \gamma x$, $x \geq 0$ to the y-axis. By $P(p_m)$ we denote the second coordinate of the projection of p_m. See Figure 2.2. Note that a 0 means going one step to the right on the broken Fibonacci halfline, and for the projection this corresponds to going down γ along the y-axis. Similarly a 1 corresponds to going up 1 along the y-axis. We have
$$\begin{aligned} P(p_m) = |\tilde{u}_m|_1 - \gamma|\tilde{u}_m|_0 \ &= \ \lfloor (m+1)\gamma^2 \rfloor - \lfloor \gamma^2 \rfloor - \gamma(m - \lfloor (m+1)\gamma^2 \rfloor + \lfloor \gamma^2 \rfloor) \\ &= (1+\gamma)(-\{(m+1)\gamma^2\} + \{\gamma^2\}) = \gamma - (1+\gamma)\{(m+1)\gamma^2\} \in (-1,\gamma]. \end{aligned}$$
Hence the projected points are all in the interval $(-1,\gamma]$ on the y-axis. Since $P(p_{m+1}) - P(p_m)$ is either 1 or $-\gamma$, it follows that $P(p_{m+1}) - P(p_m) = 1$ if $P(p_m) \in (-1, -\gamma^2]$ and $P(p_{m+1}) - P(p_m) = -\gamma$ if $P(p_m) \in (-\gamma^2, \gamma]$. Thus we have an exchange of intervals.

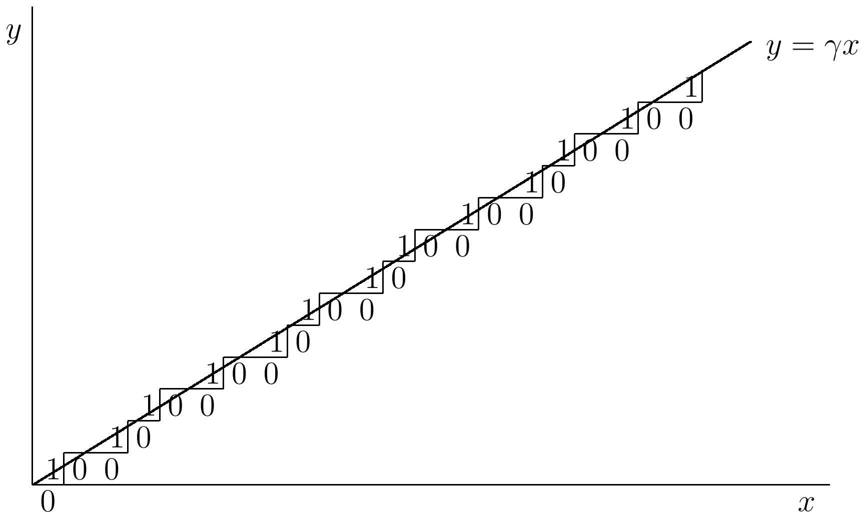


Figure 2.1: The Fibonacci sequence is a cutting sequence.

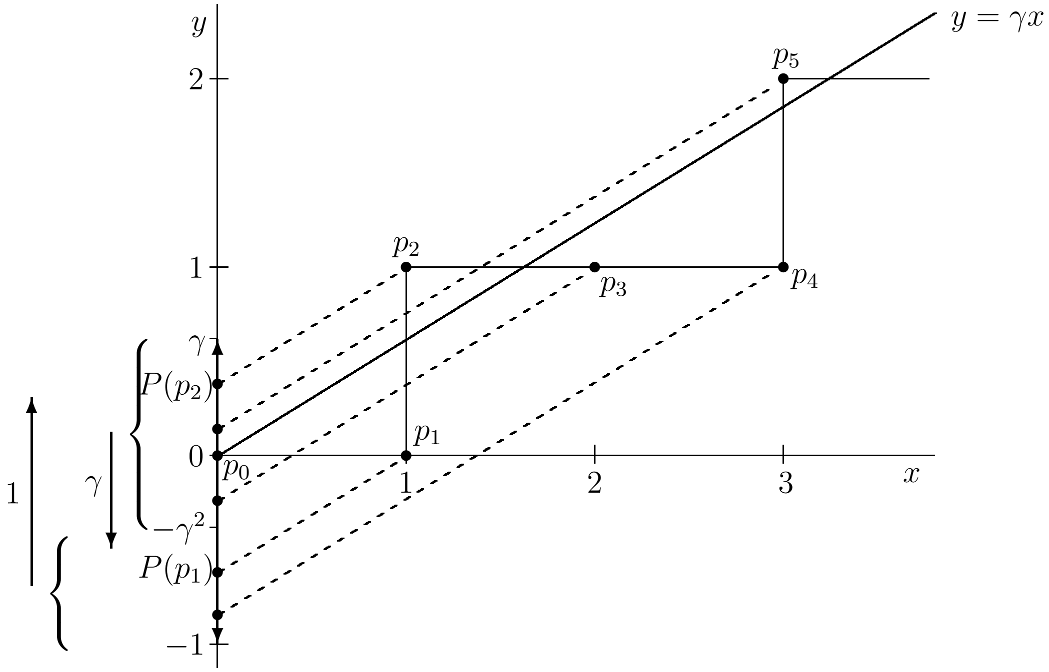


Figure 2.2: Projecting the broken Fibonacci halfline gives an exchange of intervals.

2.2.1 Incidence vectors and matrices

Let u be a finite word over a k-letter alphabet $\mathcal{A} := \{0, 1, \ldots, k-1\}$. Then we call $\vec{u} := (|u|_0, |u|_1, \ldots, |u|_{k-1})$ its *incidence vector*. If α is a substitution over $\mathcal{A}$, then the *incidence matrix* M_α belonging to α has $|\alpha(i-1)|_{j-1}$ as entry (i, j). Incidence vectors and matrices contain the global information (the numbers of each letter), but not the local information (precise order).

When applying substitution ϕ defined above to a finite word, the new incidence vector is obtained by multiplying the old one on the right by $M_\phi = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. Starting with the incidence vector $(1, 0)$ of u_0 and repeatedly multiplying with M_ϕ yields successively $(1, 1)$, $(2, 1)$, $(3, 2)$, $(5, 3)$, $(8, 5)$, $(13, 8), \ldots$. This agrees with the sequence (p_n, q_n) where p_n/q_n are the convergents of the continued fraction expansion of $\gamma^{-1} = \frac{1}{2} + \frac{1}{2}\sqrt{5}$.

2.2.2 Local behavior

We consider the broken line segment in the x-y-plane corresponding with the word u_n where we start from the origin and a 0 means going 1 in the direction of the x-axis and a 1 corresponds to going 1 in the direction of the y-axis. We project parallel to the line (depending on n) through the origin and the end point $p_{|u_n|}$ of the broken line segment, and we project to the y-axis. By $P_n(p_m)$ we denote the second coordinate of the projection of the point p_m. See Figure 2.3.

In Figure 2.3 we see that $u_2 = 010$ leads to $w'_2 = 102$ and that $u_3 = 01001$ leads to $w'_3 = 41302$, where w'_n is given by the increasing order of the projected points $P_n(p_m)$ on the y-axis. Note that $|w'_n| = |u_n| = F_{n+2}$ for every nonnegative integer n. We now write the projections w'_n not from down to up, but from left to right.

n	$\lvert u_n\rvert$	u_n									w'_n					
0	1	0	$\to$									0_0				
1	2	01	$\to$								1_1	0_0				
2	3	010	$\to$								1_1	0_0	2_0			
3	5	01001	$\to$						4_1	1_1	3_0	0_0	2_0			
4	8	01001010	$\to$					4_1	1_1	6_1	3_0	0_0	5_0	2_0	7_0	
5	13	0100101001001	$\to$	12_1	4_1	9_1	1_1	6_1	11_0	3_0	8_0	0_0	5_0	10_0	2_0	7_0

……

The subscripts refer to the corresponding values in u_n. The incidence matrix $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ of the Fibonacci substitution has determinant equal to -1. Since it is negative, for even n we reflect the w'_n in the origin and interchange the 0's and 1's in the subscripts. We call the resulting words w_n.

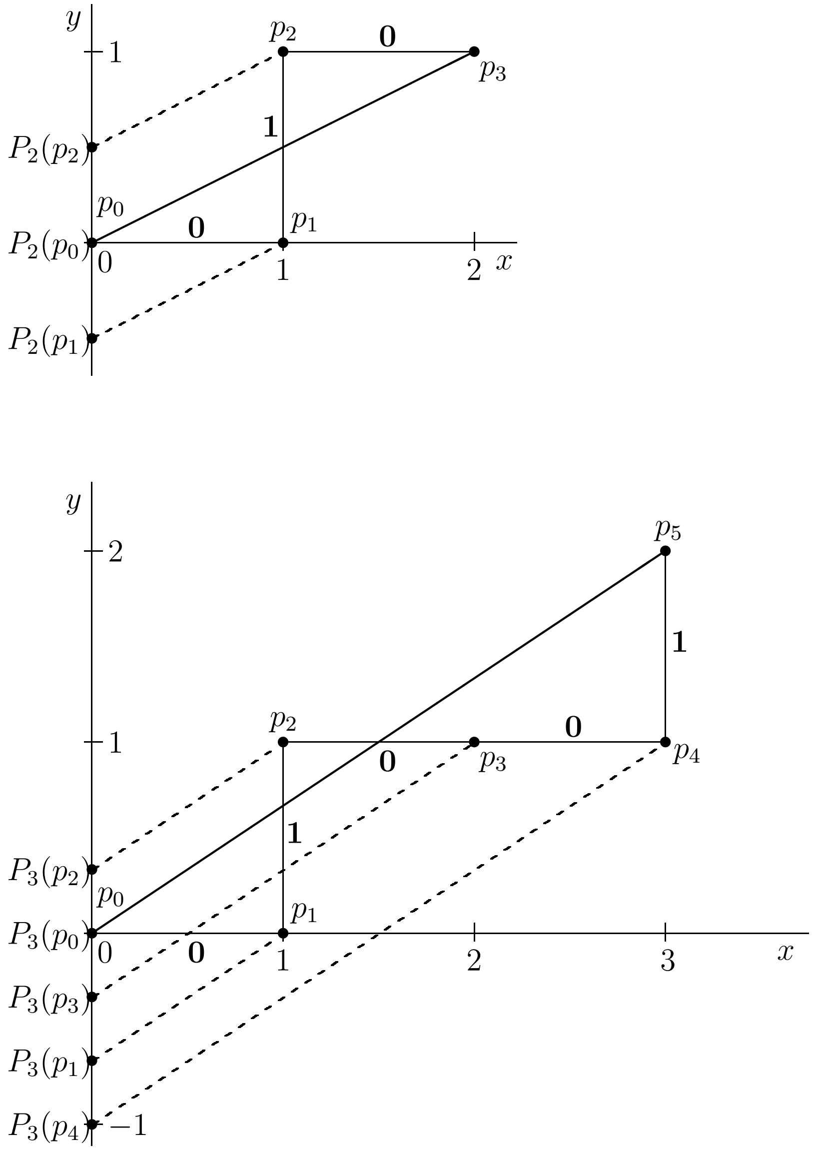


Figure 2.3: Projection of $u_2 = 010$ and $u_3 = 01001$.

$$
\begin{array}{lcl c ccccccc|cc|cccc}
n & |u_n| & u_n & & & & & & & & & \multicolumn{2}{c}{w_n} & & & & \\
0 & 1 & 0 & \rightarrow & & & & & & & & & 0_0 & & & & \\
1 & 2 & 01 & \rightarrow & & & & & & & & 1_1 & 0_0 & & & & \\
2 & 3 & 010 & \rightarrow & & & & & & & & 2_1 & 0_1 & 1_0 & & & \\
3 & 5 & 01001 & \rightarrow & & & & & & 4_1 & 1_1 & 3_0 & 0_0 & 2_0 & & & \\
4 & 8 & 01001010 & \rightarrow & & & & & & 7_1 & 2_1 & 5_1 & 0_1 & 3_1 & 6_0 & 1_0 & 4_0 \\
5 & 13 & 0100101001001 & \rightarrow & 12_1 & 4_1 & 9_1 & 1_1 & 6_1 & 11_0 & 3_0 & 8_0 & 0_0 & 5_0 & 10_0 & 2_0 & 7_0
\end{array}
$$

......

Observe that the numbers form a complete system of representatives mod $|u_n|$ = mod F_{n+2} and that the number i is placed in a position congruent to $iF_n(\text{mod } F_{n+2})$ for $i = 0, 1, \ldots, |u_n| - 1$. Moreover, the subscript is 0 if the next jump in the projection is to the left and it is 1 if the next jump is to the right. Subsequently we replace each number in w_n which is less than F_{n+1} with 0 and each number at least F_{n+1} with 1. We call the resulting word v_n. If we underline a word, it means that the letters are not only placed in order but also in the right position with respect to the origin.

$$
\begin{array}{l ccccccc|cc|cccc}
\underline{v_0} = & & & & & & & & & 0 & & & & \\
\underline{v_1} = & & & & & & & & 1 & 0 & & & & \\
\underline{v_2} = & & & & & & & & 1 & 0 & 0 & & & \\
\underline{v_3} = & & & & & & 1 & 0 & 1 & 0 & 0 & & & \\
\underline{v_4} = & & & & & & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
\underline{v_5} = & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0
\end{array}
$$

......

We shall show that the sequence of words $(\underline{v_n})_{n=0}^{\infty}$ converges to the two-sided Fibonacci word when n tends to infinity. It means that the generated broken line is the broken Fibonacci line.

Lemma 2.2.1. *The positions of w_n run from $-F_{n+1}$ up to $F_n - 1$ if n is odd, and from $-F_n$ up to $F_{n+1} - 1$ if n is even.*

Proof. For each projected point we have $P_n(p_m) = |\tilde{u}_m|_1 - \frac{F_n}{F_{n+1}}|\tilde{u}_m|_0$ with $0 \leq |\tilde{u}_m|_1 \leq F_n, 0 \leq |\tilde{u}_m|_0 \leq F_{n+1}$. Since $\gcd(F_n, F_{n+1}) = 1$, it follows that if $m < m'$ we get $P_n(p_m) = P_n(p_{m'}) \Rightarrow m = 0, m' = |u_n| = F_{n+2}$. In other words all F_{n+2} projected points $P_n(p_0), \ldots, P_n(p_{|u_n|-1})$ are distinct and of the form $\frac{x}{F_{n+1}}, x \in \mathbb{Z}$. Note that to get from one projected point to the next, we either add $\frac{F_{n+1}}{F_{n+1}}$ or subtract $\frac{F_n}{F_{n+1}}$ which is the same modulo $\frac{F_{n+2}}{F_{n+1}}$. Hence x passes through all cosets modulo F_{n+2}.

Now we construct w_n by placing index m in position $P_n(p_m)$, and for even n reflecting w_n in the origin.

We prove the lemma by induction. It is true for $n = 2, 3$. Assume the lemma is true for $n - 1$ and assume n is odd. To go from u_{n-1} to u_n in u_{n-1} every 0 is replaced with 01 and every 1 is replaced with 0. It follows that to go from w_{n-1} to w_n every jump to the right of length F_{n-2} is replaced by a jump to the left of length F_{n-1} followed by a jump to the right of length F_n and every jump to the left of length F_{n-1} in w_{n-1} leads to a jump to the left of the same length in w_n. Thus w_n consists of F_{n+1} numbers without gaps placed directly under w_{n-1} and F_n numbers on the left-hand side of them, possibly containing gaps. But because these F_n numbers on the left are in different cosets modulo F_{n+2} and are only one jump to the left of length F_n away from the F_{n+1} numbers directly under w_{n-1}, they must occupy exactly the first F_n positions left of w_{n-1}, and it follows that w_n has no gaps either. Because w_{n-1} runs from position $-F_{n-1}$ up to $F_n - 1$, it follows that w_n runs from position $-F_{n+1}$ up to $F_n - 1$. The situation for even n is similar. □

Remark. Lemma 2.2.1 implies that w_n has no gaps, so is defined on a block of integers.

Lemma 2.2.2. $\underline{v_n} = \begin{cases} \underline{v_{n-1}}v_{n-2} & \textit{if } n \textit{ is even,} \\ v_{n-2}\underline{v_{n-1}} & \textit{if } n \textit{ is odd.} \end{cases}$

Proof. Note that $p_{F_{n+1}} = (|u_{n-1}|_0, |u_{n-1}|_1) = (F_n, F_{n-1})$, so $P_n(p_{F_{n+1}}) = F_{n-1} - F_n\frac{F_n}{F_{n+1}} = \frac{(-1)^2}{F_{n+1}}$. Because for n is even w_n is reflected in the origin, w_n has F_{n+1} in position -1 for every n. Assume n is odd. Because F_{n+1} in w_n is placed directly below F_n in w_{n-1}, below every number smaller than F_n in w_{n-1} a number is placed in w_n that is smaller than F_{n+1}, and below every number larger than F_n in w_{n-1} a number is placed that is larger than F_{n+1}. It follows that the part of $\underline{v_n}$ placed directly below $\underline{v_{n-1}}$ is equal to $\underline{v_{n-1}}$. Because of the way w_n is constructed from w_{n-1} in the proof of Lemma 2.2.1, the left F_n numbers of $\underline{v_n}$ are an exact copy of the left F_n numbers of v_{n-1}, which by induction are an exact copy of the F_n numbers of v_{n-2}. This proves the lemma for n is odd. The case n is even is similar. □

We still need another lemma. By overlining we indicate that the word should be reversed. The lemma states that $01u_n$ for even n and $10u_n$ for odd n are palindromes.

Lemma 2.2.3. $\begin{array}{rcl} \overline{0\ 1\ u_n} & = & 0\ 1\ u_n \ \ \textit{if } n \textit{ is even,} \\ \overline{1\ 0\ u_n} & = & 1\ 0\ u_n \ \ \textit{if } n \textit{ is odd.} \end{array}$

Proof. The equalities hold for $n = 0$ and $n = 1$. Let n be even. Then, by induction on n,

$$\begin{aligned} 01u_n &= 01u_{n-1}u_{n-2} = 01u_{n-2}u_{n-3}u_{n-2} = \overline{u_{n-2}}10u_{n-3}u_{n-2} = \overline{u_{n-2}u_{n-3}}01u_{n-2} \\ &= \overline{u_{n-2}u_{n-3}u_{n-2}}10 = \overline{01u_{n-2}u_{n-3}u_{n-2}} = \overline{01u_{n-1}u_{n-2}} = \overline{01u_n}. \end{aligned}$$

For odd n the proof is similar. □

Now we can prove the theorem.

Theorem 2.2.4.

$$\lim_{n\to\infty} \underline{v_n} = \overline{f} \mid 1\ 0 \mid f$$

Proof. It suffices to show by induction on n that

$$0\underline{v_n}1 = \begin{cases} \overline{u_{n-2}}|10|u_{n-1} \text{ if } n \text{ is even,} \\ \overline{u_{n-1}}|10|u_{n-2} \text{ if } n \text{ is odd.} \end{cases}$$

We have $0\underline{v_2}1 = 0|10|01 = \overline{u_0}|10|u_1$ and $0\underline{v_3}1 = 010|10|01 = \overline{u_2}|10|u_1$ indeed. Suppose the statement is true for all integers less than n. Then, for even n, indicating by subscript p that the last digit is missing and by subscript s that the first digit is missing, from Lemma 2.2.2 and the induction hypothesis we get

$$0\underline{v_n}1 = 0\underline{v_{n-1}}v_{n-2}1 = \overline{u_{n-2}}|10|(u_{n-3})_p(\overline{u_{n-4}})_s10u_{n-3}.$$

Hence, by Lemma 2.2.3 and the facts that u_{n-4} starts with a 0 and u_{n-3} ends with a 1,

$$0\underline{v_n}1 \;=\; \overline{u_{n-2}}|10|(u_{n-3})_p(\overline{01u_{n-4}})_s u_{n-3} \;=\; \overline{u_{n-2}}|10|(u_{n-3})_p 1 u_{n-4}u_{n-3} \;=$$

$$\overline{u_{n-2}}|10|u_{n-3}u_{n-4}u_{n-3} \;= \overline{u_{n-2}}|10|u_{n-2}u_{n-3} \;=\; \overline{u_{n-2}}|10|u_{n-1}.$$

The proof in case n is odd is similar. $\square$

2.3 The Tribonacci word

We can ask ourselves what happens in higher dimensions. Does a substitution over a 3-letter alphabet generate a discretisation of a line or of a plane? Rauzy [41] considered as a generalisation of the Fibonacci substitution the substitution $\sigma : \begin{cases} 0 & \rightarrow & 01 \\ 1 & \rightarrow & 02 \\ 2 & \rightarrow & 0 \end{cases}$ over the alphabet $\{0, 1, 2\}$. Starting with 0 and repeatedly applying σ we get successively

$$\begin{array}{ll}
u_0 = 0 & \vec{u}_0 = (1,0,0) \\
u_1 = 01 & \vec{u}_1 = (1,1,0) \\
u_2 = 0102 & \vec{u}_2 = (2,1,1) \\
u_3 = 0102010 & \vec{u}_3 = (4,2,1) \\
u_4 = 0102010010201 & \vec{u}_4 = (7,4,2) \\
u_5 = 010201001020101020100102 & \vec{u}_5 = (13,7,4) \\
u_6 = 010201001020101020100102010201\ldots & \vec{u}_6 = (24,13,7) \\
\ldots &
\end{array}$$

On the right-hand side the incidence vectors are given. Note that $u_n = u_{n-1}u_{n-2}u_{n-3}$ for every integer $n > 2$. The limit word is called the *Tribonacci word* $t = (t_m)_{m=1}^{\infty}$. Note that if we define $T_0 = 0, T_1 = T_2 = 1, T_n = T_{n-1} + T_{n-2} + T_{n-3}$ for $n > 2$, then the number of symbols of u_n equals $T_{n+2}, |u_n|_0 = T_{n+1}, |u_n|_1 = T_n, |u_n|_2 = T_{n-1}$.

We check which properties of the Fibonacci word mentioned in Section 2.1 have analogues for the Tribonacci word.

- The frequency of 0 equals $\lim_{n\to\infty} \frac{T_{n+1}}{T_{n+2}} = \tau$, the frequency of 1 equals $\lim_{n\to\infty} \frac{T_n}{T_{n+2}} = \tau^2$, the frequency of 2 equals $\lim_{n\to\infty} \frac{T_{n-1}}{T_{n+2}} = \tau^3$, where $\tau+\tau^2+\tau^3 = 1$. Since the frequencies are irrational, the Tribonacci sequence is non-periodic.

- The Tribonacci word is not balanced with respect to each letter. The word u_5 contains subwords of length 3 without and a subword of length 3 with two 1's. The word u_6 contains subwords of length 5 without 2's and equally long subwords with two 2's. By applying σ to the subword 201020 of u_6 we get 0010201001, and it follows that u_7 contains subwords of length 9 with four 0's and with six 0's. However, it is true that for all subwords u, v of equal lengths of the Tribonacci word $||u|_i - |v|_i| \leq 2$ for $i = 0, 1, 2$, [10] sect.3.

- The Tribonacci word is *episturmian* on three letters which implies that $P(n) = 2n + 1$ for every n [8]. Because a word on three letters is periodic if the frequencies of its

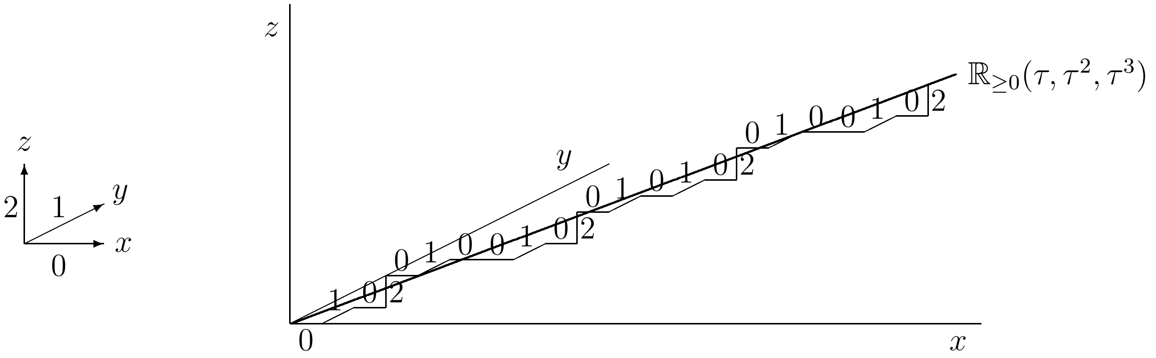


Figure 2.4: The Tribonacci sequence is a cutting sequence.

letters are independent over $\mathbb{Q}$ and there exists an n for which $P(n) \leq 2n$ [60], an episturmian word is in this sense a most regular non-periodic word on three letters with independent frequencies.

- The Tribonacci word is doubly rotational, as we shall specify in Remark 1 of Subsection 2.4.6.
- There is not a natural extension of the concept of Beatty sequence for 3-letter words.
- The Tribonacci word can be described as a cutting sequence, see [22]. In fact, in the x-y-z-space the *broken Tribonacci halfline* that we get by starting in $(0, 0, 0)$ and going 1 in the direction of the x-axis if $t_m = 0$, 1 in the direction of the y-axis if $t_m = 1$ and 1 in the direction of the z-axis if $t_m = 2$ provides a very good discrete approximation of the halfline $\mathbb{R}_{\geq 0}(\tau, \tau^2, \tau^3)$, see Figure 2.4.

2.4 Discretisation of the plane

In this section we study projections of the broken Tribonacci halfline to the y-z-plane. The integer points p_m for $m \in \mathbb{Z}_{\geq 0}$ on this line are given by $p_0 = (0, 0, 0),\ p_m = p_{m-1} + \vec{e}_{t_m}$ $(m > 0)$ where $\vec{e}_0, \vec{e}_1, \vec{e}_2$ denote the unit vectors in the direction of the x-axis, y-axis, z-axis, respectively. Now we project each integer point parallel to the halfline $\mathbb{R}_{\geq 0}(\tau, \tau^2, \tau^3)$ to the y-z-plane. Note that for the projection to the y-z-plane, a 0 means moving over $(-\tau, -\tau^2)$, a 1 corresponds to going 1 to the right and a 2 to going 1 upwards.

2.4.1 Local behavior

For $n = 1, 2, \ldots$ we consider the broken line segment corresponding with the word u_n in the x-y-z-space which is obtained by taking the segment of the broken Tribonacci halfline from p_0 to p_k where $k = |u_n|$. We project the integer points on this line segment parallel to the line through the origin and the end point of the broken line segment to the y-z-plane. Hence a 1 means for the projection going 1 in the direction of the y-axis and a 2 means for the projection going 1 in the direction of the z-axis. Since there are T_{n+1} 0's, T_n 1's and T_{n-1} 2's and the end point is projected to the origin, we see that for the projection a 0 means a translation over $(-\frac{T_n}{T_{n+1}}, -\frac{T_{n-1}}{T_{n+1}})$. Observe that projecting to any other generic plane would change the projection by some linear transformation only. We number the projections $P_n(p_m)$ of the points p_m according to the index m. See Figure 2.5.

We see that, after suitable linear transformations, $u_2 = 0102$, $u_3 = 0102010$ and $u_4 = 0102010010201$ lead to the configurations

$$\begin{array}{cccccccccc}
w_2 & & & & w_3 & & & & w_4 & \\
 & & & & & & & & 8_1 & 12_1 \\
 & & & & & & & 10_2 & 1_1 & 5_1 \\
2_0 & 3_2 & & & 4_0 & 6_0 & & 3_2 & 7_0 & 11_0 \\
0_0 & 1_1 & & 5_1 & 0_0 & 2_0 & & 9_0 & 0_0 & 4_0 \\
 & & & 1_1 & 3_2 & & & 2_0 & 6_0 &
\end{array}$$

respectively. The subscripts refer to the corresponding values in u_n. Observe that the subscript indicates which jump has to be made to reach the next number.

Subsequently we replace in w_n every number less than T_{n+1} with 0, every number at least T_{n+1} but less than $T_{n+1} + T_n$ with 1 and every number at least $T_{n+1} + T_n$ but less than T_{n+2} with 2 to obtain v_n. We underline the 0 which is at the origin. This yields

$$\begin{array}{cccccccccc}
v_2 & & & & v_3 & & & & v_4 & \\
 & & & & & & & & 1 & 2 \\
 & & & & & & & 1 & 0 & 0 \\
1 & 2 & & & 1 & 2 & & 0 & 1 & 2 \\
\underline{0} & 0 & & 1 & \underline{0} & 0 & & 1 & \underline{0} & 0 \\
 & & & 0 & 0 & & & 0 & 0 &
\end{array}$$

respectively. Observe that v_n is an extension of v_{n-1} for $n = 3, 4$ and that the new part of v_3 is obtained by translating v_2 over $(-1, -1)$ and the new part of v_4 by translating v_3 over $(0, 2)$. We shall study the behavior of the sequence $(v_n)_{n=1}^{\infty}$ and show that there are many similarities with the one-dimensional case of the broken Fibonacci line.

2.4.2 Incidence vectors and matrices

Incidence vectors and matrices have been introduced in Subsection 2.2.1. The incidence vector of u_2 is $\vec{u}_2 = (2, 1, 1)$. The incidence matrix of the substitution σ is given by $M_\sigma = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$. Hence $\sigma(u)$ has incidence vector $\vec{u}M_\sigma$. In particular, the incidence vector of the word u_n is given by $(1, 0, 0)M_\sigma^n$ for $n = 0, 1, \ldots$. By induction we find that

$$M_\sigma^n = \begin{pmatrix} T_{n+1} & T_n & T_{n-1} \\ T_n + T_{n-1} & T_{n-1} + T_{n-2} & T_{n-2} + T_{n-3} \\ T_n & T_{n-1} & T_{n-2} \end{pmatrix}.$$

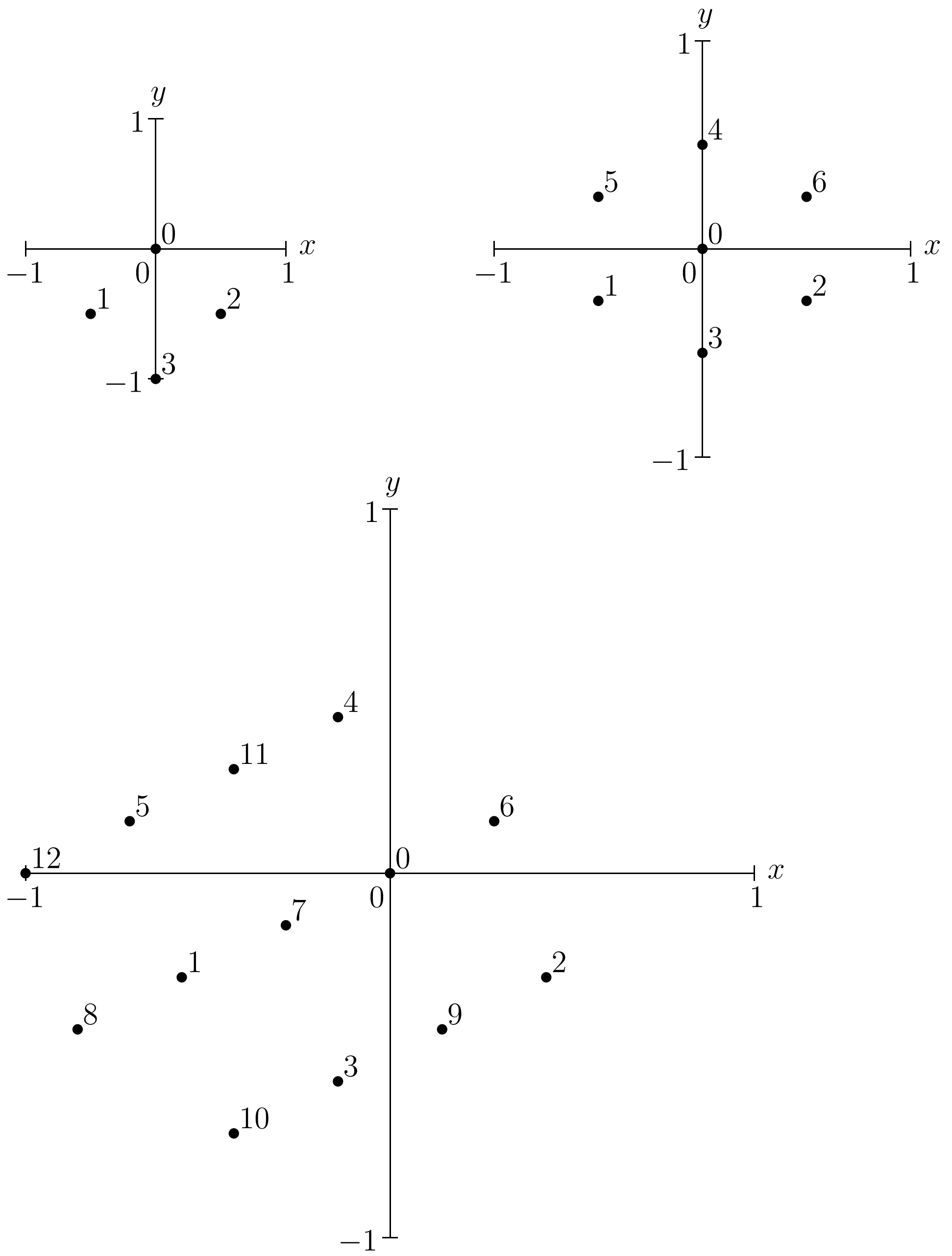


Figure 2.5: Projection w'_2 of $u_2 = 0102$, projection w'_3 of $u_3 = 0102010$ and projection w'_4 of $u_4 = 0102010010201$. We number the projections $P_n(p_m)$ of the points p_m according to m.

It follows that for $n \geq 2$ the word $u_n = \sigma^{n-2}(u_2)$ originates from the word $u_2 = 0102$ where in u_n the first T_n letters come from the first 0 in u_2, the next $T_{n-1} + T_{n-2}$ letters come from 1, the next T_n letters have their origin in the second 0 and the last T_{n-1} letters are generated by the 2. The total number is T_{n+2} indeed.

Suppose that the first m letters of u_{n-1} have incidence vector (a, b, c), so $p_m = (a, b, c)$. The letter in position m is mapped by the substitution σ to one or two letters in u_n the last of which has incidence matrix $(a + b + c, a, b) = (a, b, c)M_\sigma$.

Lemma 2.4.1. *For $n = 2, 3, \ldots$ we can choose the linear transformation applied to w'_n to get w_n such that in w_n the number at the origin is 0, the number in position $(1, 0)$ is T_n, the number in position $(0, 1)$ is T_{n+1} and every number m with $0 \leq m < T_{n+2}$ is in the lattice $\mathbb{Z}^2$.*

Proof. Without loss of generality we can place 0 in w_n at the origin for $n = 2, 3, \ldots$. We use induction on n. The lemma is true for $n = 2$. Suppose it is true for $n - 1 \geq 2$. Then the number in position $(1, 0)$ in w_{n-1} is the first number which originates from the 1 in $u_2 = 0102$, and the number in position $(0, 1)$ in w_{n-1} is the first number in u_{n-1} which comes from the second 0 in $u_2 = 0102$. Moreover, every number in w_{n-1} is in a position in the lattice $\mathbb{Z}^2$. If the position of a letter in w_{n-1} is (i, j), then $(i, j) = i * P_{n-1}((T_{n-1}, T_{n-2}, T_{n-3})) + j * P_{n-1}((T_n, T_{n-1}, T_{n-2}))$. When we apply σ to the word u_{n-1} to obtain u_n, we apply M_σ to the endpoint of broken line segment corresponding to u_{n-1} to obtain the endpoint of the broken line segment corresponding to u_n and therefore $P_n M_\sigma$ to get their projections. If (a, b, c) is the integer point on the broken line segment corresponding to u_{n-1} which is projected to (i, j), then

$$(a, b, c) \in i * (T_{n-1}, T_{n-2}, T_{n-3}) + j * (T_n, T_{n-1}, T_{n-2}) + \mathbb{R}(T_{n+1}, T_n, T_{n-1}).$$

Applying M_σ on the right we obtain

$$(a, b, c)M_\sigma \in i * (T_n, T_{n-1}, T_{n-2}) + j * (T_{n+1}, T_n, T_{n-1}) + \mathbb{R}(T_{n+2}, T_{n+1}, T_n).$$

If we now apply P_n we get that the image of (i, j) equals

$$i * P_n((T_n, T_{n-1}, T_{n-2})) + j * P_n((T_{n+1}, T_n, T_{n-1})).$$

Thus the number in position (i, j) in w_{n-1} is kept in w_n in the same position (i, j). In case the corresponding letter in u_{n-1} is a 2, then the image is the letter 0 and the next letter in u_n corresponds also with a number at a lattice point in w_n. It follows that a step 0 along the broken line segment corresponding to u_n is projected by P_n to the difference of two lattice points. Since all new points are obtained by translating a known lattice point by the projection of a step 0, all the new lattice points are also in the lattice and can be found by translating the corresponding old lattice points by the vector $P_n((1, 0, 0))$. This completes the proof of the lemma. $\square$

2.4.3 Continued lattices

Note that in w_4 the jump to reach the next number is $(0,2), (-1,-3), (2,-1)$, if the subscript is 0,1,2, respectively. For w_3 the translation vectors are $(-1,-1), (2,1), (0,2)$, respectively. We shall study the relation between the translation vectors of consecutive w_n's.

Lemma 2.4.2. *Let $n > 2$. Let $\vec{a}_0^{(n)}, \vec{a}_1^{(n)}, \vec{a}_2^{(n)}$ be the translation vectors corresponding to w_n for $n = 2, 3, \ldots$. Then, for $n = 3, 4, \ldots$,*

$$\vec{a}_0^{(n)} = \vec{a}_2^{(n-1)},\ \vec{a}_1^{(n)} = \vec{a}_0^{(n-1)} - \vec{a}_2^{(n-1)},\ \vec{a}_2^{(n)} = \vec{a}_1^{(n-1)} - \vec{a}_2^{(n-1)}.$$

Moreover, the domain of w_n is a fundamental domain of the lattice

$$\Lambda_n := \mathbb{Z}(\vec{a}_1^{(n)} - \vec{a}_0^{(n)}) + \mathbb{Z}(\vec{a}_2^{(n)} - \vec{a}_0^{(n)}).$$

If $d_0\vec{a}_0^{(n)} + d_1\vec{a}_1^{(n)} + d_2\vec{a}_2^{(n)} = \vec{0}$ for integers d_0, d_1, d_2 then $d_i = kT_{n-i+1}$ for some $k \in \mathbb{Z}$ and $i = 0, 1, 2$.
The number m in position (i, j) in w_n is congruent to $iT_n + jT_{n+1} \pmod{T_{n+2}}$ and (i, j) is congruent to $m\vec{a}_0^{(n)} \pmod{\Lambda_n}$.

Proof. The statements are correct for $n = 3$ with $\vec{a}_0^{(2)} = (1,0), \vec{a}_1^{(2)} = (-1,1), \vec{a}_2^{(2)} = (-1,-1)$ and $\vec{a}_0^{(3)} = (-1,-1), \vec{a}_1^{(3)} = (2,1), \vec{a}_0^{(3)} = (0,2)$. The lattice $\Lambda_3 = \mathbb{Z}(3,2) + \mathbb{Z}(1,3)$ has determinant $t_5 = 7$ and the domain of w_3 is a fundamental domain of Λ_3. The number 6, for example, is in position $(1,1)$ in w_3 and indeed $T_3 + T_4 \equiv 6 \pmod 7$ and $(1,1) \equiv (-6,-6) \pmod{\Lambda_3}$, since $(7,7) = 2(3,2) + (1,3)$.

Suppose the statements are true for values smaller than n. According to the substitution σ each jump 0 in w_{n-1} is replaced with a jump 0 followed by a jump 1 in w_n, each jump 1 in w_{n-1} is replaced with a jump 0 followed by a jump 2 in w_n, and each jump 2 is replaced with a jump 0. Hence $\vec{a}_0^{(n-1)} = \vec{a}_0^{(n)} + \vec{a}_1^{(n)}, \vec{a}_1^{(n-1)} = \vec{a}_0^{(n)} + \vec{a}_2^{(n)}, \vec{a}_2^{(n-1)} = \vec{a}_0^{(n)}$. This implies the first statement of the lemma.

Suppose $d_0\vec{a}_0^{(n)} + d_1\vec{a}_1^{(n)} + d_2\vec{a}_2^{(n)} = \vec{0}$ for integers d_0, d_1, d_2. Then

$$d_0\vec{a}_2^{(n-1)} + d_1(\vec{a}_0^{(n-1)} - \vec{a}_2^{(n-1)}) + d_2(\vec{a}_0^{(n-1)} - \vec{a}_2^{(n-1)}) = \vec{0}.$$

By the induction hypothesis there is an integer k such that $d_1 = kT_n, d_2 = kT_{n-1}$,
$d_0 - d_1 - d_2 = kT_{n-2}$, which implies $d_0 = k(T_n + T_{n-1} + T_{n-2}) = kT_{n+1}$. This proves the third statement.

In w_n the number $m + 1$ is reached by a jump $\vec{a}_0^{(n)}, \vec{a}_1^{(n)}$ or $\vec{a}_2^{(n)}$ from the number m. Since each vector is congruent to $\vec{a}_0^{(n)} (\text{mod } \Lambda_n)$ and the number 0 is at the origin, it is immediate that m is in a position congruent to $m\vec{a}_0^{(n)} (\text{mod } \Lambda_n)$. This is the last assertion.

Suppose $P_n(p_{m_1})$ and $P_n(p_{m_2})$ with $0 \leq m_1 < m_2 < T_{n+2}$ are in positions congruent modulo Λ_n. Then $(m_2 - m_1)\vec{a}_0^{(n)} \in \mathbb{Z}(\vec{a}_1^{(n)} - \vec{a}_0^{(n)}) + \mathbb{Z}(\vec{a}_2^{(n)} - \vec{a}_0^{(n)})$. Let i and j be integers such that

$$(m_2 - m_1)\vec{a}_0^{(n)} = i(\vec{a}_1^{(n)} - \vec{a}_0^{(n)}) + j(\vec{a}_2^{(n)} - \vec{a}_0^{(n)}).$$

Then there exists an integer k such that $m_2 - m_1 + i + j = kT_{n+1}, -i = kT_n, -j = kT_{n-1}$. Hence $m_2 - m_1 = kT_{n+2}$, which yields a contradiction. Thus the domain of w_n is a fundamental domain of Λ_n. This is the second assertion.

Suppose there is an m in position (i, j) of w_n. Then $(i, j) \equiv m\vec{a}_0^{(n)} (\text{mod } \Lambda_n)$. We know from Lemma 2.4.1 and the last assertion of Lemma 2.4.2 that $(1, 0) \equiv T_n\vec{a}_0^{(n)} (\text{mod } \Lambda_n)$ and $(0, 1) \equiv T_{n+1}\vec{a}_0^{(n)} (\text{mod } \Lambda_n)$. Thus $m\vec{a}_0^{(n)} \equiv (iT_n + jT_{n+1})\vec{a}_0^{(n)} (\text{mod } \Lambda_n)$. As in the preceding paragraph it follows that $T_{n+2} \mid m - iT_n - jT_{n+1}$. This completes the proof. $\square$

Corollary 2.4.3. *If μ is in v_n in position (i, j), then*

$$\mu = \begin{cases} 0 \text{ if } iT_n + jT_{n+1} \ (\text{mod } T_{n+2}) \in [0, T_{n+1}) \\ 1 \text{ if } iT_n + jT_{n+1} \ (\text{mod } T_{n+2}) \in [T_{n+1}, T_{n+1} + T_n) \\ 2 \text{ if } iT_n + jT_{n+1} \ (\text{mod } T_{n+2}) \in [T_{n+1} + T_n, T_{n+2}). \end{cases}$$

Moreover, the value in position (i, j) will remain μ in v_h for all $h > n$.

Proof. According to Lemma 2.4.2 the number m in position (i, j) in w_n is congruent to $iT_n + jT_{n+1}$ (mod T_{n+2}). By definition of v_n, m is replaced with a 0 if $0 \leq m < T_{n+1}$, by a 1 if $T_{n+1} \leq m < T_{n+1} + T_n$, by a 2 if $T_{n+1} + T_n \leq m < T_{n+2}$. Since $0 \leq m < T_{n+2}$ this proves the first assertion.

The value μ in v_n is 0, 1, 2, depending on whether the original point on the broken line segment corresponding to u_n is among the first T_{n+1} integer points, among the next T_n integer points, among the remaining T_{n-1} integer points of the broken line segment, respectively. If we apply σ to the word u_n, hence M_σ to the broken line segment, then the images of the integer points are among the first T_{n+2} integer points, among the next T_{n+1} integer points, among the remaining T_n integer points of the broken line segment corresponding to u_{n+1}, respectively. Therefore the resulting value m in w_{n+1} satisfies $0 \leq m < T_{n+1}, T_{n+1} \leq m < T_{n+1} + T_n, T_{n+1} + T_n \leq m < T_{n+2}$ in the respective cases. Hence, the corresponding values in v_{n+1} are 0, 1, 2, respectively. Thus the value of μ remains unchanged. $\square$

2.4.4 The Tribonacci number system

The preceding theory implies that $\lim_{n\to\infty} v_n$ exists. In this subsection we study the Tribonacci number system to be able to show that every integer point is contained in the domain of v_n for sufficiently large n. Therefore the limit function will be "space-filling". Lemma 2.4.4 and its consequences can be derived from more general results on so-called beta-expansions by Frougny and Solomyak [28] and by Akiyama [2]. Since we want to keep the paper self-contained we present a direct proof.

The Tribonacci number system is obtained by writing the nonnegative integers successively in the two-letter alphabet $\{0, 1\}$ while at the same time not allowing three consecutive digits 1. So $0 \to 0, 1 \to 1, 2 \to 10, 3 \to 11, 4 \to 100, 5 \to 101, 6 \to 110, 7 \to 1000, 8 \to 1001, 9 \to 1010, 10 \to 1011, 11 \to 1100, 12 \to 1101, 13 \to 10000, \ldots$. It follows by induction on n that for $n > 0$ the number T_n is expressed by a 1 followed by $n - 2$ 0's.

Let τ be the positive root of the polynomial $x^3 + x^2 + x - 1$. Hence $\tau \approx 0.5437$. We shall apply the following result. The condition $k_n k_{n+1} k_{n+2} = 0$ says that $k_N k_{N-1} \dots k_1$ is a Tribonacci representation of some nonnegative integer.

Lemma 2.4.4. *a) Every number of the form* $\sum_{n=1}^{N} k_n \tau^n$ *with* $k_n \in \{0,1\}, k_N \neq 0$ *and* $k_n k_{n+1} k_{n+2} = 0$ *for* $n = 1, 2, \dots, N-2$, *can be uniquely expressed as* $a + b\tau + c\tau^2$ *with* $a, b, c \in \mathbb{Z}$ *and* $0 < a + b\tau + c\tau^2 < 1$.
b) Every number $a + b\tau + c\tau^2$ *with* $a, b, c \in \mathbb{Z}$ *and* $0 < a + b\tau + c\tau^2 < 1$ *can be uniquely expressed as a finite sum* $\sum_{n=1}^{N} k_n \tau^n$ *with* $k_n \in \{0,1\}, k_N \neq 0$ *and* $k_n k_{n+1} k_{n+2} = 0$ *for* $n = 1, 2, \dots, N-2$.

Proof. a) Let $\sum_{n=0}^{N} k_n \tau^n$ be a number of the specified form. Then $0 \leq \sum_{n=0}^{N} k_n \tau^n < 1$. We can replace $k_N \tau^N$ with $-k_N \tau^{N-1} - k_N \tau^{N-2} - k_N \tau^{N-3}$ to transform it into an expression $\sum_{n=0}^{M} k'_n \tau^n$ with $k'_n \in \mathbb{Z}$ for $n = 0, 1, \dots, M$ and $M < N$. By iterating this reduction we eventually arrive at the wanted representation $a + b\tau + c\tau^2$. All such values are distinct, since τ is an algebraic number of degree 3.

b) For a number $\sum_{n=0}^{N} k_n \tau^n$ with integer coefficients k_n we define its *weight* as $\sum_{n=1}^{N} |k_n|$. Let $k_i \tau^i + k_{i+1} \tau^{i+1} + k_{i+2} \tau^{i+2}$ be some number with $k_i, k_{i+1}, k_{i+2} \in \mathbb{Z}$ and $0 < k_i \tau^i + k_{i+1} \tau^{i+1} + k_{i+2} \tau^{i+2} < \tau^{i-1}$. We claim that if $k_i \tau^i + k_{i+1} \tau^{i+1} + k_{i+2} \tau^{i+2} < \tau^i$, then we can find integers $k'_{i+1}, k'_{i+2}, k'_{i+3}$ such that $k_i \tau^i + k_{i+1} \tau^{i+1} + k_{i+2} \tau^{i+2} = k'_{i+1} \tau^{i+1} + k'_{i+2} \tau^{i+2} + k'_{i+3} \tau^{i+3}$ and $|k_i| + |k_{i+1}| + |k_{i+2}| \geq | k'_{i+1}| + |k'_{i+2}| + |k'_{i+3}|$ and if $k_i \tau^i + k_{i+1} \tau^{i+1} + k_{i+2} \tau^{i+2} \geq \tau^i$, then we can find integers $k'_{i+1}, k'_{i+2}, k'_{i+3}$ such that $k_i \tau^i + k_{i+1} \tau^{i+1} + k_{i+2} \tau^{i+2} = \tau^i + k'_{i+1} \tau^{i+1} + k'_{i+2} \tau^{i+2} + k'_{i+3} \tau^{i+3}$ and $|k_i| + |k_{i+1}| + |k_{i+2}| > |k'_{i+1}| + |k'_{i+2}| + |k'_{i+3}|$.

To prove our claim, first assume $k_i \leq 0$. Then $k_{i+1} > 0$ or $k_{i+2} > 0$. If $k_{i+1} \leq -k_i$ and $k_{i+2} \leq -k_i$, then $k_i \tau_i + k_{i+1} \tau^{i+1} + k_{i+2} \tau^{i+2} \leq k_i(\tau^i - \tau^{i+1} - \tau^{i+2}) = k_i \tau^{i+3} \leq 0$ which is excluded. If $k_{i+1} > -k_i$ or $k_{i+2} > -k_i$, then it is easy to verify the claim.

Now assume $k_i = 1$. If $k_{i+1} \tau^{i+1} + k_{i+2} \tau^{i+2} \geq 0$ then we are in the second case and the claim is obviously true. Otherwise $k_{i+1} < 0$ or $k_{i+2} < 0$. Since $\tau^i + k_{i+1} \tau^{i+1} + k_{i+2} \tau^{i+2} = (k_{i+1} + 1) \tau^{i+1} + (k_{i+2} + 1) \tau^{i+2} + \tau^{i+3}$, the last expression has weight $|k_{i+1} + 1| + |k_{i+2} + 1| + 1 \leq 1 + |k_{i+1}| + |k_{i+2}|$ which is the old weight.

Finally assume $k_i \geq 2$. If $k_i + k_{i+1} \leq 0$ or $k_i + k_{i+2} \leq 0$, then the claim is clearly true. So we assume $k_i + k_{i+1} \geq 1$ and $k_i + k_{i+2} \geq 1$. Hence $k_i \tau^i + k_{i+1} \tau^{i+1} + k_{i+2} \tau^{i+2} \geq \tau^i + (k_i + k_{i+1} - 1) \tau^{i+1} + (k_i + k_{i+2} - 1) \tau^{i+2} + (k_i - 1) \tau^{i+3} \geq \tau^i$. We know that

$$(k_i + k_{i+1} - 1) \tau^{i+1} + (k_i + k_{i+2} - 1) \tau^{i+2} + (k_i - 1) \tau^{i+3} < \tau^{i+1} + \tau^{i+2}$$

and has nonnegative coefficients. Hence $(k_i + k_{i+1} - 1, k_i + k_{i+2} - 1) \in \{(0,0), (1,0), (0,1), (0,2)\}$. This leaves only few possibilities for (k_{i+1}, k_{i+2}) and it is easy to check that the claim holds in each remaining case.

We use the claim to prove the lemma. Suppose a, b, c satisfy the conditions of b). Then we start with weight $|a| + |b| + |c|$. We apply the above procedure iteratively starting with replacing a with $a\tau + a\tau^2 + a\tau^3$. In every step the weight does not increase, but each time that $k_i \tau^i + k_{i+1} \tau^{i+1} + k_{i+2} \tau^{i+2} \geq \tau^i$ the weight decreases. If $\tau^j \leq a + b\tau + c\tau^2 < \tau^{j-1}$, then at step j the weight decreases by at least 1. If $a+b\tau+c\tau^2=\tau^j$, then we are finished. Otherwise there exists a j' such that $\tau^{j'} \le a+b\tau+c\tau^2-\tau^j < \tau^{j'-1}$. After at most $|a|+|b|+|c|$ such values j the tail has weight 0 and we have a representation $\sum_{j=1}^N k_j\tau^j$ for $a+b\tau+c\tau^2$ with $k_j \in \{0,1\}$ for all j and at most $|a|+|b|+|c|$ non-zero coefficients. Since we have used the greedy algorithm and $\tau^j+\tau^{j+1}+\tau^{j+2}=\tau^{j-1}$, we have $k_jk_{j+1}k_{j+2}=0$ for all j. □

The following statement follows immediately from the proof of Lemma 2.4.4 b).

Corollary 2.4.5. *Every number $a+b\tau+c\tau^2$ with $a,b,c \in \mathbb{Z}$ and $0 \le a+b\tau+c\tau^2 < 1$ can be written in Tribonacci representation $\sum_{j=1}^N k_j\tau^j$ with $k_j \in \{0,1\}$ for $j=1,\dots,N$ and $k_jk_{j+1}k_{j+2}=0$ for $j=1,\dots,N-2$ with at most $|a|+|b|+|c|$ non-zero coefficients k_j.*

Corollary 2.4.5 implies that every element of $\mathbb{Z}[\tau] \cap [0,1)$ has a finite Tribonacci expansion. Therefore τ has the "finiteness property (F)". Frougny and Solomyak [28] and Hollander [30] have given conditions for roots of polynomials $X^d-a_0X^{d-1}\dots-a_{d-1}$ to have property (F). Akiyama [2] has characterised all cubic algebraic integers with property (F).

2.4.5 Discretisation of the plane

In this section we show that the limit function $\lim_{n\to\infty} v_n$ is bilinear on $\mathbb{Z}^2$. The next lemma characterises the domain of v_n. By Lemma 2.4.4 b) we can write τ^n $(n>0)$ in precisely one way as $a+b\tau+c\tau^2$ with $a,b,c \in \mathbb{Z}$. We denote the ordered pair (c,b) as (i_n,j_n). We first observe that $i_n=i_{n+1}+i_{n+2}+i_{n+3}$ and $j_n=j_{n+1}+j_{n+2}+j_{n+3}$ for $n \ge 1$. Indeed this follows immediately from $\tau^n=\tau^{n+1}+\tau^{n+2}+\tau^{n+3}$ and the fact that τ is an algebraic number of degree 3. Using these identities we infer from the assertion of Lemma 2.4.2 by induction on n that

$$a_0^{(n)}=(i_n,j_n),\quad a_1^{(n)}=(i_{n-1}-i_n,j_{n-1}-j_n),\quad a_2^{(n)}=(i_{n+1},j_{n+1}).$$

Lemma 2.4.6. *The domain of v_n consists of all $(i,j) \in \mathbb{Z}^2$ such that the fractional part $\{i\tau^2+j\tau\}$ can be written as $\sum_{i=1}^n k_i\tau^i$ with $k_i \in \{0,1\}$ for $i=1,\dots,n$ and $k_ik_{i+1}k_{i+2}=0$ for $i=1,\dots n-2$.*

Proof. By induction on n. The statement is true for $n=1$. Suppose it is true for all values below n. Then the domain of v_{n-1} consists of all pairs $(i,j) \in \mathbb{Z}^2$ such that $\{i\tau^2+j\tau\}$ is of the form $\sum_{i=1}^{n-1} k_i\tau^i$ with $k_i \in \{0,1\}$ for $i=1,\dots,n-1$ and $k_ik_{i+1}k_{i+2}=0$ for $i=1,\dots,n-3$. According to the proof of Lemma 2.4.1 the domain of v_n (which is the domain of w_n) is the union of the domain of v_{n-1} and the domain of v_{n-1} translated over $a_2^{(n-1)}=a_0^{(n)}$. By the above formula for $a_0^{(n)}$ we know that $a_0^{(n)}=(i_n,j_n)$. Since $\{i_n\tau^2+j_n\tau\}=\tau^n$, points (i,j) in $v_n \setminus v_{n-1}$ are such that $\{i\tau^2+j\tau\}=\tau^n+\sum_{i=1}^{n-1}k_i\tau^i$ with $k_i \in \{0,1\}$ for $i=1,\dots,n-1$ and $k_ik_{i+1}k_{i+2}=0$ for $i=1,\dots,n-3$. Put $k_n=1$. Suppose $k_nk_{n-1}k_{n-2} \ne 0$. Then $k_n=k_{n-1}=k_{n-2}=1$ and $\sum_{i=n-2}^n k_i\tau^i \ge \tau^{n-3}$. This implies that the greedy algorithm has not been applied properly. Thus $k_n=0$ if $k_{n-1}=k_{n-2}=1$ and $k_nk_{n-1}k_{n-2}=0$. □

In fact the construction with continued lattices provides an efficient way to compute the Tribonacci expansions of numbers $i\tau^2+j\tau$ (mod 1) with $|i|,|j|$ below some bound. In Figure 2.6 the Tribonacci expansions are given for $|i| \le 3, |j| \le 4$. We use that

$$a_0^{(1)}=(0,1),\ a_0^{(2)}=(1,0),\ a_0^{(3)}=(-1,-1),\ a_0^{(4)}=(0,2),\ a_0^{(5)}=(2,-1),$$

$i\backslash j$	-3	-2	-1	0	1	2	3
4	1101100	1100001	1100011	1000100	1000110	1010101	1010000
3	1100101	1100000	1100010	1001	1011	1010100	1010110
2	1100100	1100110	1101	1000	1010	11001	11011
1	101001	101011	1100	1	11	11000	11010
0	101000	101010	101	0	10	10001	10011
-1	100001	100011	100	110	10101	10000	10010
-2	100000	100010	110001	110011	10100	10110	10001101
-3	100110	110101	110000	110010	10101001	10101011	10001100
-4	101000100	110100	110110	10101101	10101000	10101010	10000101

Figure 2.6: The Tribonacci expansions of $i\tau^2 + j\tau \pmod 1$ for $|i| \leq 3, |j| \leq 4$.

$$a_0^{(6)} = (-3,-2),\ a_0^{(7)} = (1,5),\ a_0^{(8)} = (4,-4),\ a_0^{(9)} = (-8,-3).$$

Now we arrive at the main result of this paper.

Theorem 2.4.7. *We have* $\lim_{n\to\infty} v_n = V := (V_{i,j})_{(i,j)\in\mathbb{Z}^2}$ *where*

$$V_{i,j} = \begin{cases} 0 \text{ if } \{i\tau^2 + j\tau\} \in [0,\tau) \\ 1 \text{ if } \{i\tau^2 + j\tau\} \in [\tau, \tau+\tau^2) \\ 2 \text{ if } \{i\tau^2 + j\tau\} \in [\tau+\tau^2, 1). \end{cases}$$

Proof. It follows from Lemma 2.4.6 that every integer point (i,j) is in the domain of V. Let (i,j) be in the domain of v_n. Then its value at (i,j) is determined in Corollary 2.4.3. Moreover it remains unchanged if $n \to \infty$. The condition in Corollary 2.4.3 can be rewritten as

$$i\frac{T_n}{T_{n+2}} + j\frac{T_{n+1}}{T_{n+2}} (\text{mod } 1) \in [0, \frac{T_{n+1}}{T_{n+2}}), [\frac{T_{n+1}}{T_{n+2}}, \frac{T_{n+1}+T_n}{T_{n+2}}), [\frac{T_{n+1}+T_n}{T_{n+2}}, 1),$$

respectively. By letting n tend to ∞, these intervals tend to $[0,\tau), [\tau, \tau+\tau^2), [\tau+\tau^2, 1)$, respectively. Since τ is an algebraic number of degree 3, the number $i\tau^2 + j\tau$ is a boundary point if and only if (i,j) equals $(0,0), (0,1)$ or $(1,1)$. Other points (i,j) are in a fixed half-open interval from some point on. However, it is easy to check that the numbers in v_n in the positions $(0,0), (0,1), (1,1)$ have the right values 0, 1, 2, respectively. $\square$

We have proved that the limit word V is a doubly rotational sequence, and even a BV-sequence. (Cf. Section 2.4.6)

2.4.6 Remarks

Remark 1. A *doubly rotational sequence* is a sequence $f : \mathbb{Z}^2 \to \{0,1,2\}$ for which numbers α, β, γ with $0 < \alpha < \alpha+\beta < 1$ exist such that, maybe after permuting the function values 0, 1 and 2 and changing f to $-f$,

$$f_{m,n} = \begin{cases} 0 \text{ if } \{m\alpha + n\beta + \gamma\} \in [0,\alpha) \\ 1 \text{ if } \{m\alpha + n\beta + \gamma\} \in [\alpha, \alpha+\beta) \\ 2 \text{ if } \{m\alpha + n\beta + \gamma\} \in [\alpha+\beta, 1) \end{cases}$$

We call a doubly rotational sequence a *BV-sequence*, if $1, \alpha, \beta$ are linearly independent over the rationals. These sequences were investigated by Berthé and Vuillon [16], [17] in their study of discretisations of a 2-dimensional plane in a three-dimensional space. BV-sequences can be considered as two-dimensional analogues of Sturmian sequences. Berthé and Tijdeman [14] showed that BV-sequences are not balanced in the sense that there are rectangles of the same size where the numbers of one letter differ more than 1. Berthé and Vuillon [17] showed that BV-sequences have the property that $P(m,n) = mn + m + n$ for all integers m, n. They also considered projections of such BV-words to words on two letters and obtained such (uniform recurrent) words with complexity $P(m,n) = mn + m$. There exists a conjecture by Nivat [39] stating that if there exist m, n such that $P(m,n) \leq mn$ then there is periodicity. Partial results to this conjecture can be found in [47], [26], [40].

Remark 2. The theory of continued lattices has been developed in much more general form than in Subsection 2.4.3 by Berthé and Tijdeman [15], [61]. Simpson and Tijdeman [52] applied it to obtain a multi-dimensional Fine and Wilf theorem.

2.5 Acknowledgements

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Chapter 3

Substitutions on two letters, cutting segments and their projections

3.1 Introduction

The history of Sturmian words goes back to J. Bernoulli in 1772 and Christoffel [23] (1875). The first in depth study of Sturmian words was made by Morse and Hedlund [38] in 1940. A Sturmian word induces a broken half-line, the so-called cutting line, which approximates a half-line through the origin quite well. See Series [50] (1985). We call a substitution σ Sturmian if σ maps every Sturmian word to a Sturmian word. In 1991 Séébold [49] showed that Sturmian substitutions that have a fixed point are exactly those substitutions that have Sturmian words as fixed points. For further information on Sturmian words and substitutions we refer to Lothaire [35], Ch. 2 and Pytheas Fogg [27], Ch. 6.

In 1982 Rauzy [41] introduced a fractal which is defined as the closure of the projection of the cutting line corresponding to the Tribonacci substitution $0 \to 01,\ 1 \to 02,\ 2 \to 0$. The analogues of this so-called Rauzy fractal have been studied for many other substitutions, see [35], [27]. An important question is whether the projection of the cutting line generates a tiling or not. This question was the motivation for the author and Tijdeman [46] to have a closer look at the structure of the projections of the finite cutting segments corresponding to $\sigma^n(0)$ in case σ is the Fibonacci or Tribonacci substitution. They found in the Fibonacci case that the projections of the integer points of the cutting line generate a two-sided Fibonacci word. In the Tribonacci case they found a close connection with number systems. The latter connection was generalized by Fuchs and Tijdeman in [29]. In the present paper we generalize the former property to unimodular substitutions defined over two letters.

We examine for which substitutions the projected points form a (doubly-infinite) word and which properties these words have. In Section 3.2 we start with some notation and definitions. Next in Section 3.3 we give some properties of the points that we get after projecting the cutting segment corresponding to a substitution. We show that the order of the projected points of $\sigma^n(0)$ is preserved in the projection of $\sigma^{n+1}(0)$. Then in Section 3.4 we define Sturmian substitutions, Sturmian matrices and prove some of their properties.

In Section 3.5 we consider substitutions for which the incidence matrix is unimodular, and we show that the projected points form a central word if and only if the substitution is Sturmian. We show how the number of 0's and 1's in these central words can be calculated from the incidence matrix of the original substitution.

In the final Section 3.6 we consider a special class of Sturmian substitutions, that we call Christoffel substitutions. We show that when one starts with a Christoffel substitution the projected points form Christoffel words, and that the relation between these words is again given by a Christoffel substitution. Moreover, if one applies the procedure of projecting the cutting segment corresponding to these Christoffel words again, the result will be the original substitution. This induces a duality on the set of Christoffel substitutions.

3.2 Notation and definitions

An *alphabet* $\mathcal{A}$ is a finite set of elements that are called *letters*. In this article we always assume $\mathcal{A} = \{0, 1\}$. A *word* is a function u from a finite or infinite block of integers to $\mathcal{A}$. If this block of integers contains 0 we call u a *central word*. If $a \in \mathcal{A}$ and $u(k) = a$ we say u has the letter a at position k, denoted by $u_k = a$. If the block of integers is finite we call u a finite word, otherwise it is an infinite word. If $v = v_0 \dots v_m$ is a finite word and if $u = u_0 u_1 \dots$ is a finite or infinite word, and there exists a k such that $v_l = u_{k+l}$ for $l = 0, \dots m$, then v is called a *subword* of u. If a word u is finite, we denote by $|u|$ the number of letters in u, and by $|u|_a$ the number of occurrences of the letter a in u.

A *substitution* σ is an application from an alphabet $\mathcal{A}$ to the set of finite words. It extends to a morphism by concatenation, that is, $\sigma(uv) = \sigma(u)\sigma(v)$. It also extends in a natural way to a map over infinite words u. A *fixed point* of a substitution σ is an infinite word u with $\sigma(u) = u$.

A substitution over the alphabet $\mathcal{A}$ is *primitive* if there exists a positive integer k such that, for every a and b in $\mathcal{A}$, the letter a occurs in $\sigma^k(b)$.

We denote the largest integer y such that $y \leq x$ by $\lfloor x \rfloor$, the smallest integer y such that $y \geq x$ by $\lceil x \rceil$ and we put $\{x\} = x - \lfloor x \rfloor$.

Definition. Let $u = u_0 \dots u_{m-1}$ be a finite word. The *cutting segment* in the x-y-plane corresponding to u consists of $m + 1$ integer points p_i given by $p_i = (|u_0 \dots u_{i-1}|_0, |u_0 \dots u_{i-1}|_1)$ for $i = 0, \dots, m$, connected by line segments of lengths 1.

In Figure 3.1 we show the cutting segment corresponding to $u = 01001$.

Let $u = u_0 \dots u_{m-1}$ be a finite word containing at least one zero. Consider the cutting segment corresponding to u, and draw the line through the origin and the end point of the segment, given by $y = \frac{|u|_1}{|u|_0} x$. We project each integer point p_i on the cutting segment parallel to this line to the y-axis. By $P(p_i)$ we denote the second coordinate of the projection of p_i. It is clear that $P(p_0) = P(p_m)$. See Figure 3.2 for an example.

Lemma 3.2.1. *Let $u = u_0 \dots u_{m-1}$ be a finite word containing at least one zero and let $P(p_i)$ for $i = 0, \dots, m$ be defined as above. Then for every $i = 0, \dots, m$ we have $P(p_i) = (|u_0 \dots u_{i-1}|_1 |u|_0 - |u_0 \dots u_{i-1}|_0 |u|_1)/|u|_0 \in 1/|u|_0 \mathbb{Z}$.*

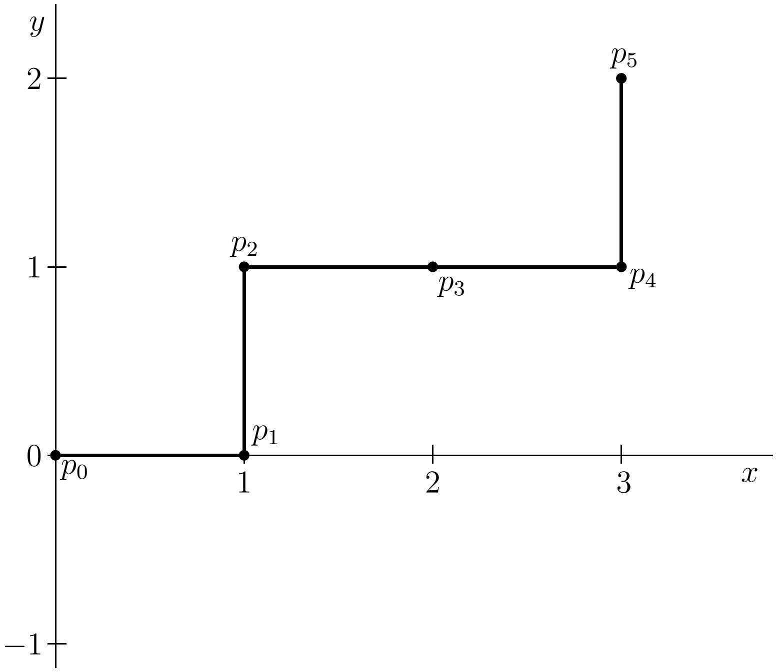

Figure 3.1: The cutting segment corresponding to $u = 01001$.

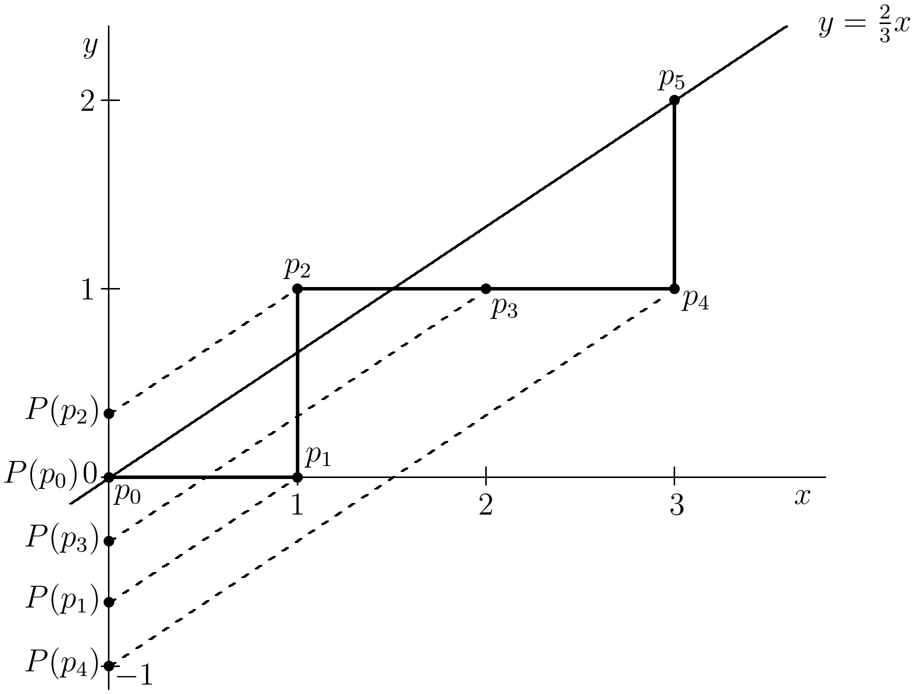

Figure 3.2: Projecting the cutting segment corresponding to $u = 01001$.

Proof. We use induction on i. Obviously $P(p_0) = 0$. Assume the lemma is valid for $i \geq 0$. If $u_i = 0$ then $P(p_{i+1}) = P(p_i) - |u|_1/|u|_0 = (|u_0 \ldots u_i|_1|u|_0 - |u_0 \ldots u_i|_0|u|_1)/|u|_0$, and if $u_i = 1$ then $P(p_{i+1}) = P(p_i) + 1 = (|u_0 \ldots u_i|_1|u|_0 - |u_0 \ldots u_i|_0|u|_1)/|u|_0$. ☐

The following lemma says that when the numbers of 0's and 1's in u are relatively prime, the points p_i are projected to distinct points.

Lemma 3.2.2. *Let $u = u_0 \ldots u_{m-1}$ be a finite word containing at least one zero, let $|u|_1 > 0$ if $m > 1$, let $\gcd(|u|_0, |u|_1) = 1$ and let $P(p_i)$ for $i = 0, \ldots, m$ be defined as above. Then for $i, j \in \{0, \ldots, m-1\}$ and $i \neq j$ we have $P(p_i) \neq P(p_j)$.*

Proof. Suppose $P(p_i) = P(p_j)$. Put $x = |u_0 \ldots u_{i-1}|_1$, $y = |u_0 \ldots u_{i-1}|_0$ and $x' = |u_0 \ldots u_{j-1}|_1$, $y' = |u_0 \ldots u_{j-1}|_0$. Then $P(p_i) = (x|u|_0 - y|u|_1)/|u|_0$ and $P(p_j) = (x'|u|_0 - y'|u|_1)/|u|_0$, hence $x|u|_0 - y|u|_1 = x'|u|_0 - y'|u|_1$. Since $\gcd(|u|_0, |u|_1) = 1$ and $|x - x'| \leq| u|_1, |y - y'| \leq| u|_0$ with at least one strict inequality, we have $x = x'$, $y = y'$, hence $i = j$. ☐

Let $D = \left\{-|u|_0 P(p_i) \middle| i \in \{0, \ldots, m-1\}\right\}$, hence D is a subset of $\mathbb{Z}$ containing m elements under the hypotheses of Lemma 3.2.2. We define the *central function* $w : D \to \{0, \ldots, m-1\}$ as follows. If $P(p_i) = k/|u|_0$ then $w(-k) = i$. We say w has number i at position $-k$. If $k_1 < k_2 < k_3$ are integers and w has numbers at positions k_1, k_3 but not at k_2, we say w has a gap at k_2. If the central function w has no gap we call w a *central block*. Note that a central function always has the number 0 at position 0. By $|w|$ we mean the number of positions on which w is defined, hence $|w| = |u|$ if w is the central function corresponding to u. The following lemma shows that $w(k+1) - w(k)$ is constant modulo $|u|$.

Lemma 3.2.3. *Let u be a finite word with $\gcd(|u|_0, |u|_1) = 1$ and w the central function corresponding to u. If w has a number at position k, then $w(k) = k|u|_1^{-1} \pmod{|u|}$, where the inverse is taken modulo $|u|$.*

Proof. Suppose w has the number i at position k. Then we obtain successively from $|u|_0 + |u|_1 = |u|$ that

$$|u_0 \ldots u_{i-1}|_0|u|_1 - |u_0 \ldots u_{i-1}|_1|u|_0 = k,$$

$$|u_0 \ldots u_{i-1}|_0|u|_1 + |u_0 \ldots u_{i-1}|_1|u|_1 \equiv k \pmod{|u|},$$

$$i|u|_1 \equiv k \pmod{|u|}.$$

☐

Remark. If $w(k) = i$ then $w(k + |u|_1) = i + 1$ in case $u_i = 0$, and $w(k - |u|_0) = i + 1$ in case $u_i = 1$. We say that to move from number i to $i + 1$ in w we either "jump" $|u|_1$ positions to the right or $|u|_0$ positions to the left. It follows that the $|w|$ positions on which w is defined, represent exactly the cosets modulo $|w|$.

Example 1. If $u = 01001$, the central function w associated with u is a central block given by $w = 2\underline{0}314$, where we have underlined the number at position 0. See Figure 3.2.

3.3 The central functions w_n

In this section for every n we define a central function w_n corresponding to $u_n = \sigma^n(0)$, where σ is a substitution. We give conditions on σ so that the construction of w_n is well-defined, and we show that the order of the numbers in w_n is preserved in w_{n+1}.

Let σ be a primitive substitution that has a fixed point. Without loss of generality we may assume the fixed point starts with 0. Let $M_\sigma = \begin{pmatrix} |\sigma(0)|_0 & |\sigma(0)|_1 \\ |\sigma(1)|_0 & |\sigma(1)|_1 \end{pmatrix}$ be its incidence matrix. Define $u_n = \sigma^n(0)$ for $n \in \mathbb{Z}_{\geq 0}$ where we use the convention that $\sigma^0(v) = v$ for every word v. We assume $\gcd(|u_n|_0, |u_n|_1) = 1$ for every $n \in \mathbb{Z}_{\geq 0}$ and consider for each u_n the corresponding central function w_n.

Example 2. Let ϕ be the substitution defined by $\phi(0) = 01001$, $\phi(1) = 01$. If we start with 0 and repeatedly apply ϕ we get successively

$$\begin{aligned} u_0 &= 0 \\ u_1 &= 01001 \\ u_2 &= 0100101010010100101 \\ &\dots \end{aligned}$$

This yields the following table of central functions w_n, where we have underlined the number at position 0. Note that the central functions are central blocks, that is, have no gap.

n					w_n														
0					$\underline{0}$														
1				2	$\underline{0}$	3	1	4											
2	9	2	14	7	$\underline{0}$	12	5	17	10	3	15	8	1	13	6	18	11	4	16

(3.1)

......

We use the following lemmas to derive conditions on the incidence matrix M_σ under which the central functions w_n exist for all n.

Lemma 3.3.1. *If M is a 2×2-matrix with integer coefficients and $\det(M) \neq 0$, then $\operatorname{trace}(M^n) - (\operatorname{trace}(M))^n$ is divisible by $\det(M)$ for every $n > 0$.*

Proof.

$$\operatorname{trace}(M^n) - (\operatorname{trace}(M))^n = \lambda_1^n + \lambda_2^n - (\lambda_1 + \lambda_2)^n = -\lambda_1\lambda_2 \sum_{k=1}^{n-1} \binom{n}{k} \lambda_1^{k-1} \lambda_2^{n-k-1}$$

where λ_1, λ_2 are the two eigenvalues of M, and $\det(M) = \lambda_1\lambda_2$. □

Lemma 3.3.2. *Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a matrix with $a, b, c, d \in \mathbb{R}$. Then $M^n = (a + d)M^{n-1} - (ad - bc)M^{n-2}$ for $n \geq 2$.*

Proof. We have $M^2 = (a + d)M - (ad - bc)M^0$. □

Proposition 3.3.3. *Let σ be a primitive substitution with a fixed point starting with 0, let $M_\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be its incidence matrix and let $u_n = \sigma^n(0)$. Then $\gcd(|u_n|_0, |u_n|_1) = 1$ for every $n \in \mathbb{Z}_{\geq 0}$ if and only if $\gcd(a, b) = \gcd(a + d, ad - bc) = 1$.*

Proof. Assume $\gcd(|u_n|_0, |u_n|_1) = 1$ for every $n \in \mathbb{Z}_{\geq 0}$. We define a_n, b_n, c_n, d_n by $M_\sigma^n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$ for $n \in \mathbb{Z}_{\geq 0}$. Since $M_{\sigma^n} = M_\sigma^n$, we have $|u_n|_0 = a_n, |u_n|_1 = b_n$. Hence $\gcd(a, b) = \gcd(|u_1|_0, |u_1|_1) = 1$. By the previous lemma, $|u_2|_0 = (a + d)a - (ad - bc), |u_2|_1 = (a + d)b$. Since $\gcd(|u_2|_0, |u_2|_1) = 1$, we have $\gcd(a + d, ad - bc) = 1$.

We now prove the other implication. Assume that $\gcd(a, b) = \gcd(a + d, ad - bc) = 1$. From Lemma 3.3.1 and our assumption that $\gcd(\text{trace}(M_\sigma), \det(M_\sigma)) = 1$ it follows that

$$\gcd(a_n + d_n, a_1d_1 - b_1c_1) = \gcd(\text{trace}(M_\sigma^n), \det(M_\sigma)) = 1 \text{ for every } n > 0. \tag{3.2}$$

We prove by induction on n that $\gcd(a_n, b_n) = 1$. We know that $\gcd(a_1, b_1) = 1$. Assume $\gcd(a_m, b_m) = 1$ for $m = 1, \ldots, n$. Note that

$$M_\sigma^{n+1} = \begin{pmatrix} a_{n+1} & b_{n+1} \\ c_{n+1} & d_{n+1} \end{pmatrix} = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} aa_n + cb_n & ba_n + db_n \\ ac_n + cd_n & bc_n + dd_n \end{pmatrix}.$$

Assume p is prime and $p|a_{n+1}$ and $p|b_{n+1}$. Then $p|ab_{n+1} - ba_{n+1} = aba_n + adb_n - aba_n - bcb_n = b_n(ad - bc)$. Hence $p|b_n$ or $p|(ad - bc)$. Suppose $p|b_n$. From $p|b_{n+1} = ba_n + db_n$ it follows that $p|b$ and from $p|a_{n+1} = aa_n + cb_n$ that $p|a$, but $\gcd(a, b) = 1$. Thus $p \nmid b_n$ and

$$p|(ad - bc) = \det(M). \tag{3.3}$$

By (3.2) this gives $p \nmid a_{n+1} + d_{n+1}$, hence $p \nmid d_{n+1}$. By (3.3) we have $p|\det(M^n) = a_nd_n - b_nc_n$. Thus $p|b(a_nd_n - b_nc_n)$ and because $p|d_nb_{n+1} = ba_nd_n + db_nd_n$ we obtain $p|db_nd_n + bb_nc_n = b_nd_{n+1}$. This contradiction implies that $\gcd(a_{n+1}, b_{n+1}) = 1$. ☐

Consider the sequence of words w_n in (3.1) from Example 2. We see that the numbers $0, \ldots, 4$ in w_1 are ordered in the same way as the numbers $0, 5, 7, 12, 17$ in w_2 directly below them. This observation illustrates the following proposition.

Proposition 3.3.4. *Let σ be a primitive substitution with incidence matrix $M_\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and a fixed point starting with 0, let $\gcd(a, b) = \gcd(a + d, ad - bc) = 1$ and let w_n be the corresponding central functions for $n \in \mathbb{Z}_{>0}$. If w_n has numbers at positions k and l then w_{n+1} has numbers at positions $\det(M_\sigma) \cdot k$ and $\det(M_\sigma) \cdot l$. If the number at position k of w_n is larger than the number at position l of w_n then the number at position $\det(M_\sigma) \cdot k$ of w_{n+1} is larger than the number at position $\det(M_\sigma) \cdot l$ of w_{n+1}.*

Proof. First note that since

$$\begin{aligned} |u_{n+1}|_0 &= a|u_n|_0 + c|u_n|_1 \\ |u_{n+1}|_1 &= b|u_n|_0 + d|u_n|_1 \end{aligned}$$

we have $a|u_{n+1}|_1 - b|u_{n+1}|_0 = (ad - bc)|u_n|_1 = \det(M_\sigma)|u_n|_1$, and similarly $c|u_{n+1}|_1 - d|u_{n+1}|_0 = -\det(M_\sigma)|u_n|_0$. We prove the proposition by induction. Since the first letter of u_n is a 0, the first jump in w_n starting at position 0 is to the right and places the number 1 at position $|u_n|_1$. When going from u_n to u_{n+1} this 0 is replaced by a 0's and b 1's. Therefore the number $a + b$ in w_{n+1} is placed at position $a|u_{n+1}|_1 - b|u_{n+1}|_0 = \det(M_\sigma)|u_n|_1$. We see that the statements of the proposition hold for the positions of the numbers 0 and 1 in w_n. Assume that in w_n the number m is placed at position k and in w_{n+1} the number m' is placed at position $\det(M_\sigma)k$. The next jump is either to the right or to the left, depending on the value of the $(m+1)$-th letter in u_n. If the next jump in w_n is to the right, then the number $m+1$ is placed at position $k+|u_n|_1$ in w_n, and it follows that the number $m'+a+b$ is placed at position $\det(M_\sigma)(k+|u_n|_1)$ in w_{n+1}. If the next jump in w_n is to the left, then the number $m+1$ is placed at position $k-|u_n|_0$ in w_n, and it follows that the number $m'+c+d$ is placed at position $\det(M_\sigma)(k-|u_n|_0)$ in w_{n+1}. The assertions of the proposition follow now easily. $\square$

Example 3. Let the substitution σ be given by $\sigma(0) = 011$, $\sigma(1) = 0$ so that it has incidence matrix $M_\sigma = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$ and $\det(M_\sigma) = -2$. We get the following table for $u_n = \sigma^n(0)$:

$$\begin{aligned} u_0 &= 0 \\ u_1 &= 011 \\ u_2 &= 01100 \\ u_3 &= 01100011011 \\ &\dots \end{aligned}$$

This gives the following sequence of w_n's, where $\sqcup$ indicates a gap.

n					w_n														
0					$\underline{0}$														
1					$\underline{0}$	2	1												
2	3	$\sqcup$	4	2	$\underline{0}$	$\sqcup$	1												
3	3	$\sqcup$	$\sqcup$	$\sqcup$	$\underline{0}$	2	4	$\sqcup$	8	10	1	$\sqcup$	5	7	9	$\sqcup$	$\sqcup$	$\sqcup$	6

......

Comparing w_2 with w_3 illustrates Proposition 3.3.4.

3.4 Sturmian matrices and Sturmian substitutions

In this section we define Sturmian matrix, Sturmian substitution, reduced matrix, dual matrix and we list some properties of these objects.

A word u is called *balanced* if $||v|_0 - |w|_0| < 2$ for all subwords v, w of equal length. A finite word u is called *strongly balanced* if u^2 is balanced. Here u^2 is the concatenation of u with u. It is easy to see that a finite word u is strongly balanced if and only if u^n is balanced for some $n \geq 2$. A one-sided infinite word is *Sturmian* if it is balanced and not ultimately periodic. We call a 2×2-matrix *Sturmian* if it has determinant equal to ± 1 and has entries in $\mathbb{Z}_{\geq 0}$. We call a substitution σ over two letters *Sturmian* if $\sigma(u)$ is a Sturmian word for every Sturmian word u. [35] Th 2.3.7 says that a substitution σ is Sturmian if and only if there exists a Sturmian word v such that $\sigma(v)$ is Sturmian. It follows that a primitive substitution σ is Sturmian if and only if its fixed point is Sturmian.

Lemma 3.4.1. *A Sturmian substitution maps every finite balanced word to a finite balanced word.*

Proof. See [42] Cor.9. $\square$

Every Sturmian substitution is composed of the following three substitutions, cf. [35] Th 2.3.7.

$$E : \left\{ \begin{array}{ccc} 0 & \to & 1 \\ 1 & \to & 0 \end{array} \right. \qquad \phi : \left\{ \begin{array}{ccc} 0 & \to & 01 \\ 1 & \to & 0 \end{array} \right. \qquad \tilde{\phi} : \left\{ \begin{array}{ccc} 0 & \to & 10 \\ 1 & \to & 0 \end{array} \right. .$$

Their incidence matrices are $M_E = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $M_\phi = M_{\tilde{\phi}} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, respectively. From this it follows that each Sturmian substitution has a Sturmian matrix as incidence matrix. On the other hand we have the following result.

Theorem 3.4.2. *If M is a Sturmian matrix then there exists a Sturmian substitution that has M as its incidence matrix.*

Proof. Let M be a Sturmian matrix. It can be written as a product of factors M_E and M_ϕ, cf. [27] Sec.6.5.5. By replacing each matrix with the corresponding substitution the composition of these substitutions is a Sturmian substitution that has M as incidence matrix. ☐

Lemma 3.4.3. *Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a Sturmian matrix.*

(i) If $a + b = c + d$, then M is the identity matrix or M_E.

(ii) If $a > c$ and $b < d$, then M is equal to the identity matrix.

(iii) If $a < c$ and $b > d$, then $M = M_E$.

Proof. (i) Assume $a + b = c + d$. Then $M = \begin{pmatrix} a & b \\ a - \delta & b + \delta \end{pmatrix}$ for some $\delta \in \mathbb{Z}$. Since $\det(M) = (a + b)\delta = \pm 1$ we have $a + b = 1$ and $\delta = \pm 1$. It follows that M is of the form $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

(ii) Assume $a > c$ and $b < d$. Using that $ad - bc = \pm 1$ we get $bc \leq (d - 1)(a - 1) \leq bc - (a + d) + 2$. It follows that $a + d \leq 2$, hence $a = d = 1$ and $b = c = 0$.

(iii) The proof is similar to the proof of (ii). ☐

Definition. Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a Sturmian matrix. If $a + b > c + d$, then we call M an *upper matrix*, if $a + b < c + d$ a *lower matrix*.

Corollary 3.4.4. *Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a Sturmian matrix. If M is an upper matrix then $a \geq c$ and $b \geq d$ with at least one of the two inequalities strict. If M is a lower matrix then $a \leq c$ and $b \leq d$ with at least one of the two inequalities strict.*

Proof. This follows immediately from Lemma 3.4.3. ☐

Corollary 3.4.5. *Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a Sturmian matrix with $ab \neq 0$. Then there exists a unique nonnegative integer k such that $\begin{pmatrix} a & b \\ c - ka & d - kb \end{pmatrix}$ is an upper matrix.*

Proof. Let $k = \min(\lfloor c/a \rfloor, \lfloor d/b \rfloor)$ and let $N = \begin{pmatrix} a & b \\ c-ka & d-kb \end{pmatrix}$. Then $\det(N) = \det(M)$ hence N is Sturmian. Since $c - ka < a$ or $d - kb < b$ it follows from Lemma 3.4.3 that N is an upper matrix. ☐

Definition. We call the upper matrix that is constructed from M as described in Corollary 3.4.5 the *reduced matrix* of M.

It follows directly from Corollary 3.4.4 that this definition of reduced matrix is unique.

Lemma 3.4.6. *Let* $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *be an upper matrix. Then there exists a unique upper matrix* $N = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ *such that* $\det(N) = \det(M)$, $\alpha + \beta = a + b$ *and* $\alpha = c + d$. *Moreover* $\gamma + \delta = a$ *and* $trace(M) = trace(N)$.

Proof. Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an upper matrix. We assume $\det(M) = 1$, the case for $\det(M) = -1$ is similar. We define $N := \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} c+d & a+b-(c+d) \\ c & a-c \end{pmatrix}$. We have to show that N has nonnegative entries. This is clear for α, β, γ. Assume $a < c$. Then $b = (ad-1)/c < d$. This implies $c + d \le a + b - 1 < c + d$. It follows that $\delta \ge 0$. Next we show that N is unique. Because $\alpha + \beta = a + b$ and $\alpha = c + d$, the variables α and β are uniquely defined. It is well known from number theory that the equation $\alpha x - \beta y = 1$ in unknowns x, y has a unique solution in nonnegative integers x, y with $x + y < \alpha + \beta$. Therefore γ and δ are also uniquely defined. For $\alpha + \beta > \gamma + \delta$, observe that if $b = 0$ then $a = d = 1$, hence $a + b = 1 \le c + d$. The other properties are obvious. ☐

Definition. For an upper matrix M we call the matrix N as defined in the previous lemma the *dual matrix* of M. If $N = M$ we say that M is *self-dual.*

It follows directly from the proof of Lemma 3.4.6 that if N is the dual matrix of M, then M is the dual matrix of N, and that an upper Sturmian matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is self-dual if and only if $a = c + d$.

Lemma 3.4.7. *Let* $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *be a Sturmian matrix with* $a, b > 1$. *Then* $\lfloor c/a \rfloor = \lfloor d/b \rfloor$.

Proof. From $ad - bc = \pm 1$ it follows that $\frac{d}{b} - \frac{c}{a} = \pm\frac{1}{ab}$. From $\frac{c+\frac{1}{b}}{a} = \frac{d}{b}$ and $b > 1$ we get $\lfloor \frac{c}{a} \rfloor = \lfloor \frac{d}{b} \rfloor$. From $\frac{d+\frac{1}{a}}{b} = \frac{c}{a}$ and $a > 1$ we also get $\lfloor \frac{c}{a} \rfloor = \lfloor \frac{d}{b} \rfloor$. ☐

The statement of Lemma 3.4.7 is not valid if $a = 1$ and $ad - bc = -1$ and if $b = 1$, $ad - bc = 1$.

Lemma 3.4.8. *Let* $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *be a Sturmian matrix with* $ab \neq 0$ *and let* $k = \min(\lfloor c/a \rfloor, \lfloor d/b \rfloor)$. *Put* $M^n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$ *and* $N_n = \begin{pmatrix} a_n & b_n \\ c_n - ka_n & d_n - kb_n \end{pmatrix}$ *for* $n > 0$. *Then* N_n *is the reduced matrix of* M^n.

Proof. The case where $n = 1$ has been proven in Corollary 3.4.5. Assume N_n is an upper matrix for $n > 1$. Then it follows from Corollary 3.4.4 that $c_n \leq (k+1)a_n,\ \ d_n \leq (k+1)b_n$ with at least one of the inequalities strict. From this we derive $ac_n + cd_n \leq (k+1)(aa_n + cb_n)$ and $bc_n + dd_n \leq (k+1)(ba_n + db_n)$. Since $M^{n+1} = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} aa_n + cb_n & ba_n + db_n \\ ac_n + cd_n & bc_n + dd_n \end{pmatrix}$ it follows that N_{n+1} is an upper matrix. ☐

3.5 Unimodular substitutions

We call a substitution σ over 2 letters *unimodular* if its incidence matrix is Sturmian. We assume that all substitutions that we consider are unimodular, primitive and have a fixed point starting with 0. Note that for a unimodular substitution $\gcd(a, b) = \gcd(a + d, ad - bc) = 1$, so that it follows from Proposition 3.3.3 that the construction of w_n is well-defined for all $n > 0$.

Lemma 3.5.1. *Let $u = 0u_1u_2\ldots$ be the fixed point of a primitive unimodular substitution. Then u is not eventually periodic.*

Proof. Let ψ be the primitive unimodular substitution with fixed point u. Then ψ has a Sturmian incidence matrix M. It follows from Theorem 3.4.2 that there exists a Sturmian substitution σ with incidence matrix M. Let $u_n = \psi^n(0)$ and $u'_n = \sigma^n(0)$. Because the numbers of 0's and 1's in u_n and u'_n are completely determined by M, we have $|u_n|_0 = |u'_n|_0$ and $|u_n|_1 = |u'_n|_1$ for every $n \in \mathbb{Z}_{\geq 0}$. Therefore $\lim_{n\to\infty} |u_n|_0/|u_n|_1 = \lim_{n\to\infty} |u'_n|_0/|u'_n|_1$ and this limit is irrational since it is the fixed point of a Sturmian substitution ([35] Sec.2.1.1). It follows that u is not eventually periodic. ☐

For a finite word u we denote by $f_u^a = \frac{|u|_a}{|u|}$ the frequency of the letter a in u.

Lemma 3.5.2. *Let u be a finite word that is not balanced. Then u has a subword v with $2|v| \leq |u|$ such that $|v|_0 > \lceil |v| f_u^0 \rceil$ or $|v|_1 > \lceil |v| f_u^1 \rceil$.*

Proof. Since u is not balanced, it has subwords v, w of equal length such that $|v|_0 - |w|_0 \geq 2$. Without loss of generality we may assume that v and w are disjoint, so that $|v| = |w| \leq |u|/2$. Assume $|v|_0 \leq \lceil |v| f_u^0 \rceil$. Then $|w|_1 = |w| - |w|_0 \geq |w| + 2 - |v|_0 \geq |w| + 2 - \lceil |v| f_u^0 \rceil = |w| + 2 - \lceil |w|(1 - f_u^1) \rceil \geq 1 + \lceil |w| f_u^1 \rceil$. ☐

Lemma 3.5.3. *Let u be a finite word with $\gcd(|u|_0, |u|_1) = 1$. Then u is strongly balanced if and only if for every subword v of u we have $|v|_0 = \lfloor |v| f_u^0 \rfloor$ or $|v|_0 = \lceil |v| f_u^0 \rceil$.*

Proof. Assume u is not strongly balanced. Put $u = u_0 \ldots u_m$. Since u^2 is not balanced, there exists a subword v of u^2 with $|v| \leq |u|$ and $|v|_0 > \lceil |v| f_u^0 \rceil$ or $|v|_1 > \lceil |v| f_u^1 \rceil$, according to the previous lemma. It is obvious that $|v| \neq |u|$. If v is a subword of u we are done. Otherwise v is of the form $u_p \ldots u_m u_0 \ldots u_q$ with $q < p - 1$. Without loss of generality we assume $|v|_1 > \lceil |v| f_u^1 \rceil$, hence $-|v|_0 > \lceil -|v| f_u^0 \rceil$. Define $v' = u_{q+1} \ldots u_{p-1}$. This is a subword of u with $|v| + |v'| = |u|$. We get

$$|v'|_0 = |u|_0 - |v|_0 > \left\lceil |u|_0 - |v| f_u^0 \right\rceil = \left\lceil |v'| f_u^0 \right\rceil .$$

Assume there is a subword v of $u = u_0 \dots u_m$ with $|v| < |u|$ and $|v|_0 < \lfloor |v| f_u^0 \rfloor$ or $|v|_0 > \lceil |v| f_u^0 \rceil$. We assume the second case, the proof for the fist case is similar. Put $n = |v|$, hence $n < |u| = m + 1$. Consider the $m + 1$ subwords of u^2 of length n that start with the letters $u_0, u_1, \dots, u_m$. Each letter of u occurs in exactly n of these subwords, hence on average each of these subwords contains $n f_u^0$ zeros. Because $\gcd(|u|_0, |u|) = 1$ we know $n f_u^0 \notin \mathbb{Z}$. Assume all these subwords contain $\lceil n f_u^0 \rceil$ or more zeros. Then $|u|_0 \geq (m+1) \lceil n f_u^0 \rceil / n > (m+1) f_u^0 = |u|_0$. This contradiction implies that there exists a subword v' of u^2 of length n with fewer than $\lceil n f_u^0 \rceil$ zeros, and it follows that u^2 is not balanced. □

Proposition 3.5.4. *Let u be a finite word with* $\gcd(|u|_0, |u|_1) = 1$ *and let w be the corresponding central function. Then w forms a central block if and only if u is strongly balanced.*

Proof. Assume that u is not strongly balanced. Then using the previous lemma we may assume without loss of generality that there exists a subword v of u with $|v|_0 > \lceil |v| f_u^0 \rceil$, since if $|v|_0 < \lceil |v| f_u^0 \rceil$ then $|v|_1 > \lceil |v| f_u^1 \rceil$. Because of the way w is defined, there are two letters in w at positions that are $|u|_1 |v|_0 - |u|_0 |v|_1$ apart. We get

$$|u|_1 |v|_0 - |u|_0 |v|_1 \geq |u|_1 \left(\lceil f_u^0 |v| \rceil + 1 \right) - |u|_0 \left(|v| - \lceil f_u^0 |v| \rceil - 1 \right)$$

$$\geq |u| f_u^0 |v| + |u| - |u|_0 |v| = |u|.$$

Hence w contains a gap.

Assume w contains a gap. Then w has two numbers at positions that are $|w|$ or more apart. Call the smallest of these two numbers a and the largest b. Let v be the subword of u that starts at the $(a+1)$-th letter and ends with the b-th letter. There are two possibilities, $|u|_0 |v|_1 - |u|_1 |v|_0 \geq |u|$ or $|u|_1 |v|_0 - |u|_0 |v|_1 \geq |u|$, depending on whether the position of a in w is left or right of the position of b. We will assume the second inequality holds; the proof for the first case is similar. We get, successively,

$$|u|_1 |v|_0 \geq |u| + |u|_0 (|v| - |v|_0),$$

$$|v|_0 \geq 1 + f_u^0 |v| > \left\lceil f_u^0 |v| \right\rceil .$$

It follows from the previous lemma that u_n is not strongly balanced. □

Remark. If we replace the condition in Proposition 3.5.4 that u is strongly balanced by the condition that u is balanced the statement does not hold. Take for example $u = 0010100$ so that $\gcd(|u|_0, |u|_1) = 1$, u is balanced (but u^2 is not balanced). Then $w = 5 \sqcup 6\ 3\ \underline{0}\ 4\ 1 \sqcup 2$ contains gaps.

Theorem 3.5.5. *Let σ be a primitive unimodular substitution with fixed point starting with 0. Then σ is Sturmian if and only if w_n forms a central block for every $n > 0$.*

Proof. Because σ^n is a Sturmian substitution it follows from Corollary 3.4.1 that $u_n^2 = \sigma^n(00)$ is balanced. Applying Proposition 3.5.4 gives that w_n has no gap for any $n > 0$.

Assume w_n has no gap for any $n > 0$. Proposition 3.5.4 implies that u_n is balanced for every n, and therefore the fixed point of σ is balanced. Because it is not eventually periodic according to Lemma 3.5.1, it is a Sturmian word. Hence σ is a Sturmian substitution. □

Example 2 (continued). For $n > 0$ we define v_n as follows. Let g_n be the number at position -1 of w_n. Then v_n is obtained by replacing every number in w_n that is smaller than g_n by 0, and every other number by 1. We get the following table of central words over a two-letter alphabet.

n					v_n														
0					$\underline{0}$														
1				1	$\underline{0}$	1	0	1											
2	1	0	1	1	$\underline{0}$	1	0	1	1	0	1	1	0	1	0	1	1	0	1

......

This example illustrates the following definition. Let σ be a primitive unimodular substitution with fixed point starting with 0. For $n > 0$ put $g_n = -\det^n(M_\sigma)|u_n|_1^{-1} \pmod{|u_n|}$, where the inverse is taken with respect to $|u_n|$. Let $w'_n = w_n$ for each n if $\det(M_\sigma) = 1$, otherwise let $w'_n = w_n$ and reflect each w'_n in the origin for n odd. Note that it follows from Lemma 3.2.3 that g_n is the number at position -1 of w'_n if there is one.

Definition. Let σ be a primitive Sturmian substitution with fixed point starting with 0 and g_n, w'_n as defined above. Then for $n > 0$ we get the *central word* v_n by replacing every number in w'_n that is smaller than g_n by 0, and every other number by 1.

We already noted in Section 3.2 that w_n has in each coset modulo $|w_n|$ exactly one position on which it is defined. This means we can extend w'_n to a bi-infinite central block $\widehat{w'_n}$ with period $|w_n|$. It follows from the definition of v_n that $\widehat{w'_n}(i) < \widehat{w'_n}(i+1) \iff \widehat{w'_n}(i) < \widehat{w'_n}(-1) \iff v_n(i) = 0$. The next corollary follows directly from Proposition 3.3.4.

Corollary 3.5.6. *Let σ be a primitive Sturmian substitution with fixed point starting with 0 and assume that v_p has a letter at position -1. Then for $n \geq p$ if v_n has a letter at position k then v_{n+1} has the same letter at position k.*

We see that in this case $v := \lim_{n\to\infty} v_n$ is an infinite central word. The only primitive Sturmian substitutions with fixed point starting with 0 for which there does not exist an integer p such that v_p has a letter at position -1 are the Christoffel substitutions defined in Section 3.6.

Example 4. Let the primitive Sturmian substitution ψ be given by $\psi(0) = 01,\ \psi(1) = 011$. Then the statement of Corollary 3.5.6 does not hold since $v_1(1) = 1$ and $v_2(1) = 0$, as we see in the table below. Note that there is no letter at position -1.

n	u_n					$w_n = w'_n$					v_n				
0	0														
1	0	1				$\underline{0}$	1				$\underline{0}$	1			
2	0	1	0	1	1	$\underline{0}$	2	4	1	3	$\underline{0}$	0	1	0	1

Theorem 3.5.7. *Let σ be a primitive Sturmian substitution with fixed point starting with 0. If v_n has a letter at position i then it equals 0 if $\{-ig_n/|v_n|\} \in [0, g_n/|v_n|)$ and 1 otherwise. Moreover, v_n is strongly balanced for every $n > 0$.*

Proof. The first statement follows directly from the definition of v_n and from Lemma 3.2.3 which says $w_n(i) = i|u_n|_1^{-1} \pmod{|u_n|}$. It follows from [35] Sec.2.1.2. that words defined in this way are strongly balanced (so-called rotation words). ☐

Remark. The central words v_n need not be balanced if we would not require that g_n is the number at position 1 or -1 in w_n. We illustrate this using the substitution ϕ from Example 2 and choosing in w_2 the number at position -2 as g_2. Then

$$v_2 = 0010\underline{0}00100100001001.$$

Clearly v_2 is not balanced, since it contains subwords of length 4 that contain two 0's and subwords of length 4 that contain four 0's.

Theorem 3.5.8. *Let σ be a primitive Sturmian substitution with fixed point starting with 0 and M_σ as its incidence matrix. Then $|v_n|_0$ and $|v_n|_1$ are given by the top left and top right entry, respectively, of the dual matrix of the reduced matrix of M_σ^n for $n > 0$.*

Proof. We know that $M_\sigma^n = \begin{pmatrix} |u_n|_0 & |u_n|_1 \\ c_n & d_n \end{pmatrix}$ with $c_n, d_n \in \mathbb{Z}_{\geq 0}$ for $n > 0$. We call its reduced matrix $\begin{pmatrix} |u_n|_0 & |u_n|_1 \\ c'_n & d'_n \end{pmatrix}$. It follows from the definition of w_n that $w'_n(-1) = x_n + y_n$ where (x_n, y_n) is the unique solution of the equation $|u_n|_0 x - |u_n|_1 y = \det(M_\sigma^n)$ with $0 \leq x_n \leq |u_n|_1$ and $0 \leq y_n \leq |u_n|_0$. Therefore we must have that $(c'_n, d'_n) = (y_n, x_n)$. From the definition of v_n we get immediately that $|v_n|_0 = g_n = w'_n(-1) = x_n + y_n$. This means that the dual matrix of the reduced matrix of M_σ^n has $|v_n|_0$ as top left entry, and because the sum of the elements of the top row is the same for the dual matrix, it has $|v_n|_1$ as top right entry. ☐

Corollary 3.5.9. *Let σ be a primitive Sturmian substitution with fixed point starting with 0, M_σ its incidence matrix and let v_n be the central word corresponding to u_n for some $n > 0$. Shift v_n a number of positions to the right so that the left most letter is at position* 0. *In the same way as before we can project the cutting segment constructed from the shifted word v_n to form the central block t_n. By looking at the incidence matrix of M_σ we get t'_n from t_n in the same way as we construct w'_n from w_n. Finally we construct a new central word z_n from t'_n. Then $|z_n|_0 = |u_n|_0$ and $|z_n|_1 = |u_n|_1$.*

Proof. Let N_n be the dual matrix of the reduced matrix of M_σ^n. We know from the previous theorem that N_n is of the form $\begin{pmatrix} |v_n|_0 & |v_n|_1 \\ c_n & d_n \end{pmatrix}$ for some $c_n, d_n \in \mathbb{Z}_{\geq 0}$ and it is an upper matrix. This means that the solution (x_n, y_n) of the equation $|v_n|_0 x - |v_n|_1 y = \det(M_\sigma^n)$ is given by the bottom entries of N_n. Therefore $|z_n|_0$ and $|z_n|_1$ are given by the top left and top right entry of the dual matrix of N_n, respectively, which is the reduced matrix of M_σ^n, since the dual matrix of the dual matrix is the original matrix. Of course the reduced matrix of M_σ^n has the same top row as M_σ^n itself. ☐

Let σ be a primitive Sturmian substitution with fixed point starting with 0 and $M_\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ its incidence matrix. For $n > 0$ we define (x_n, y_n) as the unique solution of the equation $|u_n|_0 x - |u_n|_1 y = \det(M_\sigma^n)$ satisfying $0 \leq x_n \leq |u_n|_1$ and $0 \leq y_n \leq |u_n|_0$. Since $M_\sigma^n = \begin{pmatrix} |u_n|_0 & |u_n|_1 \\ c_n & d_n \end{pmatrix}$, it follows that $|u_n|_0 d_n - |u_n|_1 c_n = \det(M_\sigma^n)$. Hence $x_n =$

$d_n - k|u_n|_1,\ y_n = c_n - k|u_n|_0$ for some suitable $k \in \mathbb{Z}_{\geq 0}$. From Lemma 3.4.8 we know that the value of k is independent of n. Since according to Lemma 3.3.2 both $(c_n), (d_n)$ and $(|u_n|_0), (|u_n|_1)$ satisfy the recurrence relation $p_n = (a+d)p_{n-1} - (ad-bc)p_{n-2}$ it follows that (x_n) and (y_n) satisfy the same recurrence relation. By going backwards we define x_0 and y_0. Since $|u_0|_0 x_0 - |u_0|_1 y_0 = 1$ and $|u_0|_0 = 1, |u_0|_1 = 0$, we see that $x_0 = 1$, but y_0 is not determined by this equation.

We mentioned in the proof of Theorem 3.5.8 that $|v_n|_0 = x_n + y_n$. Hence we see that $(|v_n|_0)$ and $(|v_n|_1)$ satisfy the same recurrence relation. By going backwards we can now define $|v_0|_0$ and $|v_0|_1$, and we see that also $|v_0|_0 = x_0 + y_0$. Note that v_0 has no meaning in terms of projecting a cutting sequence, and is only formally defined. We will see that $|v_0|_0$ and $|v_0|_1$ can take negative values, but their sum is $|u_0| = 1$. Theorem 3.5.8 can be used to find $|v_n|_0$ and $|v_n|_1$ for $n > 0$. In order to calculate $|v_0|_0$ and $|v_0|_1$ we prove the following lemma.

Lemma 3.5.10. *Let σ be a primitive Sturmian substitution with fixed point starting with 0 and $M_\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ its incidence matrix. Let k be the nonnegative integer such that $\begin{pmatrix} a & b \\ c-ka & d-kb \end{pmatrix}$ is an upper matrix. Then $|v_0|_1 = k$ and $|v_0|_0 = 1 - |v_0|_1 = 1-k$.*

Proof. The reduced matrix of M_σ equals $\begin{pmatrix} a & b \\ c' & d' \end{pmatrix}$ where $c' = c - ka, d' = d - kb$ and $ad' - bc' = \det M_\sigma$. Recall $\det M_\sigma = \pm 1$. In the table below we know $|u_0|_0 = 1, |u_0|_1 = 0, |u_1|_0 = a, |u_1|_1 = b$. Using the recurrence relation on $(|u_n|_0), (|u_n|_1)$ we compute $|u_2|_0 = a(a+d) \mp 1, |u_2|_1 = b(a+d)$. We know that $x_0 = 1$, $x_1 = d'$ and $y_1 = c'$. Next we get $x_2 = (a+d)d' \mp 1$ from the recurrence relation on (x_n). Recall that $a + d \neq 0$ and $b \neq 0$. We use the equation $|u_2|_0 x_2 - |u_2|_1 y_2 = 1$ to find y_2 as follows.

$$b(a+d)y_2 = a(a+d)^2 d' \mp a(a+d) \mp (a+d)d';$$

$$by_2 = a(a+d)d' \mp (a+d') = (bc' \pm 1)(a+d) \mp (a+d') = bc'(a+d) \pm kb;$$

$$y_2 = c'(a+d) \pm k.$$

Finally we find $y_0 = -k$ from the recurrence relation on (y_n) and then $|v_0|_0 = x_0 + y_0 = 1 - k, |v_0|_1 = 1 - |v_0|_0 = k$. See the table below.

n	$\lvert u_n\rvert_0$	$\lvert u_n\rvert_1$	x_n	y_n	$\lvert v_n\rvert_0$	$\lvert v_n\rvert_1$
0	1	0	1	$-k$	$1-k$	k
1	a	b	d'	c'		
2	$a(a+d) \mp 1$	$b(a+d)$	$(a+d)d' \mp 1$	$c'(a+d) \pm k$		

□

Consider a primitive Sturmian substitution σ with fixed point starting with 0, with incidence matrix M_σ and u_n, v_n defined as before. As mentioned before we have for $(|v_n|_0)$ and $(|v_n|_1)$ the same recurrence relation as for $(|u_n|)$. The following theorem shows that there is a Sturmian matrix that we call M_τ such that $(|v_n|_0, |v_n|_1) \cdot M_\tau = (|v_{n+1}|_0, |v_{n+1}|_1)$ for every $n \in \mathbb{Z}_{\geq 0}$. Thus M_τ has the same trace and determinant as M_σ.

Theorem 3.5.11. *Let σ be a primitive Sturmian substitution with fixed point starting with 0, with incidence matrix* $M_\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *and let* $(v_n)_{n=0}^{\infty}$ *be the corresponding central words. Let k be the nonnegative integer such that* $\begin{pmatrix} a & b \\ c-ka & d-kb \end{pmatrix}$ *is an upper matrix. Then* $(|v_n|_0, |v_n|_1) \cdot M_\tau = (|v_{n+1}|_0, |v_{n+1}|_1)$ *for every* $n \in \mathbb{Z}_{\geq 0}$ *where* $M_\tau = M'_\sigma + k\begin{pmatrix} a'-c' & b'-d' \\ a'-c' & b'-d' \end{pmatrix}$ *and* $M'_\sigma = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ *is the dual matrix of the reduced matrix of* M_σ*.*

Proof. Let σ be a Sturmian substitution. We write its incidence matrix as $M_\sigma = \begin{pmatrix} a & b \\ c+ak & d+bk \end{pmatrix}$ so that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is the reduced matrix of M_σ. Then

$$M_\tau = \begin{pmatrix} c+d(k+1) & a+b-(c+d)+k(b-d) \\ c+kd & a-c+k(b-d) \end{pmatrix}.$$

Using the previous lemma we see that $|v_0|_1$ is k and we can make the following table.

n	$\lvert u_n\rvert_0$	$\lvert u_n\rvert_1$	x_n	y_n	$\lvert v_n\rvert_0$	$\lvert v_n\rvert_1$
0	1	0	1	$-k$	$1-k$	k
1	a	b	d	c	$c+d$	$a+b-(c+d)$

We see that

$$(|v_0|_0, |v_0|_1) \cdot M_\tau = (1-k, k) \cdot \begin{pmatrix} c+d(k+1) & a+b-(c+d)+k(b-d) \\ c+kd & a-c+k(b-d) \end{pmatrix}$$

$$= (c+d, a+b-(c+d)) = (|v_1|_0, |v_1|_1).$$

Straightforward calculations show that $\operatorname{trace}(M_\tau) = a+d+bk = \operatorname{trace}(M_\sigma)$ and $\det(M_\tau) = ad-bc = \det(M_\sigma)$. Since $(|v_n|_0)$ and $(|v_n|_1)$ satisfy the same recurrence relation $p_n = \operatorname{trace}(M_\tau)p_{n-1} - \det(M_\tau)p_{n-2}$ as $(|u_n|_0)$, $(|u_n|_1)$ and since $\operatorname{trace}(M)M - \det(M)Id = M^2$ for any matrix M, it follows that $(|v_n|_0, |v_n|_1) \cdot M_\tau = (|v_{n+1}|_0, |v_{n+1}|_1)$ for every $n \in \mathbb{Z}_{\geq 0}$. ☐

3.6 Christoffel substitutions

In this section we consider a special class of unimodular substitutions that we will call Christoffel substitutions.

Let u be a finite word. We call u a *Lyndon word* if $u = vw$ implies $u < wv$ in the lexicographical order. Let $0 < p < q$ and $\gcd(p,q) = 1$. The finite word $u = u_0u_1 \dots u_{q-1}$ is called a *lower Christoffel word* or just *Christoffel word* if

$$u_i = \left\lfloor (i+1)\frac{p}{q} \right\rfloor - \left\lfloor i\frac{p}{q} \right\rfloor,$$

cf. [35] Sect.2.1.2. Note that p equals the number of 1's in u. It follows from the definition that for each $a, b \in \mathbb{Z}_{>0}$ with $\gcd(a,b) = 1$ there exists a unique Christoffel word containing a zeros and b ones. We also see that every Christoffel word starts with 0 and ends with 1.

Lemma 3.6.1. *A finite non-constant word is a Christoffel word if and only if it is a balanced Lyndon word.*

Proof. See [11] Sec.4. □

This means Christoffel words are the finite balanced words in which the 0's are placed "as far as possible" to the left.

Definition. Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a Sturmian matrix with determinant 1 and $abcd \neq 0$. We call the unique substitution σ that has M as incidence matrix and for which $\sigma(0)$ and $\sigma(1)$ are Christoffel words the *Christoffel substitution* corresponding to M.

Lemma 3.6.2. *Let σ be a substitution with $\sigma(0), \sigma(1)$ Lyndon words, and $\sigma(0) < \sigma(1)$ in the lexicographical order, and let u be a Lyndon word. Then $\sigma(u)$ is a Lyndon word.*

Proof. See [43]. □

Let M be a matrix with determinant 1. M can be written in a unique way as the product of some occurrences of the matrices $X_0 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $X_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. We associate to each occurrence of X_0 the substitution $\psi_0 : \begin{cases} 0 & \rightarrow & 0 \\ 1 & \rightarrow & 01 \end{cases}$ and to each occurrence of X_1 the substitution $\psi_1 : \begin{cases} 0 & \rightarrow & 01 \\ 1 & \rightarrow & 1 \end{cases}$. Let σ be the inverse product of the associated substitutions. It is clear that ψ_0 and ψ_1 are Sturmian substitutions, therefore σ is a Sturmian substitution. Hence $\sigma(0)$ and $\sigma(1)$ are balanced. Also it follows from Lemma 3.6.2 that they are Lyndon words. Now we conclude from Lemma 3.6.1 that σ is a Christoffel substitution.

Since for each matrix with determinant 1 and positive integer entries there exists a unique Christoffel substitution, we get the following result.

Lemma 3.6.3. *Christoffel substitutions are Sturmian substitutions.*

Lemma 3.6.4. *Let M_σ be a Sturmian matrix with positive entries and determinant equal to 1 and let σ be the Christoffel substitution belonging to M_σ. If M_σ is an upper matrix there exist possibly empty words u and v such that $\sigma(0) = u0v$ and $\sigma(1) = u1$. If M_σ is a lower matrix there exist possibly empty words u and v such that $\sigma(0) = u0v$ and $\sigma(1) = (u0v)^k u1$ for some positive integer k.*

Proof. If M_σ can be written as the product of two matrices X_0, X_1. we have either $M_\sigma = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, $\sigma : \begin{cases} 0 & \rightarrow & 001 \\ 1 & \rightarrow & 01 \end{cases}$ or $M_\sigma = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$, $\sigma : \begin{cases} 0 & \rightarrow & 01 \\ 1 & \rightarrow & 011 \end{cases}$ and we see that the statement holds. Assume the statement of the lemma is true for incidence matrices that can be written as the product of n matrices X_0, X_1. Let M_τ be a Sturmian matrix with positive entries and determinant equal to 1 that can be written as the product of $n+1$ matrices X_0, X_1, and let τ be the Christoffel substitution belonging to M_τ. Then $\tau = \sigma'\psi_0$ or $\tau = \sigma'\psi_1$, where σ' is of the form stated in the lemma. It is easy to check that τ is also of this form. □

Corollary 3.6.5. *Let σ be a Christoffel substitution. Then $\sigma^n(0)$ is a Christoffel word for every $n > 0$.*

Proof. Because according to Lemma 3.6.3 a Christoffel substitution is a Sturmian substitution, $\sigma^n(0)$ is balanced. It follows from Lemma 3.6.4 that $\sigma(0) < \sigma(1)$ in the lexicographical order, and applying Lemma 3.6.2 gives that $\sigma^n(0)$ is a Lyndon word. According to Lemma 3.6.1, $\sigma^n(0)$ is a Christoffel word. ☐

Proposition 3.6.6. *Let u be a Christoffel word, let $w' = w$ be the central block that we get after projecting the cutting segment of u, and let v be the corresponding central word, as defined in Section 5. Then v is a Christoffel word.*

Proof. According to [11] Sec.4 the cutting segment of a Christoffel word lies below the line connecting the origin and the endpoint of the cutting segment. It follows that when the integer points on the cutting segment are projected parallel to this line, they all lie below the origin, except the origin which is projected to itself. Since Christoffel words are Sturmian words according to Lemma 3.6.3, there is no gap by Theorem 3.5.5. Hence the projected points form the word $w = w(0) \dots w(|u|-1)$ with $w(i) = i|u|_1^{-1} \pmod{|u|}$. Put $g = |u|_1^{-1}$ $\pmod{|u|}$. Then

$$v(i) = 0 \iff w(i) < w(i+1) \iff ig \pmod{|u|} < (i+1)g \pmod{|u|}$$

$$\iff \left\lfloor i\frac{g}{|u|} \right\rfloor = \left\lfloor (i+1)\frac{g}{|u|} \right\rfloor$$

and we see that v is a Christoffel word containing $|u|_1^{-1} \pmod{|u|}$ ones. ☐

Theorem 3.6.7. *Let σ be a Christoffel substitution. Let (v_n) be the words that we get by projecting the cutting segments of $u_n = \sigma^n(0)$ for $n > 0$. Then there exists a Christoffel substitution τ such that $\tau(v_n) = v_{n+1}$ for $n > 0$ and τ has M_τ as defined in Theorem 3.5.11 as incidence matrix.*

Proof. It follows from the previous proposition that the words (v_n) are Christoffel words. According to Theorem 3.5.11 the relation between the number of 0's and 1's in v_n and v_{n+1} is given by M_τ. Define τ as the Christoffel substitution that has M_τ as incidence matrix. Because τ is Sturmian, $\tau(v_n)$ is a balanced word. Because v_n is a Lyndon word, and τ is a Christoffel substitution with $\tau(0) < \tau(1)$ in the lexicographical order, $\tau(v_n)$ is also a Lyndon word. Thus $\tau(v_n)$ is a Christoffel word. Because there is only one Christoffel word of length $|v_{n+1}|$ that contains $|v_{n+1}|_0$ zeros, it follows that $\tau(v_n) = v_{n+1}$. ☐

Definition. Let σ be a Christoffel substitution with incidence matrix M_σ that is an upper matrix. Then we call τ as defined in Theorem 3.6.7 the *dual substitution* of σ.

Corollary 3.6.8. *Let τ be the dual substitution of σ. Then σ is the dual substitution of τ.*

Proof. Let σ be a Christoffel substitution and let its incidence matrix be an upper matrix. It follows from Lemma 3.5.10 that $|v_0|_0 = 1$ and $|v_0|_1 = 0$. Hence according to Theorem 3.6.7 we have $v_n = \tau^n(0)$. It follows from Theorem 3.5.11 that M_τ is an upper matrix. Define z_n as the central word that corresponds to v_n, as in Corollary 3.5.9 (note that the left most letter of v_n is already at position 0, so that v_n doesn't have to be shifted to the right). Then it follows from this corollary and the fact that τ is a Christoffel substitution that $z_n = u_n$ for $n \in \mathbb{Z}_{\geq 0}$. ☐

Example 5. The duality relation exists between the following pairs σ, τ of Christoffel substitutions.

- $\sigma : \begin{cases} 0 & \to & 00101 \\ 1 & \to & 01 \end{cases}$ and $\tau : \begin{cases} 0 & \to & 01011 \\ 1 & \to & 011 \end{cases}$
- $\sigma : \begin{cases} 0 & \to & 0010101 \\ 1 & \to & 01 \end{cases}$ and $\tau : \begin{cases} 0 & \to & 0110111 \\ 1 & \to & 0111 \end{cases}$
- $\sigma : \begin{cases} 0 & \to & 0101011 \\ 1 & \to & 01011 \end{cases}$ and $\tau : \begin{cases} 0 & \to & 0001001 \\ 1 & \to & 001 \end{cases}$

The following Christoffel substitutions are self-dual (i.e. $\sigma = \tau$).

- $\sigma : \begin{cases} 0 & \to & 001 \\ 1 & \to & 01 \end{cases}$
- $\sigma : \begin{cases} 0 & \to & 0001 \\ 1 & \to & 001 \end{cases}$

The next lemma and theorem are thanks to a suggestion by Julien Cassaigne.

Lemma 3.6.9. *Let* $z = z_0 z_1 \dots z_n$ *be a finite word with* $z_i \in \{0, 1\}$. *Put* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = X_{z_0} X_{z_1} \dots X_{z_n}$. *Then* $X_{z_n} X_{z_{n-1}} \dots X_{z_0} = \begin{pmatrix} d & b \\ c & a \end{pmatrix}$.

Proof. It is easy to check that this holds for $n = 0, 1$. Assume it holds for n. Let $z = z_0 \dots z_{n+1}$ be given. Then $X_{z_0} \dots X_{z_{n+1}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} X_{z_{n+1}}$. In case $z_{n+1} = 0$ we get $\begin{pmatrix} a+b & b \\ c+d & d \end{pmatrix}$ and in case $z_{n+1} = 1$ we get $\begin{pmatrix} a & a+b \\ c & c+d \end{pmatrix}$. In the same way we find $X_{z_{n+1}} \dots X_{z_0}$ equals $\begin{pmatrix} d & b \\ c+d & a+b \end{pmatrix}$ in case $z_{n+1} = 0$, and $\begin{pmatrix} c+d & a+b \\ c & a \end{pmatrix}$ in case $z_{n+1} = 1$. We see that the lemma also holds for $n + 1$. $\square$

Let σ be a Christoffel substitution with incidence matrix M_σ that is an upper matrix. Hence there exists a finite word $z = z_0 z_1 \dots z_n$ so that $\sigma = \psi_{z_0} \dots \psi_{z_n} \psi_1$. If we write $M_\sigma = \begin{pmatrix} a+c & b+d \\ c & d \end{pmatrix}$, we see that $X_{z_n} \dots X_{z_0} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Hence using the previous lemma we get $X_{z_0} \dots X_{z_n} = \begin{pmatrix} d & b \\ c & a \end{pmatrix}$. We define the Christoffel substitution $\phi = \psi_{z_n} \dots \psi_{z_0} \psi_1$ and see that it has incidence matrix $M_\phi = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} d & b \\ c & a \end{pmatrix} = \begin{pmatrix} c+d & a+b \\ c & a \end{pmatrix}$. We find that M_ϕ is the dual matrix of M_σ and hence ϕ is the dual substitution of σ. This gives the following theorem.

Theorem 3.6.10. *Let* σ *be a Christoffel substitution with as incidence an upper matrix, so that we can write* $\sigma = \psi_{z_0} \dots \psi_{z_n} \psi_1$, *with* $z_i \in \{0, 1\}$. *Then the dual substitution of* σ *equals* $\psi_{z_n} \dots \psi_{z_0} \psi_1$.

Example 5 (continued). If we write the substitutions σ and τ from Example 5 as products of ψ_0 and ψ_1 we get the following pairs of substitutions.

- $\sigma = \psi_0\psi_1\psi_1$ and $\tau = \psi_1\psi_0\psi_1$
- $\sigma = \psi_0\psi_1\psi_1\psi_1$ and $\tau = \psi_1\psi_1\psi_0\psi_1$
- $\sigma = \psi_1\psi_0\psi_0\psi_1$ and $\tau = \psi_0\psi_0\psi_1\psi_1$
- $\sigma = \tau = \psi_0\psi_1$
- $\sigma = \tau = \psi_0\psi_0\psi_1$

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Chapter 4

Sturmian substitutions on two letters, cutting paths and their projections

4.1 Introduction

The history of Sturmian words goes back to J. Bernoulli in 1772 and Christoffel [23] (1875). The first in depth study of Sturmian words was made by Morse and Hedlund [38] in 1940. We call a substitution σ over an alphabet of two letters Sturmian if σ maps every Sturmian word to a Sturmian word. In 1991 Séébold [49] showed that Sturmian substitutions that have a fixed point are exactly those substitutions that have Sturmian words as fixed points. For further information on Sturmian words and substitutions we refer to Lothaire [35], Ch. 2 and Pytheas Fogg [27], Ch. 6.

If $u = u(0)u(1)\dots$ is a word defined over the alphabet $\mathcal{A} = \{0, \dots, n\}$, we define in the $(n+1)$-dimensional space the cutting points corresponding to u by $p_i = (|u(0)\dots u(i-1)|_0, \dots, |u(0)\dots u(i-1)|_n)$, where $|v|_a$ denotes the number of occurrences of the letter a in the word v. These cutting points approximate a half-line through the origin, that we call the cutting line, quite well. See Series [50] (1985). We can project the cutting points parallel to the cutting line, onto an n-dimensional plane through the origin. In 1982 Rauzy [41] introduced a fractal which is defined as the closure of the projection of the cutting points corresponding to the fixed point of the Tribonacci substitution $0 \to 01,\ 1 \to 02,\ 2 \to 0$. The analogues of this so-called Rauzy fractal have been studied for many other substitutions, see for example [6], [7], [21], [31]. An important question is whether the projection of the cutting points generates a tiling or not. This question was the motivation for Tijdeman and the author [46] to have a closer look at the structure of the projections of the cutting points corresponding to the set of finite words $\sigma^n(0)$ in case σ is the Fibonacci or Tribonacci substitution. In the Tribonacci case they found a close connection with number systems. This result was generalized by Fuchs and Tijdeman in [29]. In the case of the Fibonacci substitution, Tijdeman and the author projected the cutting points corresponding to $\sigma^n(0)$ onto the y-axis. By looking at the order of these projected points, they formed a two-letter word v_n, as we explain in Section 4.3 of this article. They found that the limit word that is generated in this way is a two-sided Fibonacci word. In [44] the author generalized this property to unimodular substitutions defined over two letters. He showed for a special class of Sturmian substitutions, which he called Christoffel substitutions, that the words v_n are generated by another Sturmian substitution. For more results on Christoffel substitutions see [18], [19] and [33]. In the present paper we generalize the result from [44] to the set of all Sturmian substitutions that have an incidence matrix with determinant 1.

In Section 4.2 we start with some notation and definitions. Next in Section 4.3 we derive some basic properties of the projections of the cutting points. In Section 4.4 we prove a number of results on Sturmian substitutions which we use in Section 4.5. Finally, in Theorem 4.5.8 we obtain our main result that the projection words v_n are generated by another Sturmian substitution. We give an explicit expression for this substitution in terms of the original substitution.

4.2 Notation and definitions

An *alphabet* $\mathcal{A}$ is a finite set of elements that are called *letters*. In this article we always assume $\mathcal{A} = \{0, 1\}$. A *word* is a function u from a finite or infinite block of integers B to $\mathcal{A}$. We call a word u finite when B is finite, infinite when $B = \mathbb{Z}_{\geq 0}$ and bi-infinite when $B = \mathbb{Z}$. If $0 \in B$ we call u a *central word*. If $k \in B$ and $u(k) = a$ we say u has the letter a at position k. When u is a central word, we often underline the letter at position 0. If $v = v(0) \dots v(m)$ is a finite word and if $(u(i))_{i \in B}$ is a finite or infinite word, and there exists a $k \in \mathbb{Z}$ such that $v(l) = u(k + l)$ for $l = 0, \dots m$, then v is called a *subword* of u. Moreover, if k is the smallest element of B, we say v is a *prefix* of u. On the other hand, if u is finite and $k + m$ is the largest element of B, we call v a suffix of u. If a word u is finite, we denote by $|u|$ the number of letters in u, and by $|u|_a$ the number of occurrences of the letter a in u.

A word u is called *balanced* if $||v|_0 - |w|_0| < 2$ for all subwords v, w of equal length. A finite word u is called *strongly balanced* if u^2 is balanced. Here u^2 is the concatenation of u with u. An infinite word is *Sturmian* if it is balanced and not ultimately periodic.

We call a finite word u a *Lyndon word* if every partition $u = vw$ implies $u < wv$ in the lexicographical order.

By cyclically shifting a finite word to the left (right, respectively) we mean removing the right-most (left-most, respectively) letter and placing it on the first open position on the left (right, respectively). By cyclically shifting over a number of positions we mean repeating this procedure a number of times. We call the new word a *cyclic shift* of the original one.

A *substitution* σ is an application from an alphabet $\mathcal{A}$ to the set of finite words. It extends to a morphism by concatenation, that is, $\sigma(uv) = \sigma(u)\sigma(v)$. It also extends in a natural way to a map from infinite words to infinite words. A substitution over the alphabet $\mathcal{A}$ is *primitive* if there exists a positive integer k such that, for every a and b in $\mathcal{A}$, the letter a occurs in $\sigma^k(b)$. We call $M_\sigma := \begin{pmatrix} |\sigma(0)|_0 & |\sigma(0)|_1 \\ |\sigma(1)|_0 & |\sigma(1)|_1 \end{pmatrix}$ the *incidence matrix* corresponding to σ. By M_σ^T we denote the transposed matrix. A *fixed point* of a substitution σ is an infinite word u with $\sigma(u) = u$. For each $a \in \mathcal{A}$ we specify one letter in $\sigma(a)$. If we apply σ to a central word $\dots u(-1)\underline{u(0)}u(1) \dots$, then the specified letter of $\sigma(u(0))$ will be the underlined letter at position 0 of $\dots \sigma(u(-1))\sigma(u(0))\sigma(u(1)) \dots$.

If x is a real number, and y the largest integer that is smaller than or equal to x, then we denote by $\{x\} = x - y$ the fractional part of x.

4.3 Central progressions

Definition. We call a function w a *central progression* if

- its domain is a block of integers of length m of $\mathbb{Z}$ containing 0,
- its image is the set $\{0, 1, \dots m-1\}$,
- there exists a $c \in \mathbb{Z}$ such that if k is in the domain of w, then $w(k) \equiv ck \pmod{m}$.

We denote by $|w|$ the length of the interval on which w is defined. Note that it follows from the definition that $\gcd(c, m) = 1$.

Example 1. Let the domain of w be $\{-4, -3, \dots, 1, 2\}$, so that $m = 7$, and let $c = 2$. Then $w(-4) = 6,\ w(-3) = 1,\ w(-2) = 3,\ w(-1) = 5,\ w(0) = 0,\ w(1) = 2,\ w(2) = 4$. For convenience we also use the notation $w = 6\ 1\ 3\ 5\ 0\ 2\ 4$.

Definition. Let $u = u(0) \dots u(m-1)$ be a finite word. The *cutting path* in the x-y-plane corresponding to u consists of $m+1$ integer points p_i given by $p_i = (|u(0) \dots u(i-1)|_0, |u(0) \dots u(i-1)|_1)$ for $i = 0, \dots, m$, connected by line segments of lengths 1.

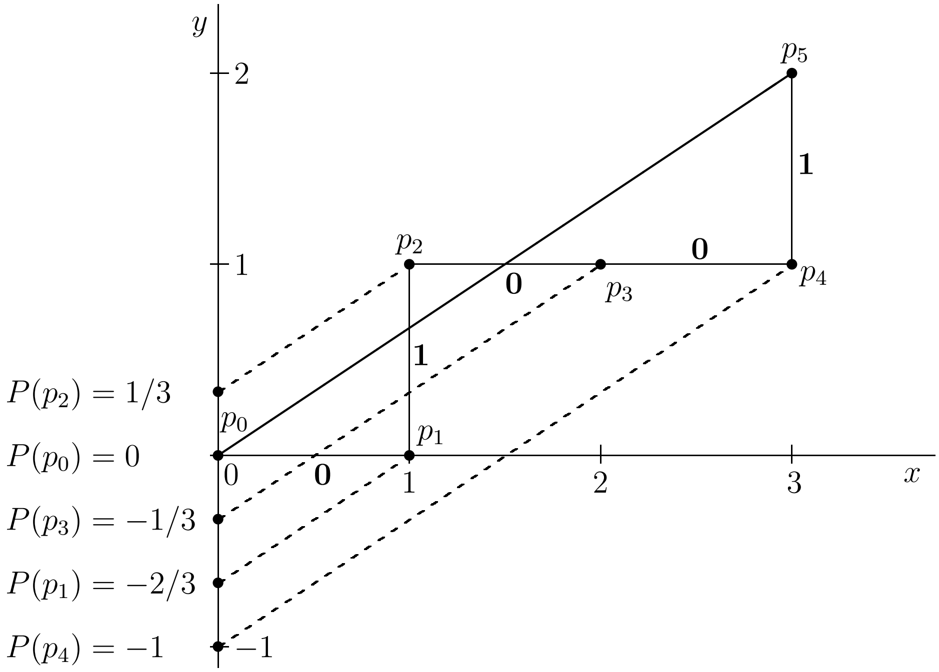
In the sequel of this section we let $u = u(0) \dots u(m-1)$ denote a finite strongly balanced word containing both zeros and ones, with $\gcd(|u|_0, |u|_1) = 1$. Consider the cutting path corresponding to u, and draw the line through the origin and the end point of the path, given by $y = \frac{|u|_1}{|u|_0} x$. We project each integer point p_i on the cutting path parallel to this line onto the y-axis. By $P(p_i)$ we denote the second coordinate of the projection of p_i. It is clear that $P(p_0) = P(p_m) = 0$.

Let

$$D_u = \left\{ -|u|_0 P(p_i) \middle| i \in \{0, \dots, m-1\} \right\}.$$

Note that D_u is a set of integers. We define $w_u : D_u \to \{0, \dots, m-1\}$ as follows. If $P(p_i) = k/|u|_0$ then $w_u(-k) = i$. We say w_u has the number i at position $-k$.

Example 2. Let $u = 01001$. Then we find that $D_u = \{-1, 0, 1, 2, 3\}$ and $w_u = 2\ 0\ 3\ 1\ 4$, as can be seen from the following picture.



Lemma 4.3.1. *For $k \in D_u$ we have $w_u(k) \equiv k|u|_1^{-1} \equiv -k|u|_0^{-1} \pmod{m}$, where the inverse is taken modulo m.*

Proof. The first congruence follows from Lemma 3.2.3. The proof of the second congruence follows from $|u|_0 + |u|_1 = m$ and $\gcd(|u|_0, |u|_1) = 1$. ☐

Corollary 4.3.2. *w_u is a central progression.*

Proof. Proposition 3.5.4 states that D_u is a block of integers of length m, if and only if u is strongly balanced. Since $P(p_0) = 0$ we have $0 \in D_u$. The third property of a central progression follows from the previous lemma. ☐

Remark. From the construction of the cutting path we see that if $w_u(k) = i$ then $w_u(k + |u|_1) = i + 1$ in case $u(i) = 0$, and $w_u(k - |u|_0) = i + 1$ in case $u(i) = 1$. We say that to move from number i to $i + 1$ in w_u we either jump $|u|_1$ positions to the right or jump $|u|_0$ positions to the left. The number of positions that we jump to the left or right is called the *length* of the jump.

From now on until the end of this section, we let w be a central progression of length m and $D = \{d, d + 1, \ldots, d + m - 1\}$ its domain. If $d < 0$ we put $z = w(-1)$, if $d = 0$ we put $z = w(m - 1)$.

Definition. We define v_w as the central word that you get by replacing every number in w smaller than z by 0, and every other number by 1.

Lemma 4.3.3. *v_w is strongly balanced.*

Proof. It follows from Lemma 4.3.1 that if $i \in D$, then $w(i)$ equals 0 if $\{-iz/m\} \in [0, z/m)$, and 1 otherwise. According to [35] Section 2.1.2 words defined in this way are strongly balanced (so-called rotation words). ☐

The following result follows from [35] Section 2.1.2.

Lemma 4.3.4. *v_w is a Lyndon word if and only if $d = 0$.*

We define $\widehat{w} : \mathbb{Z} \to \{0, 1, \dots m-1\}$ as follows. For given $k \in \mathbb{Z}$, put $\widehat{w}(k) = w(l)$, where l is such that $k \equiv l \pmod{m}$ and $l \in D$. Similarly $\widehat{v_w}$ is defined as the bi-infinite periodic continuation of v_w.

Lemma 4.3.5. *Put $x = w(1)^{-1} \pmod{m}$ and $y = (w(1)x - 1)/m$. Then every subword of $\widehat{v_w}$ of length x that starts at a position congruent to $-x$ modulo m, contains exactly $y+1$ ones, and every other subword of length x contains exactly y ones.*

Proof. Note that when we move a position to the right, the value of w increases by $w(1)$ modulo m. Because $w(1)x = ym + 1$, we see that when we start at position j in $\widehat{w}$, with j not equal to $-x$ modulo m, and move x times a position to the right, it happens exactly y times that we move to a value that is lower than the previous one. It follows from the definition of v_w that $\widehat{w}(i) > \widehat{w}(i+1) \iff \widehat{w}(i) \geq \widehat{w}(-1) \iff \widehat{v_w}(i) = 1$. Therefore the subword of $\widehat{v_w}$ of length x starting at position j contains y ones. When we start in a position equal to $-x$ modulo m, w goes down in value $y+1$ times since the last step is to a position with w-value 0, and therefore the subword of length x contains $y+1$ ones. □

Lemma 4.3.6. *Let w be a central progression taking values on -1 and 1. The difference between the number of w-values left of 0 that are larger than $w(1)$ and the number of w-values left of 0 that are smaller than $w(-1)$ is 0 or 1.*

Proof. It follows directly from the definition of central progression that $w(i+1) - w(i)$ is modulo m constant $w(1)$ for every i. Therefore every position j in w, for which $w(j) < w(-1)$ has a number directly on the right that is larger than $w(j)$, and hence at least equal to $w(1)$, except for the right-most position. Similarly, every position j in w, for which $w(j) \geq w(1)$ has a number directly on the left that is smaller than $w(j)$, and hence smaller than $w(-1)$, except for the left-most position. Hence we can form pairs of neighbouring numbers left of 0 for which the right one is larger than $w(1)$ and the left one smaller than $w(-1)$. Since every number left of 0 is either smaller or larger than $w(1)$ and therefore the only number without a companion is the left-most number in case it is larger than $w(1)$, the lemma follows. □

4.4 Sturmian substitutions

We call a 2×2-matrix *Sturmian* if it has determinant equal to ± 1 and has entries in $\mathbb{Z}_{\geq 0}$. We call a substitution ϕ over two letters *Sturmian* if $\phi(u)$ is a Sturmian word for every Sturmian word u. The following proposition follows directly from Proposition 2.3.10 of [35].

Proposition 4.4.1. *A two-letter substitution ϕ is Sturmian if and only if there exist two sequences of words $u^{(n)}, v^{(n)}$ such that*

- *$u^{(0)} = 0, v^{(0)} = 1$ or $u^{(0)} = 1, v^{(0)} = 0$,*
- *for every $n \geq 0$ we have $u^{(n+1)} \in \{u^{(n)}v^{(n)}, v^{(n)}u^{(n)}\}, v^{(n+1)} = v^{(n)}$ or $u^{(n+1)} = u^{(n)}, v^{(n+1)} \in \{u^{(n)}v^{(n)}, v^{(n)}u^{(n)}\}$,*
- *there exists an m such that $\phi(0) = u^{(m)}, \phi(1) = v^{(m)}$.*

Lemma 4.4.2. *If ϕ is a Sturmian substitution, then $\phi^n(0)$ is strongly balanced for every $n > 0$.*

Proof. It follows from Corollary 9 of [42] that a Sturmian substitution maps every finite balanced word to a finite balanced word. Hence $\phi^n(0)\phi^n(0) = \phi^n(00)$ is balanced. ☐

Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a Sturmian matrix. If $a + b > c + d$, then we call M an *upper matrix*, if $a + b < c + d$ a *lower matrix*.

Let $u = u(0) \dots u(m-1)$ denote a finite strongly balanced word containing both zeros and ones with $\gcd(|u|_0, |u|_1) = 1$ and let x be an integer with $0 \le x < |u|$. Then ${w_u}^{-1}(x)$ denotes the position of the number x in the central progression w_u.

Lemma 4.4.3. *Let ϕ be a Sturmian substitution with as incidence matrix a lower matrix $M_\phi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $abcd \neq 0$, and such that $\phi(0)$ is a prefix of $\phi(1)$. Let x, y be two different integers such that $0 \le x < |\phi(0)|$, $0 \le y < |\phi(0)|$ and ${w_{\phi(0)}}^{-1}(x) < {w_{\phi(0)}}^{-1}(y)$. Then ${w_{\phi(1)}}^{-1}(x) < {w_{\phi(1)}}^{-1}(y)$.*

Proof. It suffices to prove the statement for neighbouring numbers x, y in $w_{\phi(0)}$, with x left of y. Note that a jump in $w_{\phi(0)}$ to the left has length a and to the right length b. Similarly, a jump in $w_{\phi(1)}$ to the left has length c and to the right length d.

First assume $x < y$. We call i, j the number of jumps to the left and to the right, respectively, that we have to make in $w_{\phi(0)}$ to get from x to y. Hence $jb - ia = 1$. It is clear that $j \le a$ and it follows from Corollary 3.4.4 that $a \le c$. Since $\det(M_\phi) = \pm 1$ we get $a(jd - ic) = c(jb - ia) \pm j = c \pm j \ge 0$. If $jd - ic = 0$, then in $w_{\phi(1)}$ after j jumps to the right and i to the left, we are back in our starting position. Hence $id - jc > 0$. Since $\phi(0)$ is a prefix of $\phi(1)$, it follows that if we go from x to y in $w_{\phi(1)}$ we also make i jumps to the left and j to the right. Hence we move $jd - ic \ge 1$ positions to the right.

Now assume $x > y$. Then we find in the same way that when we go from x to y in $w_{\phi(1)}$ we move at least 1 position to the left. ☐

Lemma 4.4.4. *Let ϕ be a Sturmian substitution with as incidence matrix a lower matrix $M_\phi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $abcd \neq 0$, and such that $\phi(0)$ is a suffix of $\phi(1)$. Let x, y be two different integers such that $0 \le x < |\phi(0)|$, $0 \le y < |\phi(0)|$ and ${w_{\phi(0)}}^{-1}(x) < {w_{\phi(0)}}^{-1}(y)$. Then ${w_{\phi(1)}}^{-1}(x + c + d - (a + b)) < {w_{\phi(1)}}^{-1}(y + c + d - (a + b))$.*

Proof. The proof is analogous to the proof of the previous lemma. ☐

Lemma 4.4.5. *Let $\begin{pmatrix} a & b \\ c + ag & d + cg \end{pmatrix}$ be a Sturmian matrix with determinant 1, such that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is an upper matrix. Then $a^{-1} = c + d \pmod{a + b}$.*

Proof. This follows directly from the fact that $ad - bc = 1$. ☐

Lemma 4.4.6. *Let ϕ be a Sturmian substitution that has a fixed point starting with 0, and has an incidence matrix $M_\phi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ that has determinant equal to 1. Let x, y be two different integers such that $0 \le x < |\phi(0)|$, $0 \le y < |\phi(0)|$ and let n be a positive integer so that ${w_{\phi^n(0)}}^{-1}(x) < {w_{\phi^n(0)}}^{-1}(y)$. Then ${w_{\phi^{n+1}(0)}}^{-1}(x) < {w_{\phi^{n+1}(0)}}^{-1}(y)$.*

Proof. It suffices to prove the statement for neighbouring numbers x, y in $w_{\phi^n(0)}$, with x left of y. Put $\begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} := M_\phi^n$. Note that a jump in $w_{\phi^n(0)}$ to the left has length a_n and a jump to the right length b_n. Since

$$M_\phi^{n+1} = \begin{pmatrix} aa_n + bc_n & ab_n + bd_n \\ \cdots & \cdots \end{pmatrix},$$

a jump in $w_{\phi^{n+1}(0)}$ to the left has length $aa_n + bc_n$ and a jump to the right length $ab_n + bd_n$.

First assume $x < y$. We call i, j the number of jumps in $w_{\phi^n(0)}$ to the left and to the right, respectively, that we have to make to get from x to y. Hence $jb_n - ia_n = 1$. Since $w_{\phi^n(0)}$ is a prefix of $w_{\phi^{n+1}(0)}$, we see that to move from x to y in $w_{\phi^{n+1}(0)}$ we move $jb_{n+1} - ia_{n+1}$ positions to the right. We have

$$a_n(jb_{n+1} - ia_{n+1}) = a_n(a + jbd_n - ibc_n) = aa_n + jb + jbb_nc_n - iba_nc_n$$

$$= aa_n + bc_n + jb > 0,$$

hence the integer y is on the right of x in $w_{\phi^{n+1}(0)}$.

Now assume $x > y$. We call i, j the number of jumps in $w_{\phi^n(0)}$ to the left and to the right, respectively, that we have to make to get from y to x. Hence $jb_n - ia_n = -1$. To move from y to x in $w_{\phi^{n+1}(0)}$ we move $jb_{n+1} - ia_{n+1}$ positions to the right. We have

$$a_n(jb_{n+1} - ia_{n+1}) = a_n(-a + jbd_n - ibc_n) = -aa_n + jb + jbb_nc_n - iba_nc_n$$

$$= -aa_n - bc_n + jb.$$

If M_ϕ^n is a lower matrix, we have $j \le a_n \le c_n$ which gives $jb_{n+1} - ia_{n+1} < 0$ and we are done. Assume M_ϕ^n is an upper matrix. Note that $w_{\phi^n(0)}(-1) = c_n + d_n$, where we use Lemma 4.3.1 and Lemma 4.4.5. In case $j \le c_n$ we have $-aa_n - bc_n + jb < 0$ hence $jb_{n+1} - ia_{n+1} < 0$ and we are done. In case $j > c_n$ we have $i < d_n$, hence

$$b_n(jb_{n+1} - ia_{n+1}) = b_n(-a + jbd_n - ibc_n) = -ab_n + ib - bd_n < 0,$$

and thus $jb_{n+1} - ia_{n+1} < 0$. ☐

Definition. Let $M_\phi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a Sturmian matrix with determinant equal to 1 and $abcd \neq 0$. Let ϕ be the substitution that has M_ϕ as incidence matrix, such that $\phi(0), \phi(1)$ are balanced Lyndon words. Then we call ϕ a *Christoffel substitution.*

It follows from Section 3.6 that for every Sturmian matrix there exists a unique Christoffel substitution that has that matrix as incidence matrix. The following result is proved in Chapter 3 as Lemma 3.6.4.

Lemma 4.4.7. *Let ϕ be a Christoffel substitution. If the corresponding incidence matrix is an upper matrix, there exist words u and v, possibly empty, such that $\phi(0) = u0v$ and $\phi(1) = u1$. If it is a lower matrix, there exist words u and v, possibly empty, such that $\phi(0) = u0v$ and $\phi(1) = (u0v)^ku1$ for some positive integer k.*

Corollary 4.4.8. *Let ϕ be a Christoffel substitution. If the corresponding incidence matrix is a lower matrix, there exist possibly empty words u and v such that $\phi(0) = 0u$ and $\phi(1) = v1u$. If it is an upper matrix, there exist possibly empty words u and v such that $\phi(0) = 0u(v1u)^k$ and $\phi(1) = v1u$ for some positive integer k.*

Proof. Let ϕ have incidence matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Note that if we start with a Lyndon word, interchange the zeros and ones, and read it backwards, the result is a Lyndon word again. Let ψ be the substitution obtained from ϕ by interchanging the zeros and ones, swapping $\psi(0)$ and $\psi(1)$ and reading them backwards. Then ψ has incidence matrix $\begin{pmatrix} d & c \\ b & a \end{pmatrix}$, and the result follows directly from Lemma 4.4.7. ☐

Lemma 4.4.9. *Let ϕ be a Christoffel substitution that has an upper matrix as incidence matrix. Let k with $0 < k < |\phi(0)|$ be an integer and let ψ be the substitution such that you get $\psi(0)$ by cyclically shifting $\phi(0)$ over k positions to the left, and $\psi(1)$ by cyclically shifting $\phi(1)$ over k positions to the left. Then*

- *a) $\psi(1)$ is a prefix of $\psi(0)$,*
- *b) ψ is a Sturmian substitution.*

Proof. a) It follows from Corollary 4.4.8 that $\phi(0) = 0u(\phi(1))^l$ with u a suffix of $\phi(1)$, and from Lemma 4.4.7 that $\phi(1) = v1$ with v a prefix of $\phi(0)$. This implies the statement.
b) First let ψ be the substitution that corresponds to $k = 1$. We know from (a) that $\psi(1)$ is a prefix of $\psi(0)$. We use induction to show that $\psi^n(0)$ is a cyclic shift of $\phi^n(0)$ for every $n > 0$. For $n = 1$ this is clear. Assume it is true for $n = m - 1$. Consider the word $\psi^m(0)$. By cyclically shifting it one position to the right, we get the word $\phi(\psi^{m-1}(0))$, where we use that $\psi(0)$ and $\psi(1)$ start with the same letter. It follows from the induction hypothesis that this is a cyclic shift of $\phi^m(0)$.

Now let k be any number with $0 < k < |\phi(0)|$ and ψ the corresponding cyclic shift of ϕ. By induction on k it follows that $\psi^n(0)$ is a cyclic shift of $\phi^n(0)$ for every $n > 0$. Since ϕ is a Christoffel substitution, $\phi^n(0)$ is strongly balanced for every n, and it follows that $\psi^n(0)$ is strongly balanced. Hence the limit word that you get by letting $n \to \infty$ is a Sturmian word that is mapped to itself by ψ, and it follows from [35] Theorem 2.3.7 that ψ is a Sturmian substitution. ☐

Proposition 4.4.10. *Let $N = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a Sturmian upper matrix with determinant 1 and let v be a strongly balanced word that is not Lyndon, with $|v|_0 = a, |v|_1 = b$. Then the substitution ψ, with $\psi(0) = v$ and $\psi(1)$ the prefix of $\psi(0)$ of length $c + d$, is a Sturmian substitution that has N as incidence matrix.*

Proof. Let k be the number of times you have to cyclically shift v to the right, to get a Lyndon word. Let ϕ be the Christoffel substitution that has N as incidence matrix. Let ψ' be the substitution such that you get $\psi'(0)$ by cyclically shifting $\phi(0)$ over k positions to the left, and $\psi'(1)$ by cyclically shifting $\phi(1)$ over k positions to the left. Then according to Lemma 4.4.9 (a) $\psi'(1)$ is a prefix of $\psi'(0)$, hence $\psi' = \psi$. According to Lemma 4.4.9 (b) ψ is a Sturmian substitution. Clearly ψ' has the same incidence matrix as ϕ which is N.☐

Remark. The result of Proposition 4.4.10 does not hold if $\psi(0)$ is a Lyndon word. Take for example $N = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$ and $\psi(0) = 00101$. Then $\psi(1) := 00$ is the prefix of $\psi(0)$ of length 2, but the substitution ψ is not Sturmian and does not have N as incidence matrix.

4.5 Projecting Sturmian substitutions

From now on let $M_\sigma = \begin{pmatrix} a & b \\ c+ag & d+bg \end{pmatrix}$ be a Sturmian matrix with determinant 1, such that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is an upper Sturmian matrix, and let σ be a primitive Sturmian substitution that has M_σ as incidence matrix, has a fixed point starting with 0, and such that $\sigma(0)$ is not a Lyndon word. Note that the case that σ is a Christoffel substitution has been considered in Section 3.6.

Put $w_n = w_{\sigma^n(0)}$ and $v_n = v_{w_n}$ for $n \geq 1$, where w_u and v_w are defined in Section 3. Denote by e, f the number of values left of the 0 position in $w_{\sigma(0)}$ and $w_{\sigma(1)}$ respectively, by p the number of 0's in v_1 left of the underlined letter, and set $r = e + b(f - p - eg)$. Note that $e > 0$ since $\sigma(0)$ is not a Lyndon word, hence $p < e$ and $r > 0$.

Example 3. Let $M_\sigma = \begin{pmatrix} 3 & 2 \\ 7 & 5 \end{pmatrix}$ and $\sigma(0) = 01001$, $\sigma(1) = 010010101001$. Then $g = 2$, $w_{\sigma(0)} = 2\ 0\ 3\ 1\ 4$, $v_1 = 1\underline{0}101$ and $w_{\sigma(1)} = 9\ 2\ 7\ 0\ 5\ 10\ 3\ 8\ 1\ 6\ 11\ 4$. Hence $e = 1,\ f = 3,\ p = 0$ and $r = 3$.

Lemma 4.5.1. *Let $g = 0$. Then $f - p = 0$ or $f - p = 1$.*

Proof. By Proposition 4.4.1, we know that $\sigma(1)$ is a prefix of $\sigma(0)$, or $\sigma(1)$ is a suffix of $\sigma(0)$.

Assume $\sigma(1)$ is a prefix of $\sigma(0)$. If we apply Lemma 4.4.3 to the substitution ϕ with $\phi(0) = \sigma(1)$ and $\phi(1) = \sigma(0)$, we see that the $c + d$ numbers of $w_{\sigma(1)}$ appear in the same order in $w_{\sigma(0)}$. Hence f equals the number of w-values in $w_{\sigma(0)}$ left of 0 that are smaller than $c + d$. It follows from Lemma 4.3.1 and Lemma 4.4.5 that $w_{\sigma(0)}(-1) = c + d$. Since p is by definition the number of zeros left of the underlined letter in v_1, we get $f = p$.

Now assume $\sigma(1)$ is a suffix of $\sigma(0)$. Then the sequence of jumps to the right and left in $w_{\sigma(1)}$ starting from the number 0, are in the same order as the sequence of jumps to the right and left in $w_{\sigma(0)}$ starting from the number $a + b - (c + d)$. Hence f equals the number of w-values in $w_{\sigma(0)}$ left of $a + b - (c + d)$ that are larger than $a + b - (c + d)$. It follows from $w_{\sigma(0)}(-1) = c + d$ that $w_{\sigma(0)}(1) = a + b - (c + d)$. Recall that p equals the number of w-values in $w_{\sigma(0)}$ left of 0 that are smaller than $c + d$. It follows from Lemma 4.3.6 that $f - p$ is 0 or 1. ☐

Lemma 4.5.2. *If $g > 0$, there exist nonnegative integers x, y with $x + y = g$ and a word u such that $\sigma(1) = \sigma(0)^x u \sigma(0)^y$ and such that u is a prefix or suffix of $\sigma(0)$.*

Proof. Since g is the maximum number of times we can subtract the sum of the entries in the top row of M_σ from the sum of the entries in the bottom row, it follows from Proposition 4.4.1 that g is the maximum number of times we can remove $\sigma(0)$ as a prefix or suffix from $\sigma(1)$. It also follows from Proposition 4.4.1 that u is a prefix or suffix of $\sigma(0)$. ☐

Lemma 4.5.3. *We have*

(i) $(d+bg)^{-1} \equiv a+b \pmod{c+d+ag+bg}$*,*

(ii) $(c+d)(a^2+bc+abg) \equiv a+bg \pmod{a^2+bc+abg+ab+bd+b^2g}$*,*

(iii) $(a^2+bc+abg)^{-1} \equiv ac+cd+adg+bc+d^2+bdg \pmod{a^2+bc+abg+ab+bd+b^2g}$*.*

Proof.

(i) $(a+b)(d+bg)-1 = b(c+d+ag+bg)$,

for (ii), (iii) the proofs are similar. $\square$

Proposition 4.5.4. $0 \le f-p-eg \le g+1$.

Proof. In case $g=0$ this follows from Lemma 4.5.1.

Assume $g>0$. According to Lemma 4.5.2 there exist nonnegative integers x, y with $x+y=g$ and a word u that is a prefix or suffix of $\sigma(0)$ such that $\sigma(1) = \sigma(0)^x u \sigma(0)^y$. From (i) in Lemma 4.5.3 it follows that $|\sigma(1)|_1^{-1} \equiv |\sigma(0)| \pmod{|\sigma(1)|}$. Hence according to Lemma 4.3.1 we have $w_{\sigma(1)}(1) = |\sigma(0)| = a+b$. So when we start at position 0 in $w_{\sigma(1)}$ and make $a+b$ jumps, where the jumps follow the pattern indicated by the word $\sigma(0)$, we end up at position 1. After repeating this process x times we end up at position x. After making the next $|u|$ jumps we must end up at position $-y$, since after repeating the process of making $a+b$ jumps y times at the end, we get back at position 0. This yields $y \le g$ numbers of $w_{\sigma(1)}$ in the negative positions $-y, -y+1, \ldots, -1$.

Notice that $w_{\sigma(0)}$ contains the numbers 0 up to $a+b-1$, of which by definition e numbers are left of 0. If $w_{\sigma(0)}$ is a prefix of $w_{\sigma(1)}$ (which means $x>0$), it follows from Lemma 4.4.3 that the numbers between 0 and $a+b$ appear in the same order in $w_{\sigma(1)}$ as they do in $w_{\sigma(0)}$. Hence $w_{\sigma(1)}$ has exactly e numbers between 0 and $a+b$ that are left of position $-y$. If $w_{\sigma(0)}$ can be removed twice as a prefix of $w_{\sigma(1)}$ (which means $x>1$), the jumps from $a+b$ to $2(a+b)$ in $w_{\sigma(1)}$ are an exact copy of the jumps from 0 to $a+b$. So $w_{\sigma(1)}$ also has exactly e numbers between $a+b$ and $2(a+b)$ that are left of position $-y$. Continuing in this way up to x times, this generates exactly xe numbers left of position $-y$ in $w_{\sigma(1)}$.

In the same way, using Lemma 4.4.4, we see that ye numbers are generated left of position $-y$ in $w_{\sigma(1)}$ by the suffix $\sigma(0)^y$ of $\sigma(1)$. Together this gives $xe+ye=eg$ numbers left of position $-y$ in $w_{\sigma(1)}$.

Now we consider the remaining subword u which is a prefix or suffix of $\sigma(0)$. When we jump in $w_{\sigma(1)}$ from position x to position $-y$, where the jumps follow the pattern indicated by the word u, it generates a certain amount of values left of position $-y$ in $w_{\sigma(1)}$. We shall show that this amount is equal to p or $p+1$.

Note that $|u|_0 = c$, $|u|_1 = d$. Recall that $w_{\sigma(0)}(-1) = c+d$ and $w_{\sigma(0)}(1) = a+b-(c+d)$. Consider the substitution ϕ with $\phi(0) = u$ and $\phi(1) = \sigma(0)$, which has incidence matrix $M_\phi = \begin{pmatrix} c & d \\ a & b \end{pmatrix}$. It follows from Proposition 4.4.1 that ϕ is a Sturmian substitution.

In case u is a prefix of $\sigma(0)$, we see by applying Lemma 4.4.3 to ϕ, that the $c+d$ numbers in $w_{\phi(0)} = w_u$ are in the same order as the numbers 0 up to $c+d-1$ in $w_{\phi(1)} = w_{\sigma(0)}$. Hence the number of values left of 0 in w_u equals the number of values left of 0 in $w_{\sigma(0)}$ that are smaller than $c+d = w_{\sigma(0)}(-1)$, which is p by definition.

In case u is a suffix of $\sigma(0)$, we see by applying Lemma 4.4.4 to ϕ, that the $c+d$ numbers in $w_{\phi(0)} = w_u$ are in the same order as the numbers $a+b-(c+d)$ up to $a+b-1$ in $w_{\phi(1)} = w_{\sigma(0)}$. Hence the number of values left of 0 in w_u equals the number of values in $w_{\sigma(0)}$ left of $w_{\sigma(0)}(1) = a+b-(c+d)$ that are at least $a+b-(c+d)$. Since p equals by definition the number of values in $w_{\sigma(0)}$ left of 0 that are smaller than $w_{\sigma(0)}(-1)$, it follows from Lemma 4.3.6 that the number of w-values left of 0 in w_u is p or $p+1$.

Because the order of the $c+d$ numbers in $w_{\phi(0)} = w_u$ is preserved in $w_{\phi(1)} = w_{\sigma(0)}$, and the order of the $a+b$ numbers in $w_{\sigma(0)}$ is preserved in $w_{\sigma(1)}$, we see that when we jump in $w_{\sigma(1)}$ from position x to position $-y$, where the jumps follow the pattern indicated by the word u, it generates either p or $p+1$ numbers left of position $-y$ in $w_{\sigma(1)}$. It follows that the total number of values left of position 0 in $w_{\sigma(1)}$ is in between $p+eg+y$ and $p+eg+y+1$, hence in between $p+eg$ and $p+eg+g+1$. □

Definition. We denote by τ the substitution that has

$$M_\tau := \begin{pmatrix} 1 & g+1 \\ 1 & g \end{pmatrix} M_\sigma^T \begin{pmatrix} -g & g+1 \\ 1 & -1 \end{pmatrix}$$

as incidence matrix, and which is such that

- if we cyclically shift $\tau(0)$ over r positions to the right, we get a Christoffel word,
- the $(r+1)$th letter of $\tau(0)$ is underlined,
- $\tau(1)$ is a prefix of $\tau(0)$.

It follows from Proposition 4.4.10 that τ is a Sturmian substitution.

Example 3 (Continued). We get $M_\tau = \begin{pmatrix} 4 & 5 \\ 3 & 4 \end{pmatrix}$ and $\tau(0) = 011\underline{0}10101$, $\tau(1) = 0110101$. Note furthermore that

$$\sigma^2(0) = 010010100101010010100101001010010101001,$$

$w_2 = w_{\sigma^2(0)} = 14\ 36\ 19\ 2\ 24\ 7\ 29\ 12\ 34\ 17\ 0\ 22\ 5\ 27\ 10\ 32\ 15\ 37\ 20\ 3\ 25\ 8\ 30\ 13\ 35\ 18\ 1\ 23\ 6\ 28\ 11\ 33\ 16\ 38\ 21\ 4\ 26\ 9\ 31$ and

$$v_2 = 0110101011\underline{0}1010101101010110101010110101.$$

Lemma 4.5.5. *Let $n \geq 1$. Then the number of values left of the 0 position in the central progression w_n equals*

$$(1,0)\sum_{k=0}^{n-1} M_\sigma^k \begin{pmatrix} e \\ f \end{pmatrix}.$$

Proof. For $n = 1$ the statement follows directly from the definition of e. Assume the statement is true for some positive integer n.

According to Lemma 4.4.6 the numbers in w_n appear in the same order in w_{n+1}. Put $M_\sigma = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ and $M_\sigma^n = \begin{pmatrix} a'_n & b'_n \\ c'_n & d'_n \end{pmatrix}$. Using

$$\begin{pmatrix} a'_{n+1} & b'_{n+1} \\ c'_{n+1} & d'_{n+1} \end{pmatrix} = \begin{pmatrix} a'_n & b'_n \\ c'_n & d'_n \end{pmatrix} \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \begin{pmatrix} a'_n & b'_n \\ c'_n & d'_n \end{pmatrix}$$

it is easy to check that $a'b'_{n+1} - b'a'_{n+1} = b'_n$ and $c'b'_{n+1} - d'a'_{n+1} = -a'_n$. It follows that if there exists a jump to the right in w_n from some position x to some position y, then there exists a series of $a' + b'$ jumps in w_{n+1} from position x to position y. Similarly if there exists a jump to the left in w_n from some position x to some position y, then there exists a series of $c' + d'$ jumps in w_{n+1} from position x to position y.

It follows from the previous remarks, that the first $a' + b'$ jumps in w_{n+1}, from position 0 to position b'_n, generate e numbers that are located left of the most left number in w_n. In the same way each other jump to the right in w_n corresponds to a series of $a' + b'$ jumps in w_{n+1} which each generate e numbers at positions left of the most left position of w_n, and each jump to the left in w_n corresponds to a series of $c' + d'$ jumps in w_{n+1} which each generate f numbers at positions left of the most left position of w_n.

Note that the number of jumps to the right and left in w_n are given by the two entries in the top row of M_σ^n, respectively. Hence w_{n+1} contains the same number of values left of 0 as w_n has, augmented with $(1, 0)M_\sigma^{n+1}\begin{pmatrix} e \\ f \end{pmatrix}$, and the result follows. □

Let q be the number of zeros in $\tau(0)$ left of the underlined letter.

Lemma 4.5.6. *For $n \geq 1$ the number of letters left of the underlined letter in $\tau^{n-1}(v_1)$ equals*

$$\left((p, e-p)\sum_{k=0}^{n-1} M_\tau^k + (q-p, r-q-e+p)\sum_{k=0}^{n-2} M_\tau^k\right)\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Proof. Note that if we apply any substitution ϕ with incidence matrix M_ϕ to a word u containing x zeros and y ones, then the number of zeros and ones in $\phi(u)$ is given by the elements of $(x, y)M_\phi$, respectively. By definition v_1 has p zeros and $e - p$ ones left of the underlined letter. After applying τ^{n-1} to v_1 these zeros and ones yield $(p, e-p)M_\tau^{n-1}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ letters left of the underlined letter. Furthermore, $\tau(\underline{0})$ has by definition q zeros and $r - q$ ones left of the underlined letter. Hence $\tau^{n-1}(\underline{0})$ has $(q, r-q)(M_\tau^0 + M_\tau^1 + \ldots + M_\tau^{n-2})\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ letters left of the underlined letter. We see that the total number of letters left of the underlined letter in the central word $\tau^{n-1}(v_1)$ equals

$$\left((p, e-p)M_\tau^{n-1} + (q, r-q)\sum_{k=0}^{n-2} M_\tau^k\right)\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

From this the result follows directly. □

Proposition 4.5.7.

$$d(f - p - eg) = q - p.$$

Proof. First assume $g = 0$ (i.e. M_σ is an upper matrix). We know from Lemma 4.5.1 that $f - p$ equals 0 or 1.

If $f = p$, it follows that $r = e$. Since v_1 and $\tau(0)$ are both strongly balanced, have the same number of zeros, the same number of ones, and the same number of letters left of the underlined letter, we get $v_1 = \tau(0)$, hence $q = p$.

Assume $f - p = 1$. Then $r = e + b$. Hence $\tau(0)$ is the cyclic shift of v_1 that you get by cyclically shifting over b positions to the left. Since $b(a + b - (c + d)) = (b - d)(a + b) + 1$, it follows from Lemma 4.3.5 that the prefix of length b of $\tau(0)$ contains exactly $b-d$ ones, hence exactly d zeros. Thus $q-p=d$.

Now assume $g>0$, hence M_σ is a lower matrix. We know from Proposition 4.5.4 that $f-p-eg \leq g+1$. Since the first jump in $w_{\sigma(0)}$ is to the right of length b, we see that $w_{\sigma(0)}$ has at most $a-1$ numbers left of 0, hence $e<a$. It follows that $r<a+b(g+1)$.

Note that

$$M_\sigma^2 = \begin{pmatrix} a^2+bc+abg & ab+bd+b^2g \\ ac+a^2g+cd+bcg+adg+abg^2 & bc+abg+d^2+2bdg+b^2g^2 \end{pmatrix}.$$

It follows from Lemma 4.3.1 and Lemma 4.4.5 that $w_1(-1)=c+d$ and $w_2(-1)=ac+cd+adg+bc+d^2+bdg$.

Let s be the positive integer such that $w_2(-s)=c+d$. Since $w_2(-1)\equiv(a^2+bc+abg)^{-1} \pmod{|w_2|}$ we see that $s\equiv(c+d)(a^2+bc+abg)\equiv a+bg \pmod{|w_2|}$ according to Lemma 4.5.3 (ii). Hence $s=a+bg$.

A calculation gives $bw_2(1)=b(|w_2|-w_2(-1))=(b-d)|w_2|+a+b$. Using a similar argument as in the proof of Lemma 4.3.5, we see that every subword of b letters in v_2 contains d zeros and $b-d$ ones, provided that the number in w_2 at the position directly right of this subword is at least $a+b$. The numbers smaller than $a+b$ in w_2 are exactly the numbers in $w_{\sigma(0)}$, and since according to Lemma 4.4.6 their order is preserved in w_2, the first number left of 0 in w_2 that is smaller than $a+b$ is $w_{\sigma(0)}(-1)=c+d$ which can be found at position $-s=-(a+bg)$ in w_2.

Since $r<a+b(g+1)$, the position of the left most letter of $\tau(0)$ is larger than $-(a+b(g+1))$, and therefore the values of w_2, at the negative positions of $\tau(0)$ are each at least $a+b$. Hence each subword of length b of $\tau(0)$ left of position -1 contains exactly d zeros. Since $r-e=b(f-p-eg)$, we obtain $q-p=d(f-p-eg)$. □

The next theorem says that if $\sigma(0)$ is not a Christoffel word, the words v_n are generated by another Sturmian substitution. In case σ is a Christoffel substitution, this was proved in Theorem 3.6.7. Recall that τ is the substitution that has

$$M_\tau := \begin{pmatrix} 1 & g+1 \\ 1 & g \end{pmatrix} M_\sigma^T \begin{pmatrix} -g & g+1 \\ 1 & -1 \end{pmatrix}$$

as incidence matrix, and which is such that

- if we cyclically shift $\tau(0)$ over r positions to the right, we get a Christoffel word,
- the $(r+1)$th letter of $\tau(0)$ is underlined,
- $\tau(1)$ is a prefix of $\tau(0)$.

Theorem 4.5.8. *Let σ be a primitive Sturmian substitution that has an incidence matrix with determinant 1, that has a fixed point starting with 0, and for which $\sigma(0)$ is not a Christoffel word. Let the central words v_n for $n\geq 1$ be defined as before. Then $v_n=\tau^{n-1}(v_1)$.*

Proof. Note that

$$M_\sigma = \begin{pmatrix} 1 & 0 \\ g & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } \mathrm{M}_\tau = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & g \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}.$$

Hence for $k > 0$

$$(1,0)M_\sigma^k \begin{pmatrix} e \\ f \end{pmatrix} = (e,f)\left[\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} 1 & g \\ 0 & 1 \end{pmatrix}\right]^k \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= (e,f)\begin{pmatrix} a & c \\ b & d \end{pmatrix}\left[\begin{pmatrix} 1 & g \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix}\right]^{k-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} =$$

$$(e,f)\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}\left[\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & g \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}\right]^{k-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= (e,f)\begin{pmatrix} c & a-c \\ d & b-d \end{pmatrix} M_\tau^{k-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

By Proposition 4.5.7 and $r = e + b(f - p - eg)$ we have

$$(f-p-eg)(d, b-d) = (q-p, r-q-e+p).$$

Hence

$$(e,f)\begin{pmatrix} c & a-c \\ d & b-d \end{pmatrix} = (e,p)\begin{pmatrix} 1 & g \\ 0 & 1 \end{pmatrix}\begin{pmatrix} c & a-c \\ d & b-d \end{pmatrix} + (0, f-p-eg)\begin{pmatrix} c & a-c \\ d & b-d \end{pmatrix}$$

$$= (p, e-p)M_\tau + (q-p, r-q-e+p).$$

From the above formulas we conclude that for $n \geq 1$

$$(1,0)\sum_{k=0}^{n-1} M_\sigma^k \begin{pmatrix} e \\ f \end{pmatrix} = \left((p, e-p)\sum_{k=0}^{n-1} M_\tau^k + (q-p, r-q-e+p)\sum_{k=0}^{n-2} M_\tau^k\right)\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

According to Lemma 4.5.5, the expression on the left represents the number of letters left of the underlined letter in v_n, and according to Lemma 4.5.6 the expression on the right represents the number of letters left of the underlined letter in $\tau^{n-1}(v_1)$. Since v_n and $\tau^{n-1}(v_1)$ are strongly balanced words, and they contain according to Theorem 3.5.11 the same number of zeros and ones, the result follows. □

4.6 Final remarks

If the substitution τ has a fixed point starting with 0, we can apply the procedure of projecting the cutting paths corresponding to τ, to construct the substitution ϕ, in the same way as we constructed the substitution τ from the substitution σ. It is easy to check that if M_σ is an upper matrix then $M_\phi = M_\sigma$, hence $\phi(0), \phi(1)$ are cyclic shifts of $\sigma(0), \sigma(1)$ respectively. We have seen in Corollary 3.6.8 that if σ is a Christoffel substitution, then we get $\phi = \sigma$. This is not the case in general, as the following example shows.

Example 4. Let $M_\sigma = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}$ and $\sigma(0) = 0101101,\ \sigma(1) = 01101$. Then $e = 1,\ f = 1,\ p = 0$ and $r = 5$. We get $M_\tau = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$ and $\tau(0) = 0100100$, $\tau(1) = 010$. Repeating this process, we get $e = 4,\ f = 1,\ p = 1$ and $r = 4$, which results in

$\phi(0) = 1011010,\ \phi(1) = 10110.$

If σ is a Sturmian substitution that has an incidence matrix with determinant 1, for which $\sigma(0)$ is a Christoffel word, but that is not a Christoffel substitution, there need not exist a substitution τ such that $v_n = \tau^{n-1}(v_1)$, as the following example shows.

Example 5. Let $M_\sigma = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ and $\sigma(0) = 01,\ \sigma(1) = 101$. Then we get the following table for v_n.

n					v_n								
1					$\underline{0}$	1							
2				1	$\underline{0}$	0	1	0					
3	0	1	0	1	$\underline{0}$	0	1	0	0	1	0	1	0

……

It is clear that there is no substitution τ such that $\tau(v_1) = v_2$ and $\tau(v_2) = v_3$.

If σ is a Sturmian substitution with a fixed point starting with 0 that has an incidence matrix M_σ with determinant -1, we can still form the central words v_n, except that for odd n, we need to reflect the central progressions w_n in the origin, before we construct v_n from w_n. Since the substitution σ^2 has incidence matrix with determinant 1, it is clear that there exists a substitution τ_2 such that $v_{2n} = \tau_2(v_{2n-2})$. But as the following example shows, there does not need to exist a substitution τ such that $v_n = \tau(v_{n-1})$.

Example 6. Let $M_\sigma = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$ and $\sigma(0) = 001,\ \sigma(1) = 0$. Then we get the following table for v_n.

n													v_n				
1											1	1	$\underline{0}$				
2											1	1	$\underline{0}$	1	0	1	0
3	1	1	0	1	0	1	1	0	1	0	1	1	$\underline{0}$	1	0	1	0

……

It is easy to check that there is no substitution τ such that $\tau(v_2) = v_3$. However, this example suggests that if we define the substitution τ by $\tau(\underline{0}) = 11\underline{0},\ \tau(1) = 10$, then we have $v_n = \tau(\overline{v_{n-1}})$, where we denote by $\overline{u}$ the word u mirrored in the origin. An interesting question is if this holds for all Sturmian substitutions that have an incidence matrix with determinant -1.

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Chapter 5

Beta-substitutions, cutting paths and their projections

5.1 Introduction

In 1982 Rauzy [41] introduced the Rauzy fractal as a closure of an infinite sequence of points. These points can be seen as the projection of the integer points that lie on a cutting path that approximates the line $\mathbb{R}(\beta^2, \beta, 1)$ quite well, where β is the dominant eigenvalue of the incidence matrix corresponding to the Tribonacci substitution σ. The cutting path can be represented by the invariant word $\lim_{n\to\infty} \sigma^n(0)$.

In [46] Tijdeman and the author projected the integer points that lie on a finite cutting path corresponding to $\sigma^n(0)$, in case σ is the Fibonacci or Tribonacci substitution. After projecting and normalising, they replaced the projected points by letters. In the former case this lead to a sequence of words that expands in $\mathbb{Z}$, and in the latter case to a sequence of two-dimensional words filling up $\mathbb{Z}^2$ (without normalising we would get the Rauzy fractal). In [29] Fuchs and Tijdeman generalised the results from [46] to state under which conditions a substitution over an alphabet of k letters leads to a sequence of projections that is space-filling.

In [44] the author considered substitutions over a two-letter alphabet, and showed that Sturmian substitutions are exactly those substitutions over a two-letter alphabet for which the sequence of projected words $(v^{(n)})_{n\in\mathbb{N}}$ is space-filling. Moreover, he showed that in case σ is a Christoffel substitution, there exists a Christoffel substitution τ, so that $v^{(n+1)} = \tau(v^{(n)})$ for each n. This result was the motivation to investigate general Sturmian substitutions in [45]. There the author showed that when σ is a Sturmian substitution that has an incidence matrix with determinant 1, there exists a Sturmian substitution τ such that $v^{(n+1)} = \tau(v^{(n)})$ for each n.

In the present paper we consider substitutions over an alphabet containing arbitrary many letters, and find similar results. We limit ourselves to a specific type of substitution, the so-called β-substitution. For more information about β-expansions and β-substitutions we refer the reader to [1], [13], [28] and [56]. We use the method of forming a multi-dimensional word $v^{(n)}$ by projecting the integer points on a cutting path corresponding to $\sigma^n(0)$, that was first described in [46]. Under suitable conditions we show that if σ is a β-substitution, there exists a function ϕ so that $v^{(n+1)} = \phi(v^{(n)})$ for each n. We make extensive use of the theory that is developped in Sections 2,3,4 of [29], where it is explained how projecting the integer points on a cutting path can lead to an abstract number system.

In Section 5.2 we start with some properties of a β-substitution σ over an alphabet of $k+1$ letters. Next in Section 5.3 we explain how we can project the integer points on the corresponding cutting path to form a k-dimensional word $w^{(n)}$. Then in Section 5.4 we define the automaton and abstract number system corresponding to σ. In Section 5.5 we define a table of values $f(\vec{x})$ that is very useful to study properties of the words $w^{(n)}$, and we show what happens when we go from $w^{(n)}$ to $w^{(n+1)}$. Finally in Section 5.6 we state our main result, by defining a function ϕ that sends the k-dimensional word $v^{(n)}$ to $v^{(n+1)}$ for every $n \in \mathbb{N}$. We end with an example in Section 5.7.

5.2 Beta-substitutions

Let Σ be a finite set. By Σ^* we denote the set of finite words over the well ordered alphabet Σ where we denote the empty word by ε. A *substitution* is a map $\sigma : \Sigma \to \Sigma^* \backslash \{\varepsilon\}$. This map can be extended to a map $\sigma : \Sigma^* \to \Sigma^*$ by letting $\sigma(\varepsilon) = \varepsilon$ and $\sigma(ws) = \sigma(w)\sigma(s)$ for $w \in \Sigma^*, s \in \Sigma$ where as usual the operation is the concatenation of words. (With this operation the set Σ^* is a free monoid with identity ε generated by Σ.) We denote by $|w|$ the length of the word $w \in \Sigma^*$, i.e. $|w| = n$ if $w \in \Sigma^n$. Moreover, we set $|w|_s$ for $w \in \Sigma^*, s \in \Sigma$ to be the number of occurrences of letter s in the word w. Hence $\sum_{s\in\Sigma} |w|_s = |w|$. By w_i we denote the letter at position i in the finite or infinite word w. Let $v = v_0 \dots v_n$ and $w = w_0 \dots w_m$ be finite words with $n \leq m$. Then we say v is a *prefix* (*suffix*, respectively) of w when $v_i = w_i$ ($v_i = w_{i+m-n}$, respectively) for $i = 0, \dots, n$. When v is a prefix (suffix, respectively) of w and $v \neq w$ we call v a proper prefix (suffix, respectively). For $v, w \in \Sigma^*$ we say $v < w$ when v is smaller than w in the lexicographical ordering, which means that either v is a proper prefix of w, or there exists an integer j so that $v_i = w_i$ for $i < j$, and $v_j < w_j$. We can extend σ to $W \in \Sigma^\omega$ by concatenation. Define the *incidence matrix* M_σ of the substitution σ over the alphabet $\Sigma = \{0, 1, \dots, k\}$ by

$$M_\sigma = \Big(|\sigma(i)|_j \Big)_{i=0,\dots,k; j=0,\dots,k}.$$

A β-substitution over the alphabet $\Sigma = \{0, 1, \dots, k\}$ is a substitution of the form

$$\sigma(0) = 0^{n_0}1, \quad \sigma(1) = 0^{n_1}2, \quad \dots, \quad \sigma(k-1) = 0^{n_{k-1}}(k), \quad \sigma(k) = 0^{n_k},$$

where $n_0, n_k \geq 1$ and $n_i \geq 0$ for $i = 1, \dots, k-1$. Because we want the incidence matrix of M_σ to be unimodular, we assume in this article that $n_k = 1$. Furthermore we define the word $N := n_0 n_1 \dots n_{k-1} 1$ over the alphabet $\Delta := \{0, 1, \dots, \max_i n_i\}$ and make the following assumption.

$$\textit{Each proper suffix of } N \textit{ is smaller than } N. \tag{5.1}$$

It follows from (5.1) that $n_0 \geq n_i$ for all i. Note that σ has an invariant word starting with 0. The incidence matrix of σ is given by

$$M_\sigma = \begin{pmatrix} n_0 & 1 & 0 & \dots & 0 \\ n_1 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ n_{k-1} & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix},$$

with characteristic polynomial $f(x) = x^{k+1} - n_0 x^k - n_1 x^{k-1} - \cdots - n_{k-1}x - 1$.

Lemma 5.2.1. *M_σ has a dominant real eigenvalue β with $n_0 < \beta < n_0 + 1$.*

Proof. It follows from the Perron-Frobenius theorem for nonnegative matrices that M_σ has a real dominant eigenvalue β that is positive, see [9]. We have $f(n_0) < 0$, and it follows from $n_0 \geq n_i$ that $f(n_0 + 1) > 0$. Hence f has a zero γ with $n_0 < \gamma < n_0 + 1$. Note that it follows from $x > 0$ and $f(x) \geq 0$ that

$$f'(x) > (k+1)\left(x^k - n_0 x^{k-1} - \cdots - n_{k-1} - 1/x\right) \geq 0.$$

It follows that f is monotonically increasing for $x \geq \gamma$ so that $\gamma = \beta$. □

Put $\tau := 1/\beta$. Using the fact that

$$1 - n_0\tau - n_1\tau^2 - \cdots - n_{k-1}\tau^k - \tau^{k+1} = 0, \tag{5.2}$$

we see that

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{k-1} \\ a_k \end{pmatrix} := \begin{pmatrix} \tau \\ n_1\tau^2 + n_2\tau^3 + \cdots + n_{k-1}\tau^k + \tau^{k+1} \\ n_2\tau^2 + n_3\tau^3 + \cdots + n_{k-1}\tau^{k-1} + \tau^k \\ \vdots \\ n_{k-1}\tau^2 + \tau^3 \\ \tau^2 \end{pmatrix}$$

is an eigenvector corresponding to the eigenvalue β.

Lemma 5.2.2. *Let $b = b_1 \ldots b_l$ be a finite word over the alphabet Δ so that each suffix of b is lexicographically smaller than $N = n_0 n_1 \ldots n_{k-1}1$. Then*

$$b_1\tau + b_2\tau^2 + \cdots + b_l\tau^l < 1.$$

Proof. We use induction on l to prove the lemma. For $l = 1$ we assume $b_1 < N$, hence $b_1\tau \leq n_0\tau < 1$, where we used (5.2).

Assume the lemma is true for $l < m$, and assume $b_i \ldots b_m < N$ for $i = 1, \ldots, m$. If $b_1 \ldots b_m$ is a proper prefix of N, the lemma holds by (5.2). Otherwise let j be the smallest integer so that $b_j \neq n_{j-1}$. Then $b_j < n_{j-1}$, and we get $b_1\tau + \cdots + b_j\tau^j + \cdots + b_m\tau^m \leq n_0\tau + \cdots + (n_{j-1} - 1)\tau^j + b_{j+1}\tau^{j+1} + \cdots + b_m\tau^m < n_0\tau + \cdots + n_{j-1}\tau^j$, where we used the induction hypothesis. Now the result follows from (5.2). □

Corollary 5.2.3. *$a_i < \tau$ for $1 \leq i \leq k$.*

Proof. By applying the lemma and using (5.1) we find that

$$a_i = n_i\tau^2 + \cdots + n_{k-1}\tau^{k+1-i} + \tau^{k+2-i} < n_0\tau^2 + \cdots + n_{k-1}\tau^{k+1} + \tau^{k+2} = \tau.$$

□

5.3 Projecting cutting paths

Almost all notation and results in this section are copied from Section 3 of [29]. For more information and some examples we refer the reader to that paper.

By $\vec{e}_0^{(k+1)}, \ldots, \vec{e}_k^{(k+1)}$ we denote the unit column vectors of $\mathbb{R}^{k+1}$, and by ${}^tM$ the transpose of the matrix M. In the sequel we will consider the sequence $(u^{(n)})_{n\geq 0}$ given by

$$u^{(n)} = \sigma^n(0) \qquad (n = 0, 1, \ldots).$$

Let U be the limit word of the sequence $(u^{(n)})_{n\geq 0}$. Put

$$\vec{b} = \lim_{n\to\infty} (|u^{(n)}|_0, \ldots, |u^{(n)}|_k)/|u^{(n)}|.$$

It follows from Lemma 2.2 of [29] that this limit exists. The limit word U generates a broken halfline in $\mathbb{R}^{k+1}$ in the following way: we start in $\vec{0}$ and for $n = 0, 1, \ldots$ go 1 in the direction of the x_i-axis when $U_n = i$. The broken halfline approximates the halfline $\mathbb{R}_{\geq 0}\vec{b}$. The integer points P_m for $m \in \mathbb{N}$ on the broken halfline are given by

$$P_0 = \vec{0}, \quad P_{m+1} = P_m + \vec{e}_{U_m}^{(k+1)}.$$

For given $n \in \mathbb{N}$, we project the integer points along the broken line segment through $\vec{0}$ and $P_{|u^{(n)}|}$ to the hyperplane $x_0 = 0$ of $\mathbb{R}^{k+1}$, using the following *canonical projection* Φ_n^*. For $\vec{x} \in \mathbb{Z}^{k+1}$ we have

$$\Phi_n^*(\vec{x}) := \Pi({}^tM_\sigma^{-n}\vec{x}),$$

where Π is the projection along $\vec{e}_0^{(k+1)}$ and therefore means deletion of the zeroth entry. We define

$$\vec{a}_i^{(n)} := \Phi_n^*\left(\vec{e}_i^{(k+1)}\right) = \Pi\left({}^tM_\sigma^{-n}\vec{e}_i^{(k+1)}\right)$$

for $i = 0, 1, \ldots, k$ and call them the *transition vectors* corresponding to $u^{(n)}$. Observe that these vectors tell which step we have to make in the hyperplane $\mathbb{Z}^k$, spanned by $\vec{e}_1^{(k+1)}, \ldots, \vec{e}_k^{(k+1)}$, to get to the next projected point. We have the following lemma, see [29] Lemma 3.1.

Lemma 5.3.1 (Fuchs, Tijdeman).

$$\left(\vec{a}_0^{(n+1)}, \ldots, \vec{a}_k^{(n+1)}\right){}^tM_\sigma = \left(\vec{a}_0^{(n)}, \ldots, \vec{a}_k^{(n)}\right)$$

for all $n \geq 0$.

A *k-dimensional word* is a map from a subset of $\mathbb{Z}^k$ to some alphabet. We define the k-dimensional word $w^{(n)}$ by setting its value at position $\Phi_n^*(P_i)$ equal to i for $i = 0, \ldots, |u^{(n)}|-1$. Note that $w^{(n)}$ defines a roundwalk on its domain. For more information on roundwalks see [15], [61]. The following result is taken from [29] Theorem 3.2.

Theorem 5.3.2 (Fuchs, Tijdeman). *The sequence of words $(w^{(n)})_{n\in\mathbb{N}}$ is well-defined.*

Proposition 5.3.3. *Let $\vec{x}, \vec{y} \in \mathbb{Z}^k$ be in the domain of $w^{(n)}$ for some $n \in \mathbb{N}$. Then $w^{(n)}(\vec{x}) > w^{(n)}(\vec{y})$ if and only if $w^{(n+1)}(\vec{x}) > w^{(n+1)}(\vec{y})$.*

Proof. It follows from Lemma 5.3.1 that

$$\vec{a}_i^{(n)} = \sum_{j=0}^{k} |\sigma(i)|_j \vec{a}_j^{(n+1)}$$

for $i = 0, \ldots, k$. Note that each transition vector $\vec{a}_i^{(n)}$ corresponds to a letter i in $u^{(n)}$, and the letter i in $u^{(n)}$ is replaced by $|\sigma(i)|_0, \ldots, |\sigma(i)|_k$ letters $0, \ldots, k$, respectively, in $u^{(n+1)}$. Since $w^{(n)}$ contains a roundwalk, that starts in $\vec{0}$, and where we move from the position containing the value m to the position containing the value $m+1$ with a jump that is equal to one of the transition vectors, it follows that the roundwalk through $w^{(n+1)}$ passes by the points in the domain of $w^{(n)}$ in the same order as the roundwalk through $w^{(n)}$. □

For $\vec{x}$ in the domain of $w^{(n)}$ we define

$$\hat{w}^{(n)}(\vec{x}) := w^{(n)}/|u^{(n)}|.$$

The sequence $\left(\hat{w}^{(n)}\right)_{n\in\mathbb{N}}$ converges to some k-dimensional limit word $\widehat{W}$ over the alphabet $[0,1)$ in the following way: the domain of $\widehat{W}$ is the union of all domains of $\hat{w}^{(n)}$, and if $\vec{x} \in \mathrm{dom}\widehat{W}$ then

$$\widehat{W}(\vec{x}) := \lim_{n\to\infty} \hat{w}^{(n)}(\vec{x}).$$

For $x \in \mathbb{R}$ we denote by $\lfloor x \rfloor$ the largest integer y such that $y \le x$. Further we define $\{x\} = x - \lfloor x \rfloor$. We have the following theorem proved as [29] Theorem 3.5.

Theorem 5.3.4 (Fuchs, Tijdeman). *The word $\widehat{W}$ is well-defined on dom $\widehat{W}$ and*

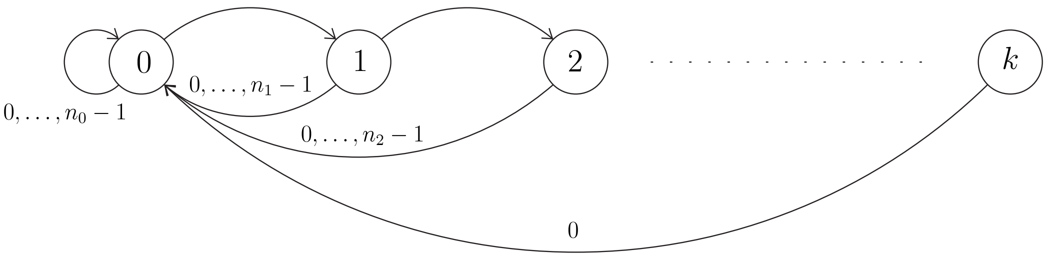
$$\widehat{W}(\vec{x}) = \left\{ \sum_{i=1}^{k} \frac{a_i x_i}{\tau} \right\}.$$

5.4 Abstract number systems

Following the notation of Section 4 of [29] we will define the automaton, the language and the abstract number system associated to σ.

The β-substitution σ defines an *automaton* in the following way:

- the set of states Q equals the letters of the alphabet Σ,
- a transition from state a to state b ($a, b \in \Sigma$) labeled i is added if b occurs in $\sigma(a)$ at position i,
- $q_0 = 0$ is the initial state,
- all states are final states.



We see that $\Delta = \{0, 1, \ldots, n_0\}$ is the set of labels. We denote by Δ^* the free monoid generated by Δ for the concatenation product, with neutral element ε. The *language* $\mathcal{L}$ associated to the substitution σ is the language of the associated automaton, that is the set of strings accepted by the automaton (all paths in the automaton from the initial state to a final state). The *genealogical ordering* on $\mathcal{L}$ is defined as follows. Let $u, v \in \mathcal{L}$, then we define $u < v$ if either $|u| < |v|$ or $|u| = |v|$ and there exist $p, u', v' \in \Delta^*, d, e \in \Delta$ such that $u = pdu', v = pev'$ and $d < e$. This ordering can be extended to the set Δ^ω of all infinite words of Δ according to the lexicographical ordering. We call the triple $\mathcal{S} = (\mathcal{L}, \Delta, <)$ an *abstract number system*.

Next we show that every real number in the interval $[0, 1)$ can be represented by a unique word from $\mathcal{L}_\infty$, where $\mathcal{L}_\infty$ is the set of infinite words which are the limit of a converging sequence of words in $\mathcal{L}$. We use the following (greedy) algorithm.
Let $x \in (0, 1)$ and set $w \leftarrow \varepsilon$. Then iterate the operations

1. Let d be the largest integer such that $x - d\tau$ is nonnegative
2. $w \leftarrow wd$
3. $x \leftarrow (x - d\tau)/\tau$

The output of the algorithm is the word w which we call the *σ-representation* of x. Conversely we say that x is the *numerical value* of w. If x becomes 0 at some stage, then it further remains 0 and we say that the original x has a finite σ-representation, where we stop after the last non-zero digit d. The σ-representation of 0 is by definition 0. The following theorem follows from [29] Theorem 4.2.

Theorem 5.4.1 (Fuchs, Tijdeman). *The above algorithm induces a bijection between* $[0, 1)$ *and the words in* $\mathcal{L}_\infty$.

We say the word $u \in \Delta^*$ is *allowed* if $u \in \mathcal{L}$, which, by the previous theorem, is the same as saying u is the σ-representation of some $x \in [0, 1)$.

Lemma 5.4.2. *The word* $u = u_0 u_1 \ldots u_m \in \Delta^{m+1}$ *is allowed if and only if every suffix of* u *is lexicographically smaller than* N.

Proof. We shall use the automaton of σ and the fact that N has property (5.1). Assume that every suffix of u is lexicographically smaller than N. In case u is a prefix of N, then $m < k$ and we see that after m steps through the automaton we are at state m. In case u is not a prefix of N, let j be the smallest number so that $u_j \neq n_j$. Then $u_j < n_j$, and after $j + 1$ steps through the automaton we are back at the initial state 0. Since $u_{j+1} \ldots u_m < N$ we can repeat this argument, and see that after m steps through the automaton we are at some state $q < m$.

Assume u is allowed. In case we end up after m steps through the automaton at state m, we see that u is a prefix of m, and it follows from (5.1) that every suffix of u is smaller than N. In case we are, after m steps, at some state $q < m$, there must be a j such that $u_i = n_i$ for $i = 0, \ldots, j - 1$, and $u_j < n_j$. Hence after $j + 1$ steps we are back at state 0. Repeating this argument, we see that $u_{j+1} \ldots u_m < N$. It follows from (5.1) that every suffix of u is smaller than N. ☐

Corollary 5.4.3. *Let* $b_0 b_1 \ldots b_l \in \Delta^{l+1}$ *be allowed. If* $b_0 \geq 1$, *then* $(b_0 - 1) b_1 \ldots b_l$ *is allowed. If* $b_0 = 0$, *then* $b_1 \ldots b_l$ *is allowed.*

Proof. The first statement follows from the previous lemma, the second statement follows from considering the automaton of σ. ☐

Theorem 5.4.4. *Let $x \in (0,1)$, and let $b = b_1 \dots b_l \in \Delta^*$ be the σ-representation of x. Then $x = b_1\tau + \cdots + b_l\tau^l$.*

Proof. Let $b = b_1 \dots b_l \in \Delta^*$ be the σ-representation of x, and put $y = b_1\tau + \dots b_l\tau^l$. Since b is allowed, it follows from Corollary 5.4.3 that every suffix of b is allowed. Hence it follows from Lemma 5.4.2 that $b_i \dots b_l < N$ for $i = 1, \dots, l$. Now it follows from Lemma 5.2.2 that for $i = 1, \dots, l$ we have

$$b_i\tau^i + \cdots + b_l\tau^l < \tau^{i-1}.$$

Hence when we apply the algorithm to y we find that b is the σ-representation of y. It follows from Theorem 5.4.1 that $y = x$. ☐

In the sequal we also refer to the σ-representation of $x \in [0,1)$ as the β-expansion of x, where we only consider the digits after the decimal point.

5.5 Beta-expansions

Recall that $k = |\Sigma| - 1$. Let $\vec{x} = (x_1, \dots, x_k) \in \mathbb{Z}^k$. As in Section 5 of [29], we define

$$f(\vec{x}) := \left\{ \sum_{i=1}^{k} \frac{a_i x_i}{\tau} \right\}.$$

By Theorem 5.3.4 the limit word $\widehat{W}$ is the restriction of f on its domain. We have the following result, proved as Theorem 5.3 of [29].

Theorem 5.5.1 (Fuchs, Tijdeman). *If $\vec{x} \in \mathbb{Z}^k$ is in the domain of $w^{(n)}$, then $f(\vec{x})$ has a finite β-expansion of length at most n. Conversely, for all elements $\vec{x}$ in the domain of $w^{(n)}$ the β-expansion of $f(\vec{x})$ is different and all words that are allowed of length at most n appear.*

Theorem 5.5.2. *Let $\vec{x}, \vec{y} \in \mathbb{Z}^k$ be in the domain of $w^{(n)}$. Then $w^{(n)}(\vec{x}) > w^{(n)}(\vec{y})$ if and only if $f(\vec{x}) > f(\vec{y})$.*

Proof. It follows from Theorem 5.4.1 and Theorem 5.5.1 that $f(\vec{x}) = f(\vec{y})$ if and only if $\vec{x} = \vec{y}$. Now the result follows from Proposition 5.3.3 and Theorem 5.3.4. ☐

Proposition 5.5.3. *Let $\vec{x} \in \mathbb{Z}^k$ be in the domain of $w^{(n)}$. Consider the σ-representation of $f(\vec{x})$, and if necessary add zeros on the right until it has length n. Let i with $0 \leq i \leq k$ be the largest integer such that the σ-representation ends with $n_0 \dots n_{i-1}$. Then the jump from $w^{(n)}(\vec{x})$ to $w^{(n)}(\vec{x}) + 1$ is given by $\vec{a}_i^{(n)}$.*

Proof. It is easy to see that the automaton of σ is in state i if and only if the largest prefix of $N = n_0 \dots n_{k-1}1$ that is a suffix of the representation that the automaton has been fed has length i. Now the result follows from Corollary 5.5 from [29] and the remark thereafter. ☐

We call the index of the transition vector to make the jump from $w^{(n)}(\vec{x})$ to $w^{(n)}(\vec{x})+1$ the *Rauzy color* of $w^{(n)}(\vec{x})$. It follows that if ε is the only prefix of $N = n_0 \dots n_{k-1}1$ that the σ-representation of $f(\vec{x})$ (with zeros added if necessary) ends with, then $w^{(n)}(\vec{x})$ has Rauzy color 0.

Remark 1. Note that

$$f(x_1, \dots, x_{k-1}, x_k + 1) = \left\{ \sum_{i=1}^{k} \frac{a_i x_i}{\tau} + \frac{a_k}{\tau} \right\} = \left\{ \sum_{i=1}^{k} \frac{a_i x_i}{\tau} + \tau \right\}.$$

Hence it follows from Theorem 5.4.4 that as long as $f(\vec{x}) < 1 - \tau$, when the σ-representation of $f(\vec{x})$ equals $b_1 b_2 \dots b_l$, then the σ-representation of $f(\vec{x} + \vec{e}^{(k)}_{k-1})$ equals $(b_1+1)b_2 \dots b_l$.

We define the function $g : \mathbb{Z}^k \to \mathbb{Z}^k$ as

$$g(\vec{x}) := \left(x_2, \dots, x_k, -\left\lfloor \sum_{i=1}^{k} \frac{a_i x_i}{\tau} \right\rfloor - \sum_{i=1}^{k} n_{i-1} x_i \right).$$

Theorem 5.5.4. *For $n \in \mathbb{N}$ and $\vec{x}$ in the domain of $w^{(n)}$ we have*

$$\tau f(\vec{x}) = f(g(\vec{x})).$$

Proof.

$$f(g(\vec{x})) = \left\{ \sum_{i=1}^{k-1} \frac{a_i x_{i+1}}{\tau} - \frac{a_k}{\tau} \left\lfloor \sum_{i=1}^{k} \frac{a_i x_i}{\tau} \right\rfloor - \frac{a_k}{\tau} \sum_{i=1}^{k} n_{i-1} x_i \right\}$$

$$= \left\{ \sum_{i=1}^{k-1} (a_{i+1} + n_i \tau) x_{i+1} - \tau \left\lfloor \sum_{i=1}^{k} \frac{a_i x_i}{\tau} \right\rfloor - \tau \sum_{i=1}^{k-1} n_i x_{i+1} - \tau n_0 x_1 \right\}$$

$$= \left\{ \sum_{i=2}^{k} a_i x_i - \tau \left\lfloor \sum_{i=1}^{k} \frac{a_i x_i}{\tau} \right\rfloor + a_1 x_1 - x_1 \right\} = \{\tau f(\vec{x})\} = \tau f(\vec{x}).$$

□

It follows from Theorem 5.4.4 that when $f(\vec{x})$ has σ-representation $b_1 \dots b_l$, then $\tau f(\vec{x})$ has σ-representation $0b_1 \dots b_l$. It follows from the previous theorem that this value can be found at position $g(\vec{x})$. We will use this property to define the function ϕ in the next section.

5.6 Multi-dimensional rotation words

Consider the k proper suffixes of $N = n_0 \dots n_{k-1}1$. Order them by their numerical values in non-decreasing order, and let these values divide the interval $[0, 1)$ into $k+1$ consecutive left-closed, right-open intervals $I_0, \dots, I_k$.

Lemma 5.6.1. *Let $x \in I_i$ and $\tau x + q\tau \in I_j$ for some $q \in \mathbb{Z}_{\geq 0}$. Then for every $y \in I_i$ we have $\tau y + q\tau \in I_j$.*

Proof. By $0.n_p n_{p+1} \dots n_{k-1}1$ we mean the numerical value of the word $n_p n_{p+1} \dots n_{k-1}1 \in \mathcal{L}$ (where the letters are seen as digits in a β-numeration system). Put $I_i = [0.n_s n_{s+1} \dots n_{k-1}1, 0.n_t n_{t+1} \dots n_{k-1}1)$ and let $y \in I_i$. It follows from Theorem 5.4.4 that $\tau y + q\tau \in [0.qn_s n_{s+1} \dots n_{k-1}1, 0.qn_t n_{t+1} \dots n_{k-1}1)$. Assume there exists an r such that

$$0.qn_s n_{s+1} \dots n_{k-1}1 < 0.n_r n_{r+1} \dots n_{k-1}1 < 0.qn_t n_{t+1} \dots n_{k-1}1.$$

Then $n_r = q$ and $0.n_s n_{s+1} \dots n_{k-1}1 < 0.n_{r+1} \dots n_{k-1}1 < 0.n_t n_{t+1} \dots n_{k-1}1$ which gives a contradiction. □

We define $v^{(n)}$ from $w^{(n)}$ as follows. The domain of $v^{(n)}$ equals the domain of $w^{(n)}$. If $\vec{x}$ is in the domain of $w^{(n)}$, and $f(\vec{x}) \in I_i$, then we put $v^{(n)}(\vec{x}) = i$.

Let $D \in \mathbb{Z}^k$ be the union of all domains of $w^{(n)}$ for $n \in \mathbb{N}$. Then we define the function $V : D \to \Sigma$ as follows. If $\vec{x} \in D$ and $f(\vec{x}) \in I_i$, then $V(\vec{x}) = i$. Note that since $f(\vec{x})$ is linear (modulo 1) in each of its coordinates, the function V is the restriction to its domain of a multi-dimensional rotation word. These words can be seen as discretisations of hyperplanes, see for example [5], [17].

When $V(\vec{x}) = c$, $f(\vec{x}) \in I_c$, and it follows from Lemma 5.6.1 that there exists an integer h_c such that $\tau I_c + h_c\tau \in [0, 1)$ and $\tau I_c + (h_c + 1)\tau \in [1, \infty)$. Clearly the value of h_c only depends on c and not on $\vec{x}$. Hence we can define a function ϕ as follows.

Definition. Let $\vec{x} \in D$ and $V(\vec{x}) = c$. Let h_c be the smallest integer such that $\tau I_c + (h_c + 1)\tau \in [1, \infty)$. Let $d_0, \dots, d_{h_c} \in \Sigma$ such that $\tau I_c + i\tau \in I_{d_i}$ for $i = 0, \dots, h_c$. Then we let ϕ be the function that maps the letter c, that has been assigned to position $\vec{x}$, to the ordered tuple of letters $d_0, d_1, \dots d_{h_c}$, which are assigned to the positions $g(\vec{x}), g(\vec{x}) + \vec{e}^{(k)}_{k-1}, \dots, g(\vec{x}) + h_c\vec{e}^{(k)}_{k-1}$, respectively.

It follows from the results in the previous sections that the positions $g(\vec{x}), g(\vec{x}) + \vec{e}^{(k)}_{k-1}, \dots, g(\vec{x}) + h_c\vec{e}^{(k)}_{k-1}$ are indeed in D. We state our final result.

Theorem 5.6.2. *Let σ be a β-substitution of the form that we consider in Section 5.2, so that (5.1) holds. Let ϕ and $v^{(n)}$ be defined as above. Then*

$$\phi(v^{(n)}) = v^{(n+1)} \text{ for every } n \in \mathbb{Z}_{\geq 1}.$$

Proof. Let $\vec{x}$ be in the domain of $v^{(n)}$, and $v^{(n)}(\vec{x}) = c$. We use the notation $c_{\vec{x}}$ to indicate that the letter c has been assigned to position $\vec{x}$. It follows from Theorem 5.5.1 that $f(\vec{x})$ has a σ-representation of length at most n. Then, by Theorem 5.5.4 and Theorem 5.4.4, $f(g(\vec{x})) = \tau f(\vec{x})$ has a σ-representation of length at most $n + 1$. Note that by definition of h_c, we have that $f(g(\vec{x})) + h_c\tau < 1$. Hence it follows from Remark 1 that each of the σ-representations of $f(g(\vec{x}), f(g(\vec{x}) + \vec{e}^{(k)}_{k-1}), \dots, f(g(\vec{x}) + h_c\vec{e}^{(k)}_{k-1})$ has the same length, which is at most $n + 1$. It follows from Theorem 5.5.1 that $\phi(c_{\vec{x}}) \subset v^{(n+1)}$.

Let $c_{\vec{y}} \in v^{(n+1)}$, and let the σ-representation of $f(\vec{y})$ be $b_0 b_1 \dots b_l$. Again by Theorem 5.5.1 it follows that $l \leq n$. We obtain from Corollary 5.4.3 that $b_1 \dots b_l$ is allowed. Hence there exists $\vec{z} \in \operatorname{dom} v^{(n)}$ such that $f(\vec{z})$ has σ-representation $b_1 \dots b_l$. Put $v^{(n)}(\vec{z}) = d$, then it follows from the definition of ϕ that $c_{\vec{y}}$ is in the image of $\phi(d_{\vec{z}})$. □

5.7 An example

Example (see [29] Example 6.1). Let the β-substitution σ over the alphabet $\Sigma = \{0, 1, 2\}$ be given by

$$\sigma(0) = 0001,\ \sigma(1) = 02,\ \sigma(2) = 0,$$

so that $N = 311$. The incidence matrix and its inverse are given by

$$M_\sigma = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad M_\sigma^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -3 \\ 0 & 1 & -1 \end{pmatrix}.$$

M_σ has characteristic polynomial $f(x) = x^3 - 3x^2 - x - 1$. We find $\beta \approx 3.383$, $\tau \approx 0.296$ and

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \tau \\ \tau^2 + \tau^3 \\ \tau^2 \end{pmatrix}.$$

We find the following transition vectors corresponding to $u^{(1)}, u^{(2)}, u^{(3)}$.

n	$\vec{a}_0^{(n)}$	$\vec{a}_1^{(n)}$	$\vec{a}_2^{(n)}$
1	${}^t(0, 1)$	${}^t(0, -3)$	${}^t(1, -1)$
2	${}^t(1, -1)$	${}^t(-3, 4)$	${}^t(-1, -2)$
3	${}^t(-1, -2)$	${}^t(4, 5)$	${}^t(-2, 6)$

Since

$$\begin{aligned} u^{(1)} &= 0001, \\ u^{(2)} &= 00010001000102, \\ u^{(3)} &= 00010001000102000100010001020001000100010200010, \end{aligned}$$

we can write down the corresponding two-dimensional words $w^{(n)}$.

$w^{(1)}$

$$\begin{array}{c} 3 \\ 2 \\ 1 \\ 0 \end{array}$$

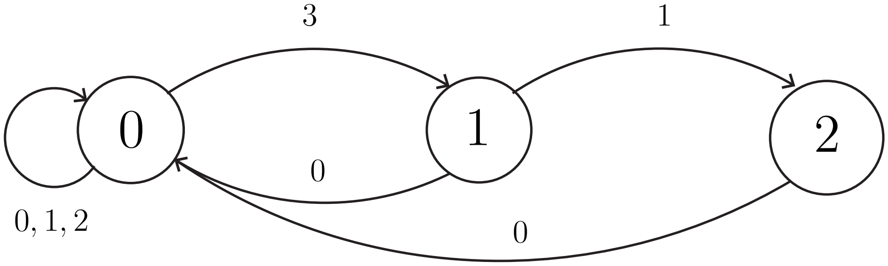
$w^{(2)}$

$$\begin{array}{cccc} 12 & & & \\ 8 & 13 & & \\ 4 & 9 & & \\ 0 & 5 & 10 & \\ & 1 & 6 & 11 \\ & & 2 & 7 \\ & & & 3 \end{array}$$

$w^{(3)}$

$$\begin{array}{ccccccc} & & & 42 & & & \\ & & & 28 & 46 & & \\ & & 43 & 14 & 32 & & \\ & & 29 & 0 & 18 & 36 & \\ & 44 & 15 & 33 & 4 & 22 & 40 \\ & 30 & 1 & 19 & 37 & 8 & 26 \\ 45 & 16 & 34 & 5 & 23 & 41 & 12 \\ 31 & 2 & 20 & 38 & 9 & 27 & \\ 17 & 35 & 6 & 24 & & 13 & \\ 3 & 21 & 39 & 10 & & & \\ & 7 & 25 & & & & \\ & & 11 & & & & \end{array}$$

The automaton associated to σ is given by



We find $f(x_1, x_2) = x_1(\tau + \tau^2) + x_2\tau \pmod 1$. The table below gives the expansions of all $f(x_1, x_2)$ with $-3 \leq x_1 \leq 3, -8 \leq x_2 \leq 3$:

2121	0111	1211	**3**	0301	20211	00111
1121	2221	0211	**2**	**31**	10211	21211
0121	1221	**301**	**1**	**21**	00211	11211
2231	0221	**201**	ε	**11**	**22**	01211
1231	**302**	**101**	**211**	**01**	**12**	**23**
0231	**202**	**001**	**111**	**221**	**02**	**13**
303	**102**	**212**	**011**	**121**	**231**	**03**
203	**002**	**112**	**222**	**021**	**131**	30301
103	**213**	**012**	**122**	300211	**031**	20301
003	**113**	**223**	**022**	200211	310211	10301
220201	**013**	**123**	301211	100211	210211	00301
120201	230201	**023**	201211	000211	110211	220211

Observe that, for $n = 1, 2, 3$, the expansions of length at most n are at the positions of the domain of $w^{(n)}$, as stated in Theorem 5.5.1. Also note that in this case the expansions that end with 3 have Rauzy color 1 for that length, the expansions that end with 31 have Rauzy color 2 for that length, and the other expansions have Rauzy color 0. This can be checked for length 3 by considering $w^{(3)}$ and adding zeros on the right until the number of symbols equals 3.

We have $g(x_1, x_2) = (x_2, -\lfloor x_1(\tau + \tau^2) + x_2\tau \rfloor - 3x_1 - x_2)$. Note that for example $f(1, -3) = \tau + 2\tau^2 + \tau^3$, $g(1, -3) = (-3, 1)$ and indeed we have $f(-3, 1) = \tau^2 + 2\tau^3 + \tau^4 = \tau f(1, -3)$.

We get that $N = 311$ has two proper suffixes, 1 and 11, which have numerical values τ and $\tau + \tau^2$, respectively. Hence we have

$$I_0 = [0, \tau), \quad I_1 = [\tau, \tau + \tau^2), \quad I_2 = [\tau + \tau^2, 1).$$

We find that

$$V(x_1, x_2) = \begin{cases} 0 \text{ if the first digit in the expansion of } f(x_1, x_2) \text{ is } 0, \\ 1 \text{ if the first two digits in the expansion of } f(x_1, x_2) \text{ are } 10, \\ 2 \text{ if the first two digits in the expansion of } f(x_1, x_2) \text{ are } 11 \text{ or more.} \end{cases}$$

For $n = 1, 2, 3$ we find the following two-dimensional words $v^{(n)}$ over the alphabet Σ, where we have underlined the letter at position $(0, 0)$.

$$\begin{array}[t]{c}
v^{(1)} \\
2 \\
2 \\
1 \\
\underline{0}
\end{array}
\qquad
\begin{array}[t]{cccc}
\multicolumn{4}{c}{v^{(2)}} \\
2 & & & \\
2 & 2 & & \\
1 & 2 & & \\
\underline{0} & 2 & 2 & \\
 & 0 & 2 & 2 \\
 & & 0 & 2 \\
 & & & 0
\end{array}
\qquad
\begin{array}[t]{ccccccc}
\multicolumn{7}{c}{v^{(3)}} \\
 & & & 2 & & & \\
 & & & 2 & 2 & & \\
 & & 2 & 1 & 2 & & \\
 & & 2 & \underline{0} & 2 & 2 & \\
 & 2 & 1 & 2 & 0 & 2 & 2 \\
 & 2 & 0 & 2 & 2 & 0 & 2 \\
2 & 1 & 2 & 0 & 2 & 2 & 0 \\
2 & 0 & 2 & 2 & 0 & 2 & \\
1 & 2 & 0 & 2 & & 0 & \\
0 & 2 & 2 & 0 & & & \\
 & 0 & 2 & & & & \\
 & & 0 & & & &
\end{array}$$

Note that when $f(\vec{x}) \in I_0 = [0, \tau)$, then $\tau f(\vec{x}) + 3\tau \in [3\tau, 3\tau + \tau^2) \in [0, 1)$, while $\tau f(\vec{x}) + 4\tau \geq 1$. When $f(\vec{x}) \in I_1 = [\tau, \tau + \tau^2)$, then $\tau f(\vec{x}) + 3\tau \in [3\tau + \tau^2, 3\tau + \tau^2 + \tau^3) \in [0, 1)$, while $\tau f(\vec{x}) + 4\tau \geq 1$. When $f(\vec{x}) \in I_2 = [\tau + \tau^2, 1)$, then $\tau f(\vec{x}) + 2\tau \in [2\tau + \tau^2 + \tau^3, 3\tau) \in [0, 1)$, while $\tau f(\vec{x}) + 3\tau \geq 1$. Hence $h_0 = 3$, $h_1 = 3$, $h_2 = 2$. This gives the following function ϕ.

$$\phi : 0_{(x_1,x_2)} \to \begin{array}{l} 2_{g(x_1,x_2)+(0,3)} \\ 2_{g(x_1,x_2)+(0,2)} \\ 1_{g(x_1,x_2)+(0,1)} \\ 0_{g(x_1,x_2)} \end{array}, \quad 1_{(y_1,y_2)} \to \begin{array}{l} 2_{g(y_1,y_2)+(0,3)} \\ 2_{g(y_1,y_2)+(0,2)} \\ 2_{g(y_1,y_2)+(0,1)} \\ 0_{g(y_1,y_2)} \end{array}, \quad 2_{(z_1,z_2)} \to \begin{array}{l} 2_{g(z_1,z_2)+(0,2)} \\ 2_{g(z_1,z_2)+(0,1)} \\ 0_{g(z_1,z_2)} \end{array}.$$

We see in the figure above that indeed for $n = 1, 2$ we have $\phi(v^{(n)}) = v^{(n+1)}$.

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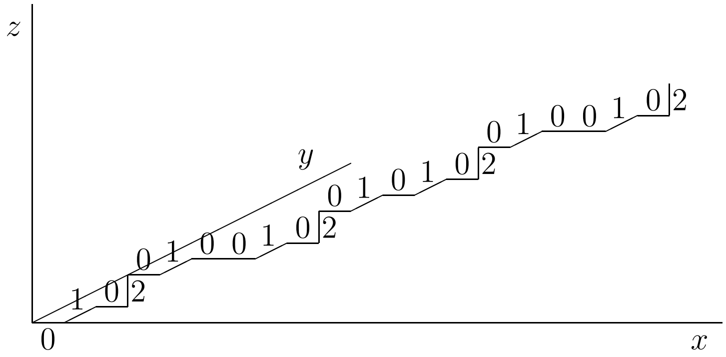
Samenvatting

In dit proefschrift bestuderen we woorden en substituties. We nemen aan dat σ een substitutie over het alfabet $\mathcal{A} = \{0, 1, \ldots, k\}$ is zodat $\sigma(0)$ begint met een 0, en definiëren $u^{(n)} = \sigma^n(0)$ voor $n = 1, 2, \ldots$. Een voorbeeld van een substitutie is de Tribonacci substitutie die gegeven wordt door $\sigma(0) = 01$, $\sigma(1) = 02$, $\sigma(2) = 0$. Laat $U = \lim_{n\to\infty} u^{(n)}$ het vaste punt zijn van σ. Voor de Tribonacci substitutie vinden we $u^{(1)} = 01$, $u^{(2)} = 0102$, $u^{(3)} = 0102010$, $u^{(4)} = 0102010010201$, en $U = 0102010010201010201001020102010010201010201001020\ldots$.

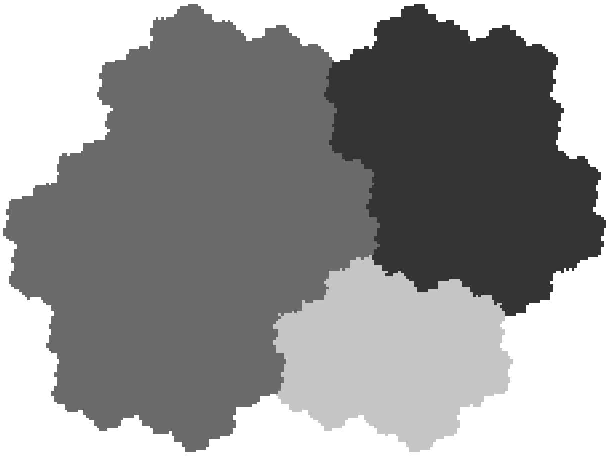
Het woord U definieert een gebroken halflijn die de gehele punten P_m gegeven door

$$P_0 = \vec{0}, \quad P_{m+1} = P_m + \vec{e}^{\,(k+1)}_{U_m},$$

waar $\vec{e}^{\,(k+1)}_i$ een eenheidsvector in $\mathbb{R}^{k+1}$ is, verbindt door lijnsegmenten van lengte 1. Voor de Tribonacci substitutie ziet het begin van de gebroken halflijn er als volgt uit.



In 1982 publiceerde G. Rauzy een artikel waarin hij in het geval van de Tribonacci substitutie de gehele punten P_m projecteerde op een vlak door de oorsprong, en aantoonde dat deze punten een fractal vormen, welke we nu kennen als de Rauzy fractal. Zie het plaatje bovenaan de volgende pagina, waarbij de grijstinten aangeven of de volgende stap in de richting van de x-as, y-as of z-as gaat.



In dit proefschrift onderzoeken we de eigenschappen van de projecties van de eerste $|u^{(n)}|$ gehele punten op de gebroken halflijn voor $n = 1, 2, \dots$. Hier is $|a|$ het aantal letters in het woord a. Deze geprojecteerde punten induceren een k-dimensionaal woord $w^{(n)}$. Voor de Tribonacci substitutie krijgen we het volgende voor $n = 1, \dots, 4$, waarbij het getal m met de projectie van P_m correspondeert.

$$
\begin{array}{c@{\qquad\qquad}cc@{\qquad\qquad}ccc@{\qquad\qquad}ccc}
w^{(1)} & \multicolumn{2}{c}{w^{(2)}} & \multicolumn{3}{c}{w^{(3)}} & \multicolumn{3}{c}{w^{(4)}} \\
 & & & & & & 10 & 8 & \\
 & & & & & & 3 & 1 & 12 \\
1 & 2 & & 5 & 4 & & 9 & 7 & 5 \\
0 & 0 & 3 & 1 & 0 & 6 & 2 & 0 & 11 \\
 & & 1 & & 3 & 2 & & 6 & 4
\end{array}
$$

Het is natuurlijker om de getallen in een bijbehorend getalsysteem te schrijven. Hiervan gebruikmakend definiëren we een k-dimensionaal woord $v^{(n)}$ over een $(k+1)$-letter alfabet $\mathcal{A}$ voor $n = 1, 2, \dots$. In geval van de Tribonacci substitutie geeft dat het volgende, waar we de letter op positie $(0, 0)$ hebben onderstreept.

$$
\begin{array}{c@{\qquad\qquad}cc@{\qquad\qquad}ccc@{\qquad\qquad}ccc}
v^{(1)} & \multicolumn{2}{c}{v^{(2)}} & \multicolumn{3}{c}{v^{(3)}} & \multicolumn{3}{c}{v^{(4)}} \\
 & & & & & & 1 & 1 & \\
 & & & & & & 0 & 0 & 2 \\
1 & 1 & & 1 & 1 & & 1 & 1 & 0 \\
\underline{0} & \underline{0} & 2 & 0 & \underline{0} & 2 & 0 & \underline{0} & 2 \\
 & & 0 & & 0 & 0 & & 0 & 0
\end{array}
$$

We laten zien dat de rij $\left(v^{(n)}\right)_{n\geq 1}$ convergeert naar een limiet woord V. Zulke woorden V representeren de beste discrete benaderingen van een k-dimensionaal hypervlak in de $(k+1)$-dimensionale reële ruimte. Voor verschillende substituties σ bestuderen we eigenschappen van de woorden $v^{(n)}$.

In Hoofdstuk 2 nemen we voor σ de Fibonacci en Tribonacci substitutie, en laten we zien dat het limiet woord V een rotatie woord is. In Hoofdstuk 3 en 4 bekijken we substituties over het alfabet $\mathcal{A} = \{0, 1\}$. In Hoofdstuk 3 laten we zien dat Sturmse substituties precies die substituties zijn waarvoor de rij $\left(v^{(n)}\right)_{n\geq 1}$ ruimte-vullend is, en in Hoofdstuk 4 laten we bovendien zien dat voor elke Sturmse substitutie σ met incidentie matrix met determinant 1, er een Sturmse substitutie τ bestaat zo dat $v^{(n+1)} = \tau\left(v^{(n)}\right)$ voor iedere n.

In Hoofdstuk 5 tenslotte behandelen we β-substituties over een alfabet van k letters, met $k \geq 2$. We tonen aan dat wanneer σ een β-substitutie is die aan bepaalde voorwaarden voldoet, er een functie ϕ bestaat zo dat $v^{(n+1)} = \phi\left(v^{(n)}\right)$ voor iedere n. In het Tribonacci geval ziet ϕ er als volgt uit.

$$\phi : 0_{(x_1,x_2)} \rightarrow \begin{array}{l} 1_{g(x_1,x_2)+(0,1)} \\ 0_{g(x_1,x_2)} \end{array} , \quad 1_{(y_1,y_2)} \rightarrow \begin{array}{l} 2_{g(y_1,y_2)+(0,1)} \\ 0_{g(y_1,y_2)} \end{array} , \quad 2_{(z_1,z_2)} \rightarrow 0_{g(z_1,z_2)} ,$$

waarbij $g : \mathbb{Z}^2 \rightarrow \mathbb{Z}^2$ expliciet in Hoofdstuk 5 gedefinieerd wordt.

Curriculum Vitae

Sierk Rosema was born on January the 30th, 1978 in Leiden. From 1990 to 1996 he attended the Stedelijk Gymnasium in Leiden. In this period he was a successful participant in mathematics and physics olympiad competitions.

From 1996 he studied mathematics at Utrecht University, where he received the propedeuse diploma in 1998 cum laude, and the doctoral diploma in 2003. His doctoral thesis was titled “Unitary monodromy groups of the Lamé equation“ and supervised by professor F. Beukers. In this period he was tutor for high school students in exact sciences and teaching assistant giving exercise classes for undergraduate students.

From 2004 to 2008 he did his Ph.D. research under the supervision of professor R. Tijdeman at Leiden University, which resulted in this thesis. He reported on his research in Leiden, Delft, Utrecht, Liège, Marseille and Graz. Furthermore he has helped the international students organisation ISN-R with organising a cultural festival in 2007.