Arithmetics and Combinatorics of Words of Low Complexity

PROEFSCHRIFT

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Chapter 1.
Introduction.

A word is a concatenation of symbols from a finite alphabet $\mathcal{A}$. In this thesis $\mathcal{A} = \{a, b\}$. A word may be finite, infinite in one direction or infinite in two directions. In the last case we speak of $\mathbf{Z}$-words. A finite pattern x of symbols which appears in w is called a finite subword or factor of w. We write $x \subset w$. In this thesis we will use number theory, combinatorics and a little graph theory to study more closely the factor set of words and, more generally, of languages. (A language is a collection of words). First we introduce two classical concepts, balanced and stiff. If x is a finite word we write $|x|$ for its length (the number of symbols of x) and $c(x)$ for its content (the number of a's in x).

Definition. *A word w is balanced if $|c(x) - c(y)| \leq 1$ for all factors $x, y \subset w$ of equal length.*

For example, $w = aabb$ is not balanced because $x = aa, y = bb$ are factors and $c(x) = 2, c(y) = 0$. Balanced words were introduced by Morse and Hedlund in [MH]. They themselves did not use the term "balanced words" but "Sturmian series", after the Swiss mathematician J. C. F. Sturm (1803-1855). The reason is that balanced words appear naturally in the theory of second order differential equations and that the crucial step in noticing this is given by the Sturmian separation theorem. For the interested reader we have given a brief account of this in the Appendix. We prefer to use the more modern terminology of balanced words. The term "sturmian" will be reserved for a subclass of the balanced $\mathbf{Z}$-words. Now the second concept.

If w is a word we write $P(w, n)$ for the number of factors of length n. The mapping $P(w, .) : \mathbf{N} \rightarrow \mathbf{N}$ is called the complexity function of w. (In this thesis $0 \in \mathbf{N}$ and $\mathbf{N}^+$ denotes the positive integers). As the name suggests, the function $P(w, n)$ indicates how "complex" the word w is. For instance, if w is a $\mathbf{Z}$-word then $P(w, n)$ is ultimately constant if and only if w is periodic. In formula,

$$\exists c, n_0 \forall n \geq n_0 \ (P(w, n) = c) \iff \exists p \in \mathbf{N}^+ \ \forall i \in \mathbf{Z} \ (w_i = w_{i+p})$$

In fact, c equals the minimal period of w, see Theorem 2.1 and the remark following it. Therefore a $\mathbf{Z}$-word w which is not periodic satisfies

$P(w, n) \geq n+1$ for all n. Hence, as noticed by Coven and Hedlund in [CH], the complexity function $P(w, n) = n+1$ is the minimal one for non-periodic w.

Definition. *A word w is called stiff if $P(w, n) \leq n+1$ for all n.*

We start, in Chapter 2, by discussing some classical results on balanced and stiff words. We show that finite balanced words x are contained in balanced **Z**-words w, likewise for stiff, and prove the well-known classification of balanced (stiff) **Z**-words. The proofs we give are more or less streamlined versions of the original ones but we have included them to keep this thesis as self-contained as possible. Then we give a new proof of the formula for $\mathrm{bal}(n)$, the number of balanced words of length n and derive a formula for $\mathrm{st}(n)$, the number of stiff words of length n. After this we discuss some other descriptions of balanced **Z**-words, in particular Beatty words, and in Sections 2.5.4, 2.5.5 we concentrate on the connection with sturmian morphisms and continued fractions. We conclude Chapter 2 with a generalized version of the Robinson equation. (The Robinson equation is $AB = Cab$ where A, B, C are required to be palindromes). The choice of material is rather pragmatic: we have included what we need later on. Many interesting parts of the theory are left out as an effect: the recurrency function for sturmian words [MH, Section 10], the three distances theorem from Diophantine approximation and the three densities theorem [B], sturmian transducers [Pa], critical exponents in sturmian words [Mi/Pi,V], sturmian words which are fixed points of substitutions [CMPS,Pa2], Fraenkel's conjecture concerning the partition of **Z** into balanced sets of distinct densities [AGH,T2], etc.

As mentioned before, the minimal complexity function for non-periodic **Z**-words w equals $P(w, n) = n + 1$. The next best thing will be $P(w, n) = n + k$ for $n \geq N$ and some constant k. Such words will be the subject of Chapter 3. If w has such a complexity function we say that w has minimal block growth (MBG). One has to differentiate between recurrent and non-recurrent w. A **Z**-word w is called recurrent if every factor x appears at least twice in w. In a sense to be made precise below **Z**-words of MBG are generalizations of stiff **Z**-words.

- If w is recurrent, then $w = T\sigma$ where T is a substitution and σ a recurrent stiff **Z**-word.
- If w is not recurrent then w is left periodic (a left tail of w is periodic), it is right periodic but not periodic as a whole.

These results are well-known but we quantify them by indicating explicitly how to calculate the constant k in the definition of MBG. We call it the stiffness of w. We compare the case $k = 1$ with the case $k > 1$ and use the obtained results to show that $P(\mathcal{S}_k, n)$, the number of words of length n contained in some w of stiffness k, is $O(n^3)$. A far as we know a similar result is not yet known for the number of k-stiff words of length n. (A word x is called k-stiff if $P(w, n) \leq n + k$ for all n). In Section 3.4 we determine which **Z**-words of MBG can be described using coding of intervals.

In Chapter 4 we take another look at the complexity function of **Z**-words. We consider the quotient $P(n)/n$ and show that it can converge to any positive integer, but not to a real value $\alpha \in (1, 2)$. The proofs rely on the notion of word graphs. We wonder whether all limit values must be integer. We also wondered whether **Z**-words exist with $P(n+1) - P(n) = 1$ for infinitely many n such that $P(n)/n$ converges to 2. It turns out that this is indeed possible, as shown in Theorem 4.4.

Chapter 2.
Balanced and stiff words: definitions and some properties.

2.1. Preliminaries.

A word is a mapping $w : I \to \Sigma$, where I is an interval of integers and where Σ is a finite set, called the alphabet. Throughout this thesis $\Sigma = \{a, b\}$ with $a \neq b$. We will identify words $x : I_1 \to \Sigma, y : I_2 \to \Sigma$ if they are shifts of each other. This means that there exists a $c \in \mathbf{Z}$ such that $I_1 + c = I_2$ and $x_i = y_{i+c}$ for all $i \in I_1$. A word is infinite if I is infinite and we will use the terminology $\mathbf{N}$-words, $\mathbf{Z}$-words for words with domain $\mathbf{N}, \mathbf{Z}$, respectively. A word w is called finite if I is finite. In that case the length $|w|$ is defined as $|I|$ and its content $c(w)$ as $|\{i \in I | w_i = a\}|$, hence as the number of a's it contains. A subword or factor of w is merely the restriction of w to a subinterval $J \subset I$. We will write $x \subset w$ if x is a factor of w.

Definition 2.1. *Let w be a word. Then w is called balanced if $|c(A) - c(B)| \leq 1$ for any two finite factors A, B of the same length.*

If w is a word and $k \in \mathbf{N}$, then $\mathcal{B}(w, k)$ is defined as the collection of all factors of length k, k-factors for short. If w is finite and $|w| = n$, then obviously $\mathcal{B}(w, k) = \emptyset$ for $k > n$. The cardinality $P(w, k) := |\mathcal{B}(w, k)|$ is a function from $\mathbf{N}$ into itself and is called the complexity function of w.

Let $p \in \mathbf{N}^+$. We call w periodic with period p if $w_i = w_{i+p}$ whenever $i, i + p \in I$. The smallest such p is called the period of w and each factor of length p is called a period cycle. If $I = \mathbf{N}$ we call w eventually periodic if there exist $N \geq 0, p \in \mathbf{N}^+$ such that $w_i = w_{i+p}$ whenever $i \geq N$. The smallest p is called the eventual period and every factor of $[N, \infty)$ of length p is called an eventual period for w. We now give some properties of $P(w, k)$ when w is infinite. We may assume $I \in \{\mathbf{N}, \mathbf{Z}\}$.

Theorem 2.1. [CH, Theorem 2.06]. *Let w be an infinite word with complexity function $P(n)$. (We omit the w). Then the following statements are equivalent:*

a*) $P(n)$ is bounded;* b*) $P(n) = P(n + 1)$ for some n;* c*) $P(n) \leq n$ for some n;* d*) w is ultimately periodic when $I = \mathbf{N}$ and purely periodic when $I = \mathbf{Z}$.*

Proof. a) $\Rightarrow$ c) is obvious. Fix $n \in \mathbf{N}$. We can define an injection $\phi : \mathcal{B}(n) \to \mathcal{B}(n+1)$ by mapping each $x \in \mathcal{B}(n)$ to a factor $x* \in \mathcal{B}(n+1)$. This shows that $P(n)$ is non-decreasing, hence c) $\Rightarrow$ b). Suppose that $P(n) = P(n+1)$ for some n. Then the injection ϕ is a bijection and for each $x \in \mathcal{B}(n)$ we have a unique symbol σ_x such that $x\sigma_x \in \mathcal{B}(n+1)$. Then we also have a unique such symbol σ_x if $x \in \mathcal{B}(k)$ where $k \geq n$. This is because we can write $x = x_1 x_2$ where $|x_2| = n$. Hence $P(n) = P(n+1) = \cdots$ and b) $\Rightarrow$ a). If b) holds, pick an n such that $P(n) = P(n+1)$ and $x \in \mathcal{B}(n)$ such that x appears at least twice in w. Say $x = w_s \cdots w_{s+n-1} = w_t \cdots w_{t+n-1}$ where $s < t$. Then induction shows $w_i = w_{i+t-s}$ for $i \geq s$. If $I = \mathbf{N}$ this proves d). If $I = \mathbf{Z}$ we use the same argument with left-extensions $*x$ to see that $w_i = w_{i+t-s}$ for all $i \in \mathbf{Z}$. We leave d) $\Rightarrow$ a) to the reader. $\square$

Remark. Let $\omega = \max P(n)$ where we assume that a) until d) hold. Let p be the period of w. In case $I = \mathbf{N}$ we pick $s \in \mathbf{N}$ minimal such that w is purely periodic on $[s, \infty)$ with period p. Then

$$\omega = \begin{cases} \text{s+p if I=}\mathbf{N} \\ \text{p if I=}\mathbf{Z} \end{cases}$$

Proof. First suppose $I = \mathbf{Z}$. Since $P(p) \leq p$, all words x of length p have a unique successor symbol σ_x as before. The words $w_i \cdots w_{i+p-1}$ with $0 \leq i < p$ are distinct since otherwise the proof of Lemma 2.1 would imply a period $< p$. Hence $P(p) = p$ and since $P(p) = P(p+1) = \cdots$ we have $\omega = p$. Now suppose $I = \mathbf{N}$. It is clear that $\omega \leq s + p$. We consider the words $x_i = w_i \cdots w_{i+s+p-1}$ where $0 \leq i < s + p$. Suppose that $x_i = x_j$ where $0 \leq i < j < p + s$. Since $P(s+p) \leq s + p$ it follows as before that all $x \in \mathcal{B}(s+p)$ have a unique successor symbol σ_x. Then the distance $j - i \geq p$, since otherwise the proof of Lemma 2.1 would yield a period smaller than p. Hence $i < s$ and we find that w is periodic on $[i, \infty)$, a contradiction. Therefore $P(s+p) = s + p = \omega$. $\square$

Now suppose that w is an infinite word for which $P(n)$ is not bounded. Then Lemma 2.1 shows that $P(n) \geq n + 1$ for all n. Hence infinite words with $P(n) = n + 1$ are as close to periodic as you can get, without actually being (ultimately) periodic. Motivated by this we have the following definition.

Definition 2.2. *Let w be a word. Then w is called stiff if $P(n) \leq n+1$ for all n.*

Hence if w is infinite and stiff, then w is (ultimately) periodic or $P(n) = n+1$ for all n. In Theorem 2.2 we reprove the fact that all balanced words are stiff. In Theorems 2.3, 2.4 we show directly from the definition that every finite balanced (stiff) word appears as factor of a balanced (stiff) **Z**-word. This will enable us to restrict ourselves to **Z**-words when we classify balanced (stiff) words in sections 2.2 and 2.3, respectively.

Theorem 2.2. [MH, Lemma 3.2]. *Every balanced word w is stiff.*

Proof. We will prove a slightly more general statement. Let $\mathcal{S}$ be a collection of words which is closed under factors, i.e. $x \subset w, w \in \mathcal{S} \Rightarrow x \in \mathcal{S}$. Furthermore, suppose that $|c(A) - c(B)| \leq 1$ for all $A, B \in \mathcal{S}$ of the same length. We will show that $P(\mathcal{S}, n) := |\mathcal{B}(\mathcal{S}, n)| \leq n + 1$ for all n and, by applying this to the collection of all factors of w, this will prove our theorem. Now suppose that for some $n \in \mathbf{N}$ we have $P(\mathcal{S}, n) > n+1$, we take n to be minimal. Then $n \geq 1$. Then $P(\mathcal{S}, n-1) \leq n, P(\mathcal{S}, n) \geq n+2$ and we see that there exist $x \neq y$ in $\mathcal{B}(\mathcal{S}, n)$ with $xa, xb, ya, yb \in \mathcal{S}$. (Then x, y are said to have multiple right extension in $\mathcal{S}$). We may write $x = x'\sigma\alpha, y = y'\overline{\sigma}\alpha$ where α is a word, possibly empty, $\sigma \in \{a, b\}$ and $\overline{\sigma}$ is the other symbol of the alphabet. Then $\sigma\alpha\sigma, \overline{\sigma}\alpha\overline{\sigma} \in \mathcal{S}$, contradicting that $\mathcal{S}$ is balanced, as was assumed. This proves our Theorem. □

For reasons of efficiency we now prove Theorem 2.3 with an extended definition of balanced.

Definition 2.1.^{bis} *Let w be a word and $k \in \mathbf{N}^+$. Then w is called k-balanced if $|c(A) - c(B)| \leq k$ for any two finite factors A, B of equal length.*

We will consider k-balanced **Z**-words with $k \geq 2$ in Chapter 3.

Theorem 2.3. *Let x be a finite k-balanced word where $k \in \mathbf{N}^+$. Then x is a factor of a k-balanced **Z**-word.*

Proof. By symmetry it suffices to show that for every finite k-balanced word x either xa or xb is k-balanced. We suppose that x is k-balanced and that xa, xb are not. There exists a minimal $n_1 \in \mathbf{N}^+$ such that xa contains n_1-factors A and A' with $c(A') > c(A)+k$. Since x is k-balanced, A' is the final n_1-factor of xa and $c(A') = c(A)+k+1$. We write $A' =: Ca,\ c(A) =: k_1$

whence $c(C) = k_1 + k$. Similarly there exists a minimal positive integer n_2 such that x contains an n_2-subword B and the last $(n_2 - 1)$-subword D with $c(D) =: k_2$ such that $c(B) = k_2 + k + 1$. Denoting the domains of A, B, C, D by $[\alpha_1, \alpha_2]$, $[\beta_1, \beta_2]$, $[\gamma_1, \gamma_2]$, $[\delta_1, \delta_2]$, respectively, we have $[\alpha_1, \alpha_2] \cap [\gamma_1, \gamma_2] = \emptyset$ and $[\beta_1, \beta_2] \cap [\delta_1, \delta_2] = \emptyset$, since n_1 and n_2 are minimal. Without loss of generality we assume $n_1 \leq n_2$: interchange a and b if necessary. In that case C is a suffix of D and $D \setminus C$ is a subword of x of length $n_2 - n_1$ and content $k_2 - k_1 - k$. Observe that A can be extended to the left or to the right to an n_2-subword $\tilde{A}$ of x, since otherwise we would have

$$\alpha_1 - (n_2 - n_1) < \beta_1 \text{ and } \alpha_2 + (n_2 - n_1) > \delta_2,$$

whence $n_1 = \alpha_2 - \alpha_1 + 1 > \delta_2 - \beta_1 - 2(n_2 - n_1) + 1 = \delta_1 - \beta_2 + 2n_1 - 2 \geq 1 + 2n_1 - 2 \geq n_1$ which is impossible. It follows that $\tilde{A} \setminus A$ is a subword of x of length $n_2 - n_1$ and content

$$c(\tilde{A}) - c(A) \geq c(B) - k - k_1 = k_2 - k_1 + 1$$

since x is k-balanced. We now arrive at the contradiction that the $(n_2 - n_1)$-subwords $\tilde{A} \setminus A$ and $D \setminus C$ of x satisfy $c(\tilde{A} \setminus A) - c(D \setminus C) > k$. □

Theorem 2.4. *Let x be a finite stiff word. Then x is a factor of a stiff* $\mathbf{Z}$*-word.*

Proof. As before it is enough to prove that xa or xb is stiff whenever x is stiff. For $\mathbf{Z}$-words x we have already seen that $P(x, n) \leq n$ for some n implies that x is periodic. We now prove a finite analogue to this, which will be Lemma 2.4.2.

Lemma 2.4.1. *Let x be a finite word with $P(x, n) \leq n$. If both the first $(n-1)$-subword and the last $(n-1)$-subword of x occur at least twice in x, then x is purely periodic with period $l \leq n$.*

Proof. For each $0 \leq i < n$ we define injections $\phi_i, \psi_i : \mathcal{B}(x, i) \hookrightarrow \mathcal{B}(x, i+1)$ by mapping each $x \in \mathcal{B}(x, i)$ to some extension $x*, *x \in \mathcal{B}(x, i+1)$, respectively. As in the proof of Theorem 2.1 we find that there exists a $0 \leq i < n$ for which ϕ_i, ψ_i are bijections and the hypothesis of this lemma implies that some i-factor of x is repeated. The remainder of the proof is similar to the proof in Theorem 2.1 and will be omitted. □

Lemma 2.4.2. [H,Vervoort]. *Let x be a finite stiff word with $P(x,k) \le k$.*
a) *Then $x = \alpha\beta\gamma$ with $|\alpha| + |\gamma| < k$ such that β has period $l \le k - |\alpha| - |\gamma|$.*
b) *Then $P(x,m) \le k$ for all $m \ge k$.*

Proof. a) Define $x_0 = x$ and l_0 as the smallest l with $1 \le l \le k$ and $P(x,l) \le l$. We define a sequence (x_t, l_t), $t \ge 0$ by induction. If the last $(l_t - 1)$-word of x_t does not occur elsewhere in x_t, then we define x_{t+1} as x_t with the last letter removed. Otherwise, if the first $(l_t - 1)$-word of x_t does not occur elsewhere in x_t, then we remove the first symbol of x_t to obtain x_{t+1}. If the first and last $(l_t - 1)$-factor are both repeated in x_t then we stop. If we don't stop, then we put $l_{t+1} = \min\{l | 1 \le l \le l_t,\ P(x_{t+1}, l\ ) \le l\}$. It follows from the definitions that $P(x_t, l_t - 1) = l_t$ for all t and that $P(x_{t+1}, l_t - 1) = P(x_t, l_t - 1) - 1 = l_t - 1$. Hence $1 \le l_{t+1} < l_t$ and there is a pair (x_τ, l_τ) for which the process ends. Applying Lemma 2.4.1 to this pair we find that $\beta := x_\tau$ has a period

$$l \le l_\tau \le l_0 - \tau \le k - \tau = k - |\alpha| - |\gamma|$$

where α, γ are the words preceding, succeeding x_τ in x respectively.
b) $P(x,m) \le |\alpha| + l_\tau + |\gamma| \le k$ by the periocity of x_τ. □

Remark. If one draws the graph $n \to P(n)$ and interpolates linearly, one may show that finite stiff words give rise to the upper boundary of a trapezoid whose upstanding sides have slope $-1, 1$ and that this trapezoid may degenerate into an isosceles triangle. This explains why one sometimes encounters the exotic name "trapezoidal" instead of "stiff".

Proof of Theorem 2.4. Put $n = |x|$. If $P(x,k) \le k$ for all $k \le n$ then xa is stiff. Otherwise, let K be the maximal $k \le n$ with $P(x,k) = k+1$. Note that $K < n$ since $P(x,n) = 1$, and that $P(x, K+1) \le K+1$. By Lemma 2.4.2 b), we infer that $P(x,m) \le K+1$ for $m > K$ whence $P(x\sigma, k) \le k+1$ for $\sigma \in \{a,b\}$ and $k > K$.

If the last $(K-1)$-word in x does not occur elsewehere as a subword of x, then $P(x', K-1) = P(x, K-1) - 1 \le K - 1$ where x' denotes x with the last symbol removed. Then Lemma 2.4.2, part b) again, implies $P(x', K) \le K - 1$ hence $P(x,K) \le K$ contradicting our choice of K. Therefore the last $(K-1)$-word appears elsewehere in x, followed by a symbol σ. For every $k \le K$ we then have that the final k-word in $x\sigma$ is repeated. Hence $P(x\sigma, k) = P(x,k) \le k+1$. Thus $x\sigma$ is stiff. □

In the next section we will classify the balanced **Z**-words.

2.2. Classification of the balanced Z-words.

Let w be a balanced **Z**-word. We fix positive integers m, n and consider subwords $A, B, C \subset w$ of length m, n, mn respectively. We denote their contents by b_m, b_n, b_{mn} respectively. We can see C as a union of m factors of length n and also as a union of n factors of length m. Since w is balanced we have $n(b_m - 1) \leq b_{mn} \leq m(b_n + 1)$, hence $n(b_m - 1) \leq m(b_n + 1)$. This result in a slightly stronger form and in a slightly different context can already be found in [MH2, Theorem 2.1]. Rewriting gives $\frac{b_m}{m} - \frac{b_n}{n} \leq \frac{1}{m} + \frac{1}{n}$ and combining this with a symmetric argument we find $|\frac{b_m}{m} - \frac{b_n}{n}| \leq \frac{1}{m} + \frac{1}{n}$. For every $n \in \mathbf{N}^+$ we choose an n-factor of w and we denote its content by b_n. The inequality shows that $(\frac{b_n}{n})_1^\infty$ is a Cauchy sequence in $\mathbf{R}$, hence it has a limit α and $\alpha \geq 0$. Since two choices for b_n can differ by at most one, it follows that α is the same for all sequences b_n. Hence we may give the following definition.

Definition 2.3. *Let w be a balanced $\mathbf{Z}$-word and let b_n be the content of an n-factor of w for every positive integer n. The density $\alpha = \alpha(w)$ of w is then defined as the limit of $\frac{b_n}{n}$ as $n \to \infty$.*

Lemma 2.5.1. *Let w be a balanced $\mathbf{Z}$-word with density α and let b_n be the content of an n-factor of w. Then $|b_n - n\alpha| \leq 1$.*

Proof. For any m-factor of w with content b_m we have $|\frac{b_m}{m} - \frac{b_n}{n}| \leq \frac{1}{m} + \frac{1}{n}$, as we saw before. Letting $m \to \infty$ we find $|\frac{b_n}{n} - \alpha| \leq \frac{1}{n}$. □

We now concentrate on balanced **Z**-words with a rational density $\alpha = \frac{k}{n}$, where $k \geq 0$ and $n \geq 1$ are integers with $(k, n) = 1$. Applying the previous lemma with this n, we find that $c(x) \in \{k-1, k, k+1\}$ for all $x \in \mathcal{B}(w, n)$. The values $k - 1$ and $k + 1$ are mutually exclusive since w is balanced. If $\alpha = 0$ then Lemma 2.5.1 shows that $b_n \leq 1$ for all n-factors of w and this means that w contains at most one a. It is easy to see that $w = \cdots bbb \cdots$ and $w = b^\infty a b^\infty$ are indeed balanced with density 0. If $\alpha = 1$, then $\alpha(\overline{w}) = 0$ where $\overline{w}$ is the **Z**-word obtained from w by interchanging a and b. It follows that $w = \cdots aaa \cdots$ or $w = a^\infty b a^\infty$. From now on we assume that $0 < \alpha < 1$.

Definition 2.4. *Let w be a balanced $\mathbf{Z}$-word of rational density $\alpha = \frac{k}{n}$*

where $(k,n)=1$ and $x \in \mathcal{B}(w,n)$. Then x is called an exceptional factor when $c(x) \neq k$. It is exceptional of max type when $c(x)=k+1$ and of min type when $c(x)=k-1$.

The correctness of the term "exceptional" follows from the following lemma.

Lemma 2.5.2 [MH, Lemma 3.1]. *Let w be a balanced $\mathbf{Z}$-word of rational density $\alpha = \frac{k}{n}$, where $(k,n)=1$. Then there is at most one exceptional factor and this factor, if it exists, appears only once in w.*

Proof. Suppose that the subwords of w with domains $X=[x,x+n)$ and $Y=[y,y+n)$ are exceptional, where $x<y$. Note that $\overline{w}$ is balanced if and only if w is balanced, where $\overline{w}$ is obtained from w by exchanging a and b. Therefore we can assume without loss of generality that the words with domain X,Y are exceptional of max-type. In that case $c(X)=c(Y)=k+1$. We consider two cases.

a) $\mathbf{x}+\mathbf{n} \leq \mathbf{y}$. Then $c_{[x,x+n)}+c_{[x+n,y)}+c_{[y,y+n)}=c_{[x,y+n)}$, hence

$$c_{[x+n,y)}+2k+2=c_{[x,y+n)}$$

Applying Lemma 2.5.1 twice we have $(y-x)\alpha+k+1=(y-x-n)\alpha+2k+1 \leq c_{[x+n,y)}+2k+2=c_{[x,y+n)} \leq (y+n-x)\alpha+1=(y-x)\alpha+k+1$. Apparently we have equality everywhere, in particular $c_{[x+n,y)}=(y-x-n)\alpha-1$. It follows that $y-x \equiv 0 \bmod n$ and we may partition $[x+n,y)$ into $\lambda := \frac{y-x-n}{n}$ intervals of length n. From $c_{[x+n,y)}=\lambda k-1$ it follows that one of these intervals has content $<k$, a contradiction.

b) $\mathbf{x}<\mathbf{y}<\mathbf{x}+\mathbf{n}$. By Lemma 2.5.1 we have $2k+2=c_{[x,x+n)}+c_{[y,y+n)}=c_{[x,y+n)}+c_{[y,x+n)} \leq (y+n-x)\alpha+1+(x+n-y)\alpha+1=2(k+1)$. Because equality holds everywhere we have $(y-x)\alpha \in \mathbf{Z}$. This, however, is a contradiction since $0<y-x<n$ and $(k,n)=1$. □

We note that a $\mathbf{Z}$-word w can be identified with a subset $W \subset \mathbf{Z}$ in an obvious way, namely by defining $w_i = a \iff i \in W$. We call a subset $W \subset \mathbf{Z}$ balanced if and only if the corresponding word w is balanced. The following lemma by Tijdeman explicitly describes subintervals of a balanced set W.

Lemma 2.5.3. [T, Lemma 3]. *Let W be a balanced set of rational density $\alpha = \frac{k}{n} > 0$ and let $x_1 < \cdots < x_p$ be consecutive elements with $p \le k$. Then there exists an integer s such that $x_i = \lfloor \frac{in+s}{k} \rfloor$ for $1 \le i \le p$.*

Proof. It is no restriction to take $p = k$. Define $T_s = \{\lfloor \frac{in+s}{k} \rfloor\}_{i=1}^k$ for all $s \in \mathbf{Z}$. Increasing s by 1 increases $\sum_{i=1}^k T_s(i)$ by 1 since only the term with $in + s \equiv -1 \bmod k$ changes. Hence we can choose an $s \in \mathbf{Z}$ with $\sum_{i=1}^k T_s(i) = \sum_{i=1}^k x_i$. Denote this T_s by T. If $(T)_{i=1}^k \neq (x_i)_{i=1}^k$ then we have $T(i) < x_i, T(j) > x_j$ for some indices i and j. First we suppose $i < j$. Then $T(i) < x_i < x_j < T(j)$. Now

$$T(j) - T(i) = \lfloor \frac{jn+s}{k} \rfloor - \lfloor \frac{in+s}{k} \rfloor \le \lfloor \frac{(j-i)n}{k} \rfloor + 1 \le \frac{(j-i)n}{k} + 1 (*),$$

which gives $x_j - x_i \le \frac{(j-i)n}{k} - 1$. The interval $[x_i, x_j + 1)$ contains $j - i + 1$ elements of W, hence $j - i + 1 \le (x_j + 1 - x_i)\alpha + 1 \le j - i + 1$, by Lemma 2.5.1. It follows that equality holds everywhere in (*), but then $\frac{(j-i)n}{k} \in \mathbf{Z}$ and $\lfloor \frac{jn+s}{k} \rfloor - \lfloor \frac{in+s}{k} \rfloor = \frac{(j-i)n}{k}$. This contradicts the fact that equality holds everywhere in (*), showing that $i < j$ is impossible. The case $j < i$ is dealt with in a similar way using the interval $[x_j, x_i - 1)$. Hence $(T_i)_{i=1}^k = (x_i)_{i=1}^k$.□

If the $\mathbf{Z}$-word w we consider has no exceptional block, then $c(x) = k$ for all $x \in \mathcal{B}(w, n)$ and it follows that w has period n. This n is in fact the minimal period, because the fraction $\frac{k}{n}$ is irreducible. Let $x_1 < \cdots < x_k$ be the elements of $W \cap [1, n]$. Then Lemma 2.5.3 gives us an integer s such that $x_i = \lfloor \frac{in+s}{k} \rfloor$ for $1 \le i \le k$. Since W and $\{\lfloor \frac{in+s}{k} \rfloor\}_{i \in \mathbf{Z}}$ both have period n and coincide on $[1, n]$, it follows that $W = \{\lfloor \frac{in+s}{k} \rfloor\}_{i \in \mathbf{Z}}$. Given s, we may choose $\lambda, \mu \in \mathbf{Z}$ such that $\lambda n + s = \mu k$. Then $\lfloor \frac{(i+\lambda)n+s}{k} \rfloor = \mu + \lfloor \frac{in}{k} \rfloor$. Hence W is equivalent, up to a shift, with $W' := \{\lfloor \frac{in}{k} \rfloor\}_{i \in \mathbf{Z}}$.

Now assume that w does have an exceptional block, say $x = w_0 \cdots w_{n-1}$. Then every n-factor of $w|_{(-\infty, n-2]}$ and $w|_{[1,\infty)}$ has content k and it follows that these infinite words have period n. The "overlapword" with domain $[1, n-2]$ has content $k-1$. Indeed, if x is of max-type, then $c(x) = k+1, w_0 = w_{n-1} = a$ and otherwise we have $c(x) = k - 1, w_0 = w_{n-1} = b$. Applying Lemma 2.5.3 we find $w|_{(-\infty,n-2]} = \{\lfloor \frac{in+s}{k} \rfloor\}_{i=-\infty}^{k-1}$ and $w|_{[1,\infty)} = \{\lfloor \frac{in+t}{k} \rfloor\}_{i=1}^{\infty}$. Here s, t are integers such that $\lfloor \frac{in+s}{k} \rfloor = \lfloor \frac{in+t}{k} \rfloor$ for $1 \le i \le k - 1$. First let us suppose that x is of max-type. Substituting $i = 0$ and $i = k$ in the respective formulas we find $\lfloor \frac{s}{k} \rfloor = 0$ and $\lfloor \frac{t}{k} \rfloor = -1$. In particular $t < s$. We have $\frac{in+s}{k}, \frac{in+t+1}{k} \notin \mathbf{Z}$ for $1 \leq i \leq k-1$, since otherwise the equality above could not hold. Since these forms are integer for some $0 \leq i < k$, it follows that they are integer for $i = 0$. Hence $s \equiv 0(k), t \equiv -1(k)$ and together with $\lfloor \frac{s}{k} \rfloor = 0, \lfloor \frac{t}{k} \rfloor = -1$, this implies that $s = 0, t = -1$. We leave it to the reader to check that $s = -1, t = 0$ if x is of min-type. We now check that all these $\mathbf{Z}$-words are indeed balanced. The periodic case is a special case of words given by $W = \{\lfloor \zeta i + \phi \rfloor\}_{i \in \mathbf{Z}}$ and $W = \{\lceil \zeta i + \phi \rceil\}_{i \in \mathbf{Z}}$. Here $\zeta \geq 1$ and $\phi \in \mathbf{R}$. These words are called Beatty words and if ζ is irrational we speak of sturmian words. In Lemma 2.5.4 we show that Beatty words are balanced, Lemma 2.5.5 is somewhat similar to Lemma 2.5.3 and in Lemma 2.5.6 we show that every balanced $\mathbf{Z}$-word of irrational density is in fact sturmian. After that we are ready to give the promised list of balanced $\mathbf{Z}$-words.

Lemma 2.5.4. *Let $\zeta \geq 1$ and $\phi \in \mathbf{R}$. Define the $\mathbf{Z}$-word w by $W = \{\lfloor \zeta i + \phi \rfloor\}_{i \in \mathbf{Z}}$ or by $W = \{\lceil \zeta i + \phi \rceil\}_{i \in \mathbf{Z}}$. Then w is balanced of density $\frac{1}{\zeta}$.*

Proof. We only deal here with the floor-case. Fix $n \geq 1$ and let $x \in \mathbf{Z}$. The number of elements of W in the interval $[x, x+n)$ equals the number of i with $x \leq \lfloor \zeta i + \phi \rfloor < x + n$. But this is equivalent to $\frac{x-\phi}{\zeta} \leq i < \frac{x+n-\phi}{\zeta}$ and the number of integer solutions is $\lceil \frac{x+n-\phi}{\zeta} \rceil - \lceil \frac{x-\phi}{\zeta} \rceil \in \lfloor \frac{n}{\zeta} \rfloor + \{0, 1\}$. Since this is true for any n, the word w is balanced. For any n-factor $y \subset w$ we have $\frac{n}{\zeta} - 1 < \lfloor \frac{n}{\zeta} \rfloor \leq c(y) \leq \lfloor \frac{n}{\zeta} \rfloor + 1 \leq \frac{n}{\zeta} + 1$, hence $|\frac{c(y)}{n} - \frac{1}{\zeta}| \leq \frac{1}{n}$. It follows that w has density $\frac{1}{\zeta}$. The ceiling case is similar. □

Lemma 2.5.5. *Let $W = \{x_i\}_{i \in \mathbf{Z}}$ with x_i strictly increasing be a balanced set of positive density α. Let $\zeta := \frac{1}{\alpha}$ and $\phi \in \mathbf{R}$ be arbitrary. Then $x_i \leq \lfloor \zeta i + \phi \rfloor$ for all i or $x_i \geq \lfloor \zeta i + \phi \rfloor$ for all i.*

Proof. Suppose that we have $i, j \in \mathbf{Z}$ with $\lfloor \zeta i + \phi \rfloor < x_i$ and $x_j < \lfloor \zeta j + \phi \rfloor$. Assume first that $i < j$, then $\lfloor \zeta i + \phi \rfloor < x_i < x_j < \lfloor \zeta j + \phi \rfloor$. We have $\lfloor \zeta j + \phi \rfloor - \lfloor \zeta i + \phi \rfloor < \zeta(j-i) + 1$, hence $x_j - x_i < \zeta(j-i) - 1$. The interval $[x_i, x_j + 1)$ contains exactly $j - i + 1$ elements of the balanced set W, hence Lemma 2.5.1 yields $j - i + 1 \leq (x_j - x_i + 1)\alpha + 1 < j - i + 1$, a contradiction. The case $j < i$ is left to the reader. □

Lemma 2.5.6. *Every balanced $\mathbf{Z}$-word w with irrational density is sturmian.*

Proof. Write $W = \{x_i\}_{i\in\mathbf{Z}}$ for the corresponding balanced set with the x_i strictly increasing and where the bisequence x_i is defined up to a shift of indices only. Let α be the density of w and $\zeta := \frac{1}{\alpha}$ as before. We introduce the set $V := \{\phi \in \mathbf{R} | x_i \leq \lfloor \zeta i + \phi \rfloor \text{ for all } i \in \mathbf{Z}\}$. The set V is not empty since any ϕ with $x_0 < \lfloor \phi \rfloor$ is contained in V by Lemma 2.5.5. Also $V \neq \mathbf{R}$ since any ϕ with $x_0 > \lfloor \phi \rfloor$ is not contained in V. Also V is monotonic in the sense that $\phi \in V, \phi < \psi \Rightarrow \psi \in V$. It follows that $\theta := \inf(V)$ is a real number and the right-continuity of $\lfloor x \rfloor$ shows that $\theta = \min(V)$, hence $V = [\theta, \infty)$. We have $x_i = \lfloor \zeta i + \theta \rfloor$ for some i, because otherwise also $\theta - 1 \in V$. If $x_j = \lfloor \zeta j + \theta \rfloor$ for all j we are done. Hence let us suppose that $x_j < \lfloor \zeta j + \theta \rfloor$ for some j. If $\zeta j + \theta \notin \mathbf{Z}$, then $x_j < \lfloor \zeta j + \theta - \epsilon \rfloor$ for some $\epsilon > 0$ and then Lemma 2.5.5 implies $\theta - \epsilon \in V$, a contradiction. Therefore such j satisfy $\zeta j + \theta \in \mathbf{Z}$ and because ζ is irrational this happens for a unique j. We have $x_j = \zeta j + \theta - 1$, because a smaller value for x_j would give $\theta - 1 \in V$ by Lemma 2.5.5. Thus $x_i = \lfloor \zeta i + \theta \rfloor - 1_j(i) = \lceil \zeta i + \theta - 1 \rceil$ for all i. Here $1_j(i)$ equals 1 if $i = j$ and 0 otherwise. □

Theorem 2.5. *The balanced* $\mathbf{Z}$*-words, modulo shift, are given by:*

a) $w = a^\infty, w = b^\infty$ *and* $W = \{\lfloor \frac{in}{k} \rfloor\}_{i\in\mathbf{Z}}$ *where* $k, n \in \mathbf{N}^+$ *satisfy* $(k, n) = 1$ *and* $\frac{k}{n} \in (0, 1)$. *(periodic case)*

b) $w = a^\infty b a^\infty, w = b^\infty a b^\infty$, $W = \{\lfloor \frac{in}{k} \rfloor\}_{-\infty}^{k-1} \cup \{\lfloor \frac{in-1}{k} \rfloor\}_1^\infty$ *and* $W = \{\lfloor \frac{in-1}{k} \rfloor\}_{-\infty}^{k-1} \cup \{\lfloor \frac{in}{k} \rfloor\}_1^\infty$, *where* $k, n \in \mathbf{N}^+$ *satisfy* $(k, n) = 1$ *and* $\frac{k}{n} \in (0, 1)$. *(skew case)*

c) $W = \{\lfloor \zeta i + \phi \rfloor\}, W = \{\lceil \zeta i + \phi \rceil\}$ *where* $\zeta > 1$ *is irrational and* $0 \leq \phi < 1$. *(sturmian case)*

Proof. We have already shown that every balanced $\mathbf{Z}$-word is of this form and that the words in classes **a** and **c** are indeed balanced. In Theorem 2.8 we will show that the classes **a**, **b**, **c** have the same collection of finite factors. It will follow that every $\mathbf{Z}$-word in **b** is balanced as well. Note that none of the $\mathbf{Z}$-words in **b** is periodic. □

Remarks. - An infinite word w is called recurrent if every finite factor appears in at least two different locations. In other words, there should be two different domains inducing it. It is easily checked that $\mathbf{Z}$-words from class **a** and **c** are recurrent and that $\mathbf{Z}$-words from class **b** are not.
- Given a sturmian $\mathbf{Z}$-word w the number ζ is unique, $\zeta = \frac{1}{\alpha(w)}$, and w has

at most one representation of each type. In the special case that $\phi \in \mathbf{Z} \oplus \zeta \mathbf{Z}$ one speaks of standard sturmian **Z**-words and we will come back to them later. The words with $\phi \in \frac{1}{2} + \mathbf{Z} \oplus \zeta \mathbf{Z}$ are known as Bernoulli words.

- Theorem 2.5. can be used to characterize the right-infinite words x such that both ax and bx are balanced. Such words have a density α and we put $\zeta = \frac{1}{\alpha}$. If $\alpha \in (0,1) \cap \mathbf{Q}$ there exist exactly two such words x. If we write $\alpha = \frac{k}{n}$ with k, n positive and coprime they are given by the subsets $\{\lfloor \frac{in}{k} \rfloor\}_1^\infty$ and $\{\lfloor \frac{in-1}{k} \rfloor\}_1^\infty$ of $\mathbf{N}^+$. If $\alpha = 0, 1$ we only have $x = b^\infty, a^\infty$, respectively. Finally, for $\alpha \in (0,1) \setminus \mathbf{Q}$ there is only one such x and it is given by the subset $\{\lfloor \zeta i \rfloor\}_1^\infty$ of $\mathbf{N}^+$. All such x are called standard Beatty words on $\mathbf{N}^+$ and we will see another characterization of them in Section 2.5.5.

2.3. Classification of the stiff Z-words.

Since the balanced **Z**-words have been given in the previous section, we need only consider those stiff **Z**-words which are not balanced. We first give the main result.

Theorem 2.6. *The stiff, not balanced,* **Z***-words w, modulo shift, are given by $W = \{\lfloor \frac{is}{l} \rfloor\}_{-\infty}^{k+l-1} \cup \{\lfloor \frac{ir-1}{k} \rfloor\}_1^\infty$ and $W = \{\lfloor \frac{is-1}{l} \rfloor\}_{-\infty}^{k+l-1} \cup \{\lfloor \frac{ir}{k} \rfloor\}_1^\infty$. Here k, l, r, s are integers with $0 \le l \le s, 0 \le k \le r$ and $lr - ks = 1$ in the first case and $lr - ks = -1$ in the second case. If $k = 0$ or $l = 0$ the corresponding set should be read as $\emptyset$.*

Proof. We start the proof with a simple lemma. A palindrome is a word which does not change when it is read from right to left. Palindromes are either finite or have domain **Z**.

Lemma 2.6.1. *Let w be any word which is not balanced. There is a unique P of minimal length such that $aPa, bPb \subset w$ and this P is a palindrome.*

Proof. Pick $A, B \subset w$ of minimal length n such that $c(A) \ge c(B) + 2$. Then $A = a\tilde{A}a, B = b\tilde{B}b$ and $c(\tilde{A}) = c(\tilde{B})$ because n is minimal. If $\tilde{A} \ne \tilde{B}$, then $\tilde{A} = C\sigma A', \tilde{B} = C\overline{\sigma}B'$ where A', B', C are words and $\sigma, \overline{\sigma}$ are complementary symbols. If $\sigma = a$, then $aCa, bCb \subset w$ and $|aCa| < |A|$, contradicting the minimality of n. If $\sigma = b$, then $A = aCbA'a, B = bCaB'b$ and cancellation gives us the factors $A'a, B'b \subset w$ of equal length with $c(A'a) = c(B'b) + 2$. Here $|A'a| < |A|$ contradicts the minimality of n. We conclude that $\tilde{A} = \tilde{B} =: P$. If P is not a palindrome, then $P = x\sigma y\overline{\sigma}\overleftarrow{x}$ for words x, y and a symbol σ, where $\overleftarrow{x}$ is the word obtained by reading x from right to left. Then $A = (ax\sigma)y(\overline{\sigma}\overleftarrow{x}a)$ and $B = (bx\sigma)y(\overline{\sigma}\overleftarrow{x}b)$. If $\sigma = a$, then the factors $axa, b\overleftarrow{x}b$ contradict the minimality of n. If $\sigma = b$ one uses the other pair. Thus P is a palindrome.

Now suppose that also $aQa, bQb \subset w$. Then $|aQa| \geq n$, hence $|Q| \geq |P|$ and we assume that $|Q| = |P|$. If $P \neq Q$, then $P = R\sigma P', Q = R\overline{\sigma}Q'$ and it follows that $\sigma R\sigma, \overline{\sigma}R\overline{\sigma} \subset w$. Since $|R| < |P|$ this contradicts the minimality of n and it follows that P is unique. □

Let w be a word. A factor x is said to have multiple right-extension (m.r.e.) in w if $xa, xb \subset w$. The collection of these factors is denoted by $\mathrm{MRE}(w)$ and $\mathrm{MRE}_i(w) = \mathrm{MRE}(w) \cap \mathcal{B}(w, i)$. If a factor doesn't have m.r.e., then we say it has unique right-extension (u.r.e.), although it might happen of course that x does not have any right-extensions at all. Similar definitions can be given with "left" instead of "right". Some authors call a factor right-special if it has m.r.e. and left-special if it has m.l.e.

Now let w be a stiff $\mathbf{Z}$-word which is not balanced, and let P be the palindrome from Lemma 2.6.1. We write $|P| =: n-2, A := aPa, B := bPb$ as in the proof of that lemma. Since w is stiff we have $|\mathrm{MRE}_i(w)| = P(w, i+1) - P(w, i) \leq 1$ for all $i \in \mathbf{N}$. In particular $\mathrm{MRE}_{n-2}(w) = \{P\}$. Then either $\mathrm{MRE}_{n-1}(w) = \emptyset$ or $\mathrm{MRE}_{n-1}(w) = \{\sigma P\}$ for some symbol σ. In the former case we find that w has period $p \leq n-1$ which is impossible, since then we could delete the first p symbols from A and from B to obtain smaller factors A', B' with $c(A') - c(B') = c(A) - c(B) = 2$. Hence $\mathrm{MRE}_{n-1}(w) = \{\sigma P\}$ and $\overline{\sigma}P\sigma \not\subset w$.

Now suppose that $\overline{\sigma}P, \sigma P$ occur in this order in w, i.e. $\overline{\sigma}P = w|_{[x,x+n-1)}$, $\sigma P = w|_{[y,y+n-1)}$ where $x < y$. We take x, y such that the distance $y - x$ is minimal. Since all $(n-1)$-factors of w different from σP have u.r.e. in w, it follows that the factors $w|_{[i,i+n-1)}$ with $x \leq i \leq y$ are all different. Hence $y - x + 1 \leq P(n-1) \leq n$ which implies $y \leq x + n - 1$. Note that $w_{x+n-1} = \overline{\sigma}$ because $\overline{\sigma}P\sigma \not\subset w$. If $y = x+n-1$, then $\sigma = w_y = w_{x+n-1} = \overline{\sigma}$, a contradiction. Hence $y \leq x + n - 2$. Writing $P = \pi_1 \cdots \pi_{n-2}$ and noting that P is a palindrome, we find $\sigma = w_y = w_{x+(y-x)} = \pi_{y-x} = \pi_{n-1-y+x} = w_{n+x-1} = \overline{\sigma}$, another contradiction. This proves that $\overline{\sigma}P$ does not occur before σP inside w. Applying this to $\overleftarrow{w}$ we find that $P\overline{\sigma}, P\sigma$ do not occcur in w in this order.

By shifting w we may assume that $w|_{[0,n-1]} = \sigma P\overline{\sigma}$ and we write $w|_{(-\infty,n-2]} = w_l, w_{[1,\infty)} = w_r$. We have $aPa, bPb \subset w$ and by the previous paragraph we have $\sigma P\sigma \subset w_l, \overline{\sigma}P\overline{\sigma} \subset w_r$. From $P\sigma \not\subset w_r$ we deduce $P(w_r, n-1) \leq P(w, n-1) - 1 \leq n-1$. Since the initial $(n-1)$-factor $P\overline{\sigma}$ of w_r is repeated, we find that w_r is purely periodic with some minimal period $r < n$. It follows that we may characterise $[1,\infty)$ as the largest right-infinite domain on which w is purely periodic. In particular we find that $\sigma P\overline{\sigma}$ appears only once in w and that w, therefore, is not recurrent. If w_r is not balanced, then Lemma 2.6.1 would give a word Q such that $aQa, bQb \subset w_r$. Then $|Q| \geq |P| = n-2$ and $Q \in \mathrm{MRE}(w_r)$, hence $P(w_r, n-2) < P(w_r, n-1)$. This means that the right-period $r \geq n$, contradicting what we saw before. Therefore w_r is balanced. It is then contained in a periodic balanced **Z**-word by periodicity and Theorem 2.5 shows that w_r is given by a rational Beatty sequence with certain density $\frac{k}{r}$, where $k \in \mathbf{N}, r \in \mathbf{N}^+$ and $(k,r) = 1$. Similarly, w_l is purely periodic with period $s < n$, balanced and given by a rational Beatty sequence of density $\frac{l}{s}$, say.

Suppose that $x \in \mathcal{B}(w_l, n-1) \cap \mathcal{B}(w_r, n-1)$ and choose integers $i \leq 0, j \geq 1$ such that $x = w|_{[i,i+n-1)} = w|_{[j,j+n-1)}$. Pick $t \geq i$ minimal such that $w|_{[t,t+n-1)} = \sigma P$. Since $\mathrm{MRE}_{n-1}(w) = \{\sigma P\}$, we find that $\sigma P = w|_{j-i+[t,t+n-1)} \subset w_r$, a contradiction. Thus for $i \geq n-1$ we have $\mathcal{B}(w_l, i) \cap \mathcal{B}(w_r, i) = \emptyset$ and the factors of w not contained in either of them are exactly those containing $\sigma P\overline{\sigma}$. There are exactly $i-n+1$ of these. Hence for $i \geq n$ we find $P(w,i) = (i-n+1)+s+r$. Since w is stiff and not periodic, we conclude $s + r = n$. We can therefore write $\sigma P\overline{\sigma} = XY$ where X, Y are period cycles for w_l and w_r respectively. Hence $c(P) = k+l-1, |P| = r+s-2$. It follows that $W_l = \{\lfloor\frac{is+\mu}{l}\rfloor\}_{-\infty}^{k+l-1}$ and that $W_r = \{\lfloor\frac{ir+\lambda}{k}\rfloor\}_1^{\infty}$. For the problem cases $k = 0, l = 0$ we adopt the convention that we read the corresponding set as $\emptyset$. The formulas give equal values for $1 \leq i \leq k+l-1$. Since $\mathcal{B}(w_l, t) \cap \mathcal{B}(w_r, t) = \emptyset$ for k large, we have $\frac{k}{r} \neq \frac{l}{s}$ and the integer $\Delta := lr - ks$ is non-zero. For the moment we assume that $\Delta > 0$ and we will show that $\Delta = 1$, as stated in the theorem. Note that $l > 0$ in this case.

If $k = 0$ then $r = 1, w_r = bbb\cdots$ and σb^∞ is not periodic, hence $\sigma = a$. Furthermore $P = b^{s-1}$, hence $w = (ab^{s-1})^\infty b^\infty = (b^{s-1}a)^\infty b^\infty$. This corresponds indeed to the first formula in Theorem 2.6 for $(k,l,r,s) = (0,1,1,s)$ and we have $\Delta = 1$. From now on we assume that $0 < l \leq s, 0 < k \leq r$.

Lemma 2.6.2 *Suppose* $k,l,r,s \in \mathbf{N}^+, \lambda, \mu \in \mathbf{Z}$ *satisfy* $(l,s) = (k,r) = 1, \frac{s}{l} < \frac{r}{k}$ *and* $\lfloor\frac{is+\mu}{l}\rfloor = \lfloor\frac{ir+\lambda}{k}\rfloor$ *for* $0 < i < k+l$*. Then* $\lambda = -\Delta + pk, \mu = pl$

for some integer p where $\Delta = lr - ks$.

Proof. Since $(l,s)=1$ we can choose $0<i<k+l$ such that $\frac{is+\mu}{l}\in\mathbf{Z}$ and since $(k,r)=1$ we can choose $0<j<k+l$ such that $\frac{jr+\lambda}{k}\in\mathbf{Z}$. Then $\frac{is+\mu}{l}\le\frac{ir+\lambda}{k}$ and $\frac{jr+\lambda}{k}\le\frac{js+\mu}{l}$. Rewriting this yields $j\Delta\le k\mu-l\lambda\le i\Delta$, hence $j\le i$. Of course this holds for all such choices i,j.

Assume $k\le l$. If two such i exist then $0<j_0\le i_0\le i_0+l<k+l$ for some i_0,j_0. But then $j_0+k\le i_0+l<k+l$ and j_0+k is also a possible choice for j. This implies that $j_0+k\le i_0$ hence $j_0,i_0+l\in\{1,\cdots,k+l-1\}$ lie at least $k+l$ apart, which is absurd. It follows that only one i_0 exists and that $i_0\le l$. If $i_0<l$ then $i_0+1,\cdots,i_0+k$ is a complete set of residues modulo k all less than $k+l$ and it would have to contain a j, which is impossible. We conclude $i_0=l$ and because $l+1,\cdots,k+l-1$ contains no j we find that l is itself a j, hence $\frac{ls+\mu}{l}=\frac{lr+\lambda}{k}\in\mathbf{Z}$. This argument, mutatis mutandis, also works for $l\le k$. In particular $\mu\equiv 0(l)$ and writing $\mu=pl$ we have $s+p=\frac{lr+\lambda}{k}$, hence $\lambda=-\Delta+pk$. □

Lemma 2.6.3. *Suppose k,l,r,s,λ,μ are as in the previous lemma. Moreover suppose that $\max(k,l)>1$. Then $\Delta=1$.*

Proof. Without loss of generality $p=0$ since increasing p by 1 shifts both sequences one place to the right. The formulas then simplify to $\lambda=-\Delta,\mu=0,s=\frac{lr-\Delta}{k}$ and $\lfloor\frac{is}{l}\rfloor=\lfloor\frac{ir-\Delta}{k}\rfloor$ for $0<i<k+l$. Suppose $k>1$. For $1\le\phi<k$ we have $s+\lfloor\frac{\phi s}{l}\rfloor=\lfloor\frac{(\phi+l)s}{l}\rfloor=\lfloor\frac{(\phi+l)r-\Delta}{k}\rfloor=s+\lfloor\frac{\phi r}{k}\rfloor$, hence $\lfloor\frac{\phi r}{k}\rfloor=\lfloor\frac{\phi s}{l}\rfloor=\lfloor\frac{\phi r-\Delta}{k}\rfloor$ and, consequently, $\mathrm{fr}(\frac{\phi r}{k})\ge\frac{\Delta}{k}$. So we have $\Delta<k$ and $\{\mathrm{fr}(\frac{r}{k}),\cdots,\mathrm{fr}(\frac{(k-1)r}{k})\}\subset\{\frac{\Delta}{k},\cdots,\frac{k-1}{k}\}$. Since all fractions on the left are distinct we find $k-1\le k-\Delta$ hence $\Delta=1$. A similar argument can be given if $l>1$. □

We have now shown that $\Delta=1$ if it is positive, except in the case where $k=l=1$, which we assume from now on. The only thing that Lemma 2.6.2 gives us in this case is that $\Delta=r-s$, which does not help very much. If x is a word, then we denote by $[x]_i,[x]^i$ the first and last factor of x of length i, respectively. If x is finite this is only defined if $i\le|x|$, if x is left-infinite only $[x]^i$ is defined and if x is right-infinite then only $[x]_i$ is defined. Suppose that $\sigma=b$, hence $bPa\subset w$. Then $w_r=(b^{r-1}a)\cdots$ and $P=[(b^{r-1}a)\cdots]_{r+s-2}$. From $0<s<r$ we deduce $r+s-2\ge s$, hence $[P]_s=b^s$. Also w_l is an extension of P to the left with period s, hence $w_l=b^\infty$ and $k=0$. This

contradicts our assumption that $k > 0$. Therefore $\sigma = a$ and a similar reasoning gives $w_l = \cdots (ab^{s-1}), P = [\cdots (ab^{s-1})]^{r+s-2}$. By $c(P) = k+l-1 = 1$ we obtain $r \leq s+1$ and we have $\Delta = r - s = 1$, as claimed.

Hence if $\Delta > 0$, then $\Delta = 1$ and Lemma 2.6.2 yields $\lambda = -1 + pk, \mu = pl$ for some integer p. Also we recall that $w|_{[0,n-1]} = aPb$, hence $\lfloor \frac{\mu}{l} \rfloor = 0$ and $p = 0$. Since $W = W_l \cup W_r$ it follows that W is indeed given by the first formula in Theorem 2.6 with $lr - ks = 1$. Now assume that $\Delta < 0$. The word $\overleftarrow{w}$ is stiff and not balanced and $\Delta(\overleftarrow{w}) = -\Delta(w) > 0$. The previous argument then shows that $\Delta(\overleftarrow{w}) = 1$, hence that $\Delta(w) = -1$, and that up to shift we have $\overleftarrow{W} = \{\lfloor \frac{ir}{k} \rfloor\}_{-\infty}^{k+l-1} \cup \{\lfloor \frac{is-1}{l} \rfloor\}_1^{\infty}$ with the same reading convention as before. Reflecting yields $W = -\overleftarrow{W}$ and the substitution $j = k+l-i$ shows that this is indeed equivalent to the second formula in Theorem 2.6.

We will now verify that the formulas for W as given in Theorem 2.6 indeed define stiff $\mathbf{Z}$-words which are not balanced. As we have seen most of the ingredients before we will be short. Suppose first that k, l, r, s are integers with $0 < l \leq s, 0 < k \leq r$ and $lr - ks = 1$. Then $\lfloor \frac{is}{l} \rfloor = \lfloor \frac{ir-1}{k} \rfloor$ for $0 < i < k+l$. For $0 < i \leq l$ this follows from $\frac{ir-1}{k} \leq \frac{is}{l} < \frac{ir}{k}$ and for $l \leq i < k+l$ this follows from $\frac{is}{l} \leq \frac{ir-1}{k} < \frac{is+1}{l}$. Since $\lfloor \frac{is}{l} \rfloor$ and $\lfloor \frac{ir-1}{k} \rfloor$ are not equal for $i = 0$ we deduce that $[1, \infty)$ is the largest right-infinite domain on which w is periodic, with period r. Similarly we find that $(-\infty, r+s-2]$ is the largest left-infinite domain on which w is periodic, with period s. We define $w|_{[0,r+s-1]} =: aPb, w_l := w|_{(-\infty,r+s-2]}$ and $w_r := w|_{[1,\infty)}$. We note that every factor of w_r of length $\geq r-1$ has u.r.e.in w_r, hence every factor of $w' := w_{[0,\infty)}$ of length $\geq r$ has u.r.e. in w'. In particular this holds for its initial factor aP. If aP were repeated in w', this would imply that w' is periodic, which is not true. Hence $aP \not\subset w_r$ and similarly we have $Pb \not\subset w_l$. In particular aPb appears only once in w. A density argument shows that $\mathcal{B}(w_l, i) \cap \mathcal{B}(w_r, i) = \emptyset$ for i large. For large i we then have $P(w, i) = (i - |aPb| + 1) + s + r = i + 1$ and we conclude that w is stiff. It is not balanced since it has no density. Now suppose that $0 < l \leq s, 0 < k \leq r, lr - ks = -1$ and let W be defined by the second formula in Theorem 2.6. Then $\overleftarrow{w}$ is given, modulo shift, by the first formula as we have already seen. Since $\overleftarrow{w}$ is stiff and not balanced the same holds for w. If $k = 0$ or $l = 0$ one uses a similar reasoning. $\square$

Remarks. - The words w from Theorem 2.6 are called infinite Hedlund words and $\Delta := lr - ks \in \{-1, 1\}$ is called the signature of w. Every triple (s, r, Δ) with s, r coprime positive integers and $\Delta \in \{-1, 1\}$ determines a unique infinite Hedlund word which we denote by $\mathrm{PER}(s, r, \Delta)$. We have seen that the palindrome P from the proof of Theorem 2.6 can be seen as the maximal overlap of the periodic parts of w. We call it the finite Hedlund word induced by w and denote it by $\mathrm{per}(s, r, \Delta)$. We denote the class of finite Hedlund words by $\mathcal{H}$; note that $\emptyset \in \mathcal{H}$. In terms of the parameters k, l, r, s one has $|P| = r+s-2, c(P) = k+l-1$ as shown in the proof.

- Given $P \in \mathcal{H}$ and $\Delta \in \{\pm 1\}$, there exists exactly one infinite Hedlund word of signature Δ inducing P.

Proof. Suppose first that two Hedlund words with period pairs $(s, r), (s', r')$ and the same Δ induce P. Writing $|P| + 2 = \alpha, c(P) + 1 = \beta$ we have $r + s = r' + s' = \alpha, k + l = k' + l' = \beta, lr - ks = l'r' - k's'$.
Then $l\alpha - s\beta = l(r + s) - s(k + l) = lr - ks = l'r' - k's' = l'\alpha - s'\beta \in \{\pm 1\}$. Hence α, β are coprime and $(l - l')\alpha = (s - s')\beta$. Since $s, s' < \alpha$ we have $s = s', r = r', l = l', k = k'$ and the infinite Hedlund words are indeed equal. Suppose now that $P \in \mathcal{H}$, then some $\mathrm{PER}(s, r, \Delta)$ induces P by definition. The word $\mathrm{PER}(r, s, -\Delta)$ is obtained from the previous one by mirroring and since P is a palindrome it follows that $\mathrm{PER}(r, s, -\Delta)$ also induces P. □

- The condition in Lemma 2.6.1 provides a second characterization of balanced words. A similar characterization can be given for stiff words.

Proposition 2.7. *A word w is stiff if and only if there exists no word x with $axa, axb, bxa, bxb \subset w$.*

Proof. If w is not stiff, then there exists an integer $k \geq 0$ and two different factors $A, B \subset w$ of length k such that $Aa, Ab, Ba, Bb \subset w$. Denoting the greatest common suffix of A and B by x we find that $axa, axb, bxa, bxb \subset w$, as required. Now suppose that w is stiff and that $axa, axb, bxa, bxb \subset w$. Applying Lemma 2.4 if necessary we find $w \subset \tilde{w}$ where $\tilde{w}$ is a stiff $\mathbf{Z}$-word. Then also $axa, axb, bxa, bxb \subset \tilde{w}$ and with $|x| + 1 =: n$ we find $P(\tilde{w}, n + 1) > P(\tilde{w}, n) + 1$, hence $P(\tilde{w}, n + 1) > n + 2$. This is a contradiction since $\tilde{w}$ is stiff. □

2.4. Counting theorems.

In this section we first show, as promised, that the three classes of balanced $\mathbf{Z}$-words have the same collection of finite factors. Combining this with the

$k = 1$ case of Theorem 2.3 we find that for finite words x we have that x is balanced iff it appears as factor of some sturmian word. This explains why numerous authors use the word "sturmian" instead of "balanced" when referring to finite words. We will not do so. We use this fact to prove a formula for $\mathrm{bal}(n)$, the number of balanced words of length n. In [D/GB, Conj. 6.4] it was conjectured that $\mathrm{bal}(n) = 1 + \sum_{i=1}^{n} (n+1-i)\phi(i)$. This formula was afterwards proved in a number of ways, see [Be/Po], [dL/Mi, Th. 7], [Mi]. As far as we know the proof we give is new. In Theorem 2.10 we prove a similar formula for $\mathrm{st}(n)$, the number of stiff words of length n. If $\mathcal{S}$ is a collection of words, we write $F(\mathcal{S})$ for its factor set. This is the collection of all finite factors of elements of $\mathcal{S}$, hence $F(\mathcal{S}) = \cup_{n\in\mathbf{N}}\mathcal{B}(\mathcal{S}, n)$.

Theorem 2.8. *Let* $\mathbf{a}, \mathbf{b}, \mathbf{c}$ *be the classes of* $\mathbf{Z}$*-words from Theorem 2.5. Then* $F(\mathbf{a}) = F(\mathbf{b}) = F(\mathbf{c})$.

Proof. First note that $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are closed under inversion, hence the same holds for their factor sets. It is trivial that $F(\mathbf{a}) \subset F(\mathbf{b})$, we now show the reverse inclusion. Finite factors of $b^\infty ab^\infty$ are easily seen to be contained in $F(\mathbf{a})$, hence the same holds for finite factors of $a^\infty ba^\infty$. By inversion, we need only consider factors of the word given by $W = \{\lfloor\frac{in-1}{k}\rfloor\}_{-\infty}^{k-1} \cup \{\lfloor\frac{in}{k}\rfloor\}_1^\infty$ where $k, n \in \mathbf{N}^+$ are coprime and $\frac{k}{n} \in (0,1)$. Pick $M \in \mathbf{N}^+$ and take $\epsilon := \frac{1}{k(M+1)}$. We have

$$\lfloor i(\frac{n}{k} + \epsilon)\rfloor = \begin{cases} [\frac{in}{k}] \text{ if } 0 \le i \le M \\ [\frac{in-1}{k}] \text{ if } -M \le i < 0 \end{cases}$$

Hence every finite factor of w is contained in $F(\mathbf{a})$ and $F(\mathbf{b}) = F(\mathbf{a})$. Had we taken ϵ irrational with $0 < \epsilon < \frac{1}{k(M+1)}$, we would have found the same conclusion. It follows that every factor of W is contained in $F(\mathbf{c})$. Since the same holds for factors of $a^\infty ba^\infty, b^\infty ab^\infty$, we find that $F(\mathbf{b}) \subset F(\mathbf{c})$. In the next lemma we show that the factor set of a sturmian word depends only on its density.

Lemma 2.8.1. *Let* w, w' *be two sturmian words of the same irrational density* α*. Then* $F(w) = F(w')$.

Proof. Let $\zeta := \frac{1}{\alpha}$. First we assume that $W = \{\lfloor\zeta i + \phi\rfloor\}, W' = \{\lfloor\zeta i + \psi\rfloor\}$ where $0 \le \phi, \psi < 1$. Let $M \in \mathbf{N}^+$ and $\epsilon := \max_{[-M,M]} \mathrm{fr}(\zeta i + \psi)$. Because ζ is irrational we can choose $\lambda, \mu \in \mathbf{Z}$ such that $\psi \le \zeta\lambda + \phi + \mu < \psi + 1 - \epsilon$.

For $|i| \leq M$ we then have $\lfloor \zeta i+\psi \rfloor \leq \zeta i+\psi \leq \zeta(i+\lambda)+\mu+\phi < \zeta i+\psi+1-\epsilon \leq \lceil \zeta i + \psi \rceil$. Hence $\lfloor \zeta i + \psi \rfloor = \lfloor \zeta(i+\lambda) + \phi \rfloor + \mu$ for these i. It follows that $F(w') \subset F(w)$ and by symmetry we have $F(w) = F(w')$. If we apply the same reasoning for $\phi = \psi$, we find that there exist $\lambda, \mu \in \mathbf{Z}$ with $\lambda, \mu \neq 0$ such that $\lfloor \zeta i + \phi \rfloor = \lfloor \zeta(i+\lambda) + \phi \rfloor + \mu$ for $|i| \leq M$. This proves that w is recurrent, i.e. that every factor of w appears in at least two different locations in w. We have already mentioned this fact after Theorem 2.5. It easily follows from this that $W = \{\lfloor \zeta i + \phi \rfloor\}, W' = \{\lceil \zeta i + \phi \rceil\}$ have the same factor set. This proves our lemma. □

By the previous lemma, to show that $F(\mathbf{c}) \subset F(\mathbf{a})$ it suffices to consider sturmian $\mathbf{Z}$-words of the form $W = \{\lfloor \zeta i \rfloor\}$. Pick $M \in \mathbf{N}^+$ and choose an irreducible fraction $\frac{n}{k} > \zeta$ with $k > M$ such that $|\zeta - \frac{n}{k}| < \frac{1}{k^2}$. Then $\lfloor \zeta i \rfloor = \lfloor \frac{in}{k} \rfloor$ for $|i| \leq M$. Hence $F(\mathbf{c}) \subset F(\mathbf{a})$ and the proof is complete. □

Let w be the standard sturmian word corresponding to $W = \{\lfloor \zeta i \rfloor\}_{i \in \mathbf{Z}}$ and w', w'' the restrictions of w to the domains $[1, \infty)$ and $(-\infty, -2]$, respectively. Let Bal be the collection of all balanced words, finite and infinite. Then $x \in \text{MLE(Bal)}$ for every prefix x of w' since ax is balanced as it is a factor of w and bx is balanced as it is a factor of the word corresponding to $\{\lceil \zeta i - 1 \rceil\}_{i \in \mathbf{Z}}$. Similarly, $x \in \text{MRE(Bal)}$ for every suffix x of w''. We now show that this yields all x in MLE(Bal) and MRE(Bal), respectively.

Lemma 2.9.1. *Let x be a finite word, $|x| =: n$. If ax, bx are balanced then $ax, bx \subset w$ where w is a standard sturmian word given by $W = \{\lfloor \zeta i \rfloor\}$ where $\zeta > 1$ is irrational. With this notation x can be found at the interval $[1, n]$. If ya, yb are balanced then $ya, yb \subset w$ with w the same as above but the interval is now $[-n-1, -2]$.*

Proof. We write $c(x) = \lambda$. Since ax is balanced, it is a factor of a sturmian $\mathbf{Z}$-word, as can be seen by combining Theorem 2.3,2.5 and 2.7. We write $\{\lfloor \zeta i + \phi \rfloor\}$ for the set associated to this $\mathbf{Z}$-word. Normalising indices, we may assume that ax is given by $\{\lfloor \zeta i + \phi \rfloor\}_0^\lambda$, seen as a subset of $[0, n]$. We will abuse notation a little and write $ax = \{\lfloor \zeta i + \phi \rfloor\}_0^\lambda \subset [0, n]$. Similarly we have

$$bx = \{\lfloor \zeta' i + \psi \rfloor\}_1^\lambda \subset [0, n].$$

We have $\lfloor \psi \rfloor < 0 = \lfloor \phi \rfloor, \lfloor \zeta i + \phi \rfloor = \lfloor \zeta' i + \psi \rfloor$ for $1 \leq i \leq \lambda$ and $\lfloor \zeta \lambda + \phi \rfloor \leq n < \lfloor \zeta(\lambda+1)+\phi \rfloor, \lfloor \zeta'(\lambda+1)+\psi \rfloor$. Let $\zeta'' := \frac{\zeta+\zeta'}{2}$ (not necessarily irrational),

$\eta := \frac{\phi+\psi}{2}$ and $w'' := \{\lfloor \zeta'' i + \eta \rfloor\}$. We have $\lfloor \eta \rfloor < 1, \lfloor \zeta i + \phi \rfloor = \lfloor \zeta' i + \psi \rfloor = \lfloor \zeta'' i + \eta \rfloor$ for $1 \le i \le \lambda$ and $\lfloor \zeta''(\lambda+1) + \eta \rfloor > n$. Hence $w''|_{[1,n]} = x$. In particular $w''|_{[0,n]} = \sigma x$ for some symbol σ. We now choose from ζ, ζ' the one belonging to $\overline{\sigma} x$ and we find a new pair (ζ, ζ'). Note that $|\zeta - \zeta'|, |\phi - \psi|$ have halved. Proceeding this way it is easy to see that $\zeta, \zeta' \to \zeta_0$ and that $\phi, \psi \to \phi_0$ where $\zeta_0 \ge 1$ and $\phi_0 \in \mathbf{R}$. Since $\psi < 0 \le \phi$ for all ϕ, ψ we have $\phi_0 = 0$ and we deduce that

$$x = \{\lfloor \zeta_0 i \rfloor\}_1^\lambda \subset [1, n].$$

If $\zeta_0 \in \mathbf{Q}$, one can make it irrational by enlarging it by a little bit. Now let w be the $\mathbf{Z}$-word $\{\lfloor \zeta_0 i \rfloor\}$. Then $ax, bx \subset w$, since they are contained in a sturmian $\mathbf{Z}$-word of density ζ_0. This proves the lemma when $x \in \mathrm{MLE(Bal)}$. The other case follows by reflection, since Bal is reversal invariant. □

Now let w be an infinite Hedlund word and P its associated finite Hedlund word. Remember that P can be characterised as the smallest word such that $aPa, bPb \subset w$. We have $aPa, bPb \subset w$ and $\sigma P \overline{\sigma} \subset w$ for some symbol σ. All these words are balanced. Indeed, if some $x \in \{aPa, bPb, \sigma P \overline{\sigma}\}$ is not balanced, then Lemma 2.6.1 gives us a word Q with $aQa, bQb \subset x$. Then $|Q| \le |P|$, hence $Q = P$, a contradiction. Therefore the three words are balanced and by reversal we find the same for $\overline{\sigma} P \sigma$. Summarising we have:

if P is a finite Hedlund word, then $aPa, aPb, bPa, bPb \in \mathrm{Bal}$

We say that P is strictly bispecial (SBS) in the language of balanced words. We now show that the finite Hedlund words can be characterised by this property.

Lemma 2.9.2. *Let x be finite with $axa, axb, bxa, bxb \in \mathrm{Bal}$. Then $x \in \mathcal{H}$.*

Proof. Let $|x| =: n, c(x) =: \lambda$. By the previous lemma there exist $\zeta, \zeta' > 1$ irrational such that $xa = \{\lfloor \zeta i \rfloor\}_1^{\lambda+1} \subset [1, n+1]$ and $xb = \{\lfloor \zeta' i \rfloor\}_1^{\lambda} \subset [1, n+1]$. Then $\lfloor \zeta i \rfloor = \lfloor \zeta' i \rfloor$ for $1 \le i \le \lambda$ and $\lfloor \zeta(\lambda+1) \rfloor = n+1 < \lfloor \zeta'(\lambda+1) \rfloor$. In particular $\zeta < \zeta'$. Now let $\zeta'' := \frac{\zeta+\zeta'}{2}$ and $w'' := \{\lfloor \zeta'' i \rfloor\}$ as before. Then $x = w|_{[1,n]}$, hence $w|_{[1,n+1]} = x\sigma$ for some symbol σ. We now choose from ζ, ζ' the one corresponding to $x\overline{\sigma}$. Then we find a new pair ζ, ζ' where $|\zeta - \zeta'|$ has halved and it clear that $\zeta, \zeta' \to \zeta_0 \ge 1$. Since $\zeta(\lambda+1) < n+2 \le \zeta'(\lambda+1)$ for all ζ, ζ' we find $\zeta_0 = \frac{n+2}{\lambda+1}$ and $x = \{\lfloor \zeta_0 i \rfloor\}_1^\lambda \subset [1, n]$. Notice that this means that there exists at most one x when n, λ are fixed. We now construct a finite Hedlund word with the same n, λ as x and then our proof will be finished.

If $(n+2, \lambda+1) =: d > 1$, there would exist an integer $1 \le t \le \lambda$ with $t\zeta_0 \in \mathbf{Z}$. Since our initial ζ was irrational we find $\lfloor \zeta t \rfloor < \zeta_0 t = \lfloor \zeta_0 t \rfloor$, a contradiction. Therefore $d = 1$ and there exist integers p, q with $0 \le p \le \lambda, 0 \le q \le n+1$ and $p(n+2) - q(\lambda+1) = 1$. We set $l := p, s := q$ and $k := \lambda+1-l, r := n+2-s$. Then $0 \le l \le s$ and $lr - ks = l(r+s) - s(k+l) = p(n+2) - q(\lambda+1) = 1$. But then $\mathrm{per}(s, r, 1)$ has length n, content λ and we are done. $\square$

Remark. It follows from the proof that

$$\mathrm{per}(s, r, 1) = \{\lfloor \frac{i(r+s)}{k+l} \rfloor\}_1^{k+l-1} \subset [1, r+s-2].$$

We will use this representation of finite Hedlund words in Section 2.5.3.

The remainder of the proof of the formula for $\mathrm{bal}(n)$ is standard and can be found, for instance, in [dL/Mi]. Since it will take us little extra effort we include it here. In the theorem ϕ denotes the Euler phi function.

Theorem 2.9. $\mathrm{bal}(n) = 1 + \sum_1^n (n+1-i)\phi(i)$.

Proof. We write $\mathrm{Sp}(n) := \mathrm{MRE}_n(\mathrm{Bal})$ and $\mathrm{sp}(n) := |\mathrm{Sp}(n)|$. Here Sp stands for "special". Since every $x \in \mathrm{Bal}(n)$ has a right-extension in $\mathrm{Bal}(n+1)$, it follows that $\mathrm{bal}(n+1) - \mathrm{bal}(n) = \mathrm{sp}(n)$ for all n. For fixed n we can define the mapping $f : \mathrm{Sp}(n+1) \to \mathrm{Sp}(n)$, given by $f(x) = [x]^n$. It follows from Lemma 2.9.1 that this mapping is surjective and the set of $y \in \mathrm{Sp}(n)$ with more than one original equals $\mathrm{SBS}_n(\mathrm{Bal})$. By Lemma 2.9.2 (and its converse) this set equals $\mathcal{B}(\mathcal{H}, n)$. We will now count the number of finite Hedlund words of length n.
By the second remark after Theorem 2.6 every finite Hedlund word P of length n can be uniquely written as $\mathrm{per}(s, r, 1)$ where $s, r \ge 1$ are coprime with $r+s = n+2$. The number of pairs (s, r) satisfying this equals $\phi(n+2)$.

Hence $\mathrm{sp}(n+1) - \mathrm{sp}(n) = \phi(n+2)$ for all n and $\mathrm{bal}(n) = 1 + \sum_{k=0}^{n-1} \mathrm{sp}(k) = 1 + \sum_{k=0}^{n-1} \sum_{i=1}^{k+1} \phi(i) = 1 + \sum_{i=1}^n (n+1-i)\phi(i)$. $\square$

Corollary 2.9.1. $\mathrm{bal}(n) = \frac{n^3}{\pi^2} + O(n^2 \ln n)$.

Proof. To prove these asymptotics it is easier to write $\mathrm{bal}(n) = 1 + \sum_1^n \Phi(k)$ where $\Phi(k) := \sum_{i=1}^k \phi(i)$. It is proved in [HW, Theorem 330] that $\Phi(k) = \frac{3k^2}{\pi^2} + O(k \ln k)$. Summing yields $\sum_1^n \Phi(k) = \frac{n^3}{\pi^2} + O(n^2 \ln n)$, which is the claimed result. □

We now turn to the number of stiff words of length n. We need only consider those which are not balanced and all of them are contained in an infinite Hedlund word. Let w be an infinite Hedlund word with corresponding finite Hedlund word P. We have seen that P could be characterised as the smallest solution x of $axa, bxb \subset w$. We will show that it is in fact the only solution. If x is a solution, then $x \in \mathrm{MRE}(w)$, hence x is a suffix of w_l, the maximal periodic left-infinite subword of w. In particular $x \subset w_l$ and, similarly, $x \subset w_r$. We have seen in the proof of Theorem 2.6, however, that $\mathcal{B}(w_l, i) \cap \mathcal{B}(w_r, i) = \emptyset$ if $i > |P|$. Hence $|x| \leq |P|$ and this implies $x = P$, as stated. Combining this with Lemma 2.6.1 we find the following lemma.

Lemma 2.10.1. *Let w be an infinite Hedlund word and P its corresponding finite Hedlund word. For a factor $x \subset w$ we have that x is not balanced iff $aPa, bPb \subset x$.*

Lemma 2.10.2. *Let $w = \mathrm{PER}(s, r, \Delta)$ be normalised such that the maximal overlap P of its periodic parts is situated at $[1, n]$. Let $S = [-s, n + r + 1]$. Then a finite subword $x \subset w$ is not balanced iff it has a domain containing S. Also, every non-balanced subword determines w completely.*

Proof of Lemma 2.10.2. We know that $w_{[0,n+1]} = \sigma P \overline{\sigma}$ for some symbol σ. Since $|\sigma P| > |P|$, it follows from what we have just seen that $\sigma P \not\subset w_r$. Therefore $[0, n]$ is the right-most domain inducing σP and the other domains are $[0, n] - s\mathbf{N}$, because w_l is periodic with minimal period $s \leq |\sigma P|$. Hence the right-most domain inducing $\sigma P \sigma$ equals $[-s, n + 1 - s]$. Similarly, the left-most domain inducing $\overline{\sigma} P \overline{\sigma}$ equals $[r, n + 1 + r]$. Now let $x \subset w$ be finite and I a domain for x. Then x not balanced $\iff$ $(aPa, bPb \subset x)$ $\iff$ $[-s, n + 1 + r] \subset I$ and this is the first part of the lemma. If $x \subset w$ is not balanced, then P is the unique solution of $aPa, bPb \subset w$. Also $\sigma P \overline{\sigma} \subset x$ for a unique symbol σ and an easy calculation shows that for infinite Hedlund words w with corresponding P we have $aPb \subset w \iff \Delta = 1$. Hence P, Δ are determined by x and the second remark after Theorem 2.6 shows that the same holds for w. □

Theorem 2.10. $\mathrm{st}(n) - \mathrm{bal}(n) = 2\sum_{2\le i\le n/2}(n+1-2i)\phi(i)$.

Proof. Note that in the previous lemma $|S| = 2(r+s)$. We want to count the stiff words x of length n, which are not balanced. To construct such an x, one chooses coprime positive integers s, r with $2(r+s) \le n$, one chooses $\Delta \in \{\pm 1\}$ and then x can be any subword of $w := \mathrm{per}(s, r, \Delta)$ containing S. It follows from Lemma 2.10.2 that no different choices for (s, r, Δ) yield the same x and writing $i = r + s$ we get our formula. □

Corollary 2.10.1. $\mathrm{st}(n) = \frac{3n^3}{2\pi^2} + O(n^2 \ln n)$.

Proof. We will write $m := \lfloor \frac{n}{2} \rfloor$ and $\delta \in \{0, 1\}$ with parity opposite to n. Then partial sommation can be used to show that $\mathrm{st}(n) - \mathrm{bal}(n) = 4\sum_1^m \Phi(i) - 2(n-1) - 2\delta\Phi_m$. The estimates from Corollary 2.8.1 then imply that $\mathrm{st}(n) - \mathrm{bal}(n) = \frac{4m^3}{\pi^2} + O(m^2 \ln m)$. In particular $\mathrm{st}(n) - \mathrm{bal}(n) = \frac{n^3}{2\pi^2} + O(n^2 \ln n)$. Now combine with Corollary 2.8.1. □

2.5. Other classical facts.

First we give some alternative descriptions of Beatty words. The first one uses cutting sequences, the second one is based on the dynamical system (X, T), where $X = \mathbf{R}/\mathbf{Z}$ is the unit circle and T a rotation. The third one connects Beatty words with Christoffel words.

2.5.1. Cutting sequences. Let $\lambda \in \mathbf{R}^+, \mu \in \mathbf{R}$ and consider the line l with equation $y = \lambda x + \mu$. Let $A = \mathbf{Z} \times \mathbf{R}, B = \mathbf{R} \times \mathbf{Z}$ and $C = A \cup B$, the integer grid. The intersection $l \cap C$ is a discrete subset of l and we write $l \cap C = \{(x_i, y_i)\}_{i\in\mathbf{Z}}$, where x_i increases strictly with i. We define $\chi : l \cap C \to \{a, b\}^*$ by

$$\chi(P) = \begin{cases} a, & \text{if } P \in A \setminus B \\ b, & \text{if } P \in B \setminus A \\ ba, & \text{if } P \in A \cap B \end{cases}$$

Hence $\chi(P) = a$ if l cuts the grid vertically, b if l cuts the grid horizontally and ba if P is an integer lattice point. We put $w_{\lambda,\mu} = \prod_{P \in l\cap C} \chi(P)$, i.e. the concatenation of all the $\chi(P)$ in ascending order.

Proposition 2.11. *$w_{\lambda,\mu}$ is the $\mathbf{Z}$-word corresponding to the subset* $\{\lfloor(\lambda +$

$1)i+\mu\rfloor\}_{i\in\mathbf{Z}}$ *of* $\mathbf{Z}$*. In particular it is a Beatty word.*

Proof. Two consecutive a's in $w_{\lambda,\mu}$ correspond to some points $(i, \lambda i+\mu)$ and $(i+1, \lambda(i+1)+\mu)$. Keeping in mind the convention for lattice points, we find that the number of b's in between equals $|\{\lfloor\lambda i+\mu\rfloor+1, \cdots, \lfloor\lambda(i+1)+\mu\rfloor\}| = \lfloor\lambda(i+1)+\mu\rfloor - \lfloor\lambda i+\mu\rfloor =: \psi_i$. Hence we have $w_{\lambda,\mu} = \prod_{i\in\mathbf{Z}}(ab^{\psi_i})$. Now consider $\tilde{w} = \{\lfloor(\lambda+1)i+\mu\rfloor\}_{i\in\mathbf{Z}}$. For all $i\in\mathbf{Z}$ we have $\lfloor(\lambda+1)(i+1)+\mu\rfloor - \lfloor(\lambda+1)i+\mu\rfloor = \psi_i+1$ hence also $\tilde{w} = \prod_{i\in\mathbf{Z}}(ab^{\psi_i})$.□

We might have taken $\chi(P) = ab$ on $A\cap B$. In that case the floor symbols should be replaced by ceiling symbols. We have considered only positive λ because (λ,μ) and $(-\lambda,-\mu)$ give rise to the same word $w_{\lambda,\mu}$ as is easily verified.

Theorem 2.11 explains the name cutting sequence for a Beatty sequence. If one considers only the halfline $l\cap\{x\geq 0\}$, the previous theorem gives a nice geometric interpretation. Consider a billiard table $[0,1]\times[0,1]$ and shoot a ball from the initial position $(0, \mathrm{fr}(\mu))$ with slope λ. Then $w_{\lambda,\mu}$ encodes the collisions of the ball with the sides of the table when opposite sides of the table are identified. For this reason Beatty sequences are also called billiard sequences and usually one restricts oneself to irrational λ. For generalisations to more dimensions see [AMST].

2.5.2. Rotations. Let $\alpha\in(0,1]$ and $\beta\in\mathbf{R}$. We define the $\mathbf{Z}$-word $w_{\alpha,\beta}$ by $w_i = a \iff \beta+i\alpha\in(0,\alpha] \bmod 1$. Then we have $w_i = a \iff \exists k\in\mathbf{Z}: k<\beta+i\alpha\leq k+\alpha \iff \exists k: i-1\leq\frac{k-\beta}{\alpha}<i \iff i\in\{\lfloor\frac{k+\alpha-\beta}{\alpha}\rfloor\}_{k\in\mathbf{Z}}$. This shows that $w_{\alpha,\beta}$ is a Beatty word of the form $\{\lfloor\zeta i+\phi\rfloor\}$. It is easy to see that every such Beatty word is a $w_{\alpha,\beta}$: just take $\alpha := \frac{1}{\zeta}, \beta := \frac{-\phi}{\zeta}$. Similarly we define $\tilde{w}_{\alpha,\beta}$, but the interval is now $[0,\alpha)$. This yields precisely the Beatty words of ceiling type.

Let α be irrational. The above reasoning shows that $w = w_{\alpha,\beta}$ is sturmian and apparently $P(w,n) = n+1$ for all $n\in\mathbf{N}$. It is instructive to see how this follows from the description above. Let $n\in\mathbf{N}$. We define $V_a = (0,\alpha], V_b = (\alpha,1]$ as subsets of the unit circle $\mathbf{T} = \mathbf{R}/\mathbf{Z}$ and for $x = x_0\cdots x_{n-1}\in\{a,b\}^n$ we define $V_x = \cap_{t=0}^{n-1}(V_{x_t}-t\alpha)$. Here $V_{x_t}-t\alpha$ is the image of V_{x_t} under a translation over $-t\alpha$. We have

$$w_i\cdots w_{i+n-1} = x \iff \beta+(i+t)\alpha\in V_{x_t}\forall(0\leq t<n) \iff \beta+i\alpha\in V_x.$$

Note that $V_x\neq\emptyset \iff x\in\mathcal{B}(w,n)$ and that $V_x\cap V_y = \emptyset$ when $x\neq y$. If $V_x\neq\emptyset$, then V_x is a finite union of intervals, all of whose endpoints are contained in $\cup_{t=-1}^{n-1}(-t\alpha)$. These points divide the unit circle into $n+1$ parts and since every non-empty V_x contains at least one of them, we find that $P(w,n) \leq n+1$. Now let $x \in \{a,b\}$ be such that $\beta \in V_x$. If w were periodic with period p, we would have $\beta + p\alpha\mathbf{Z} \subset V_x$. This is impossible because the first set lies dense in $\mathbf{T}$. Hence w is not periodic and we conclude that $P(w,n) = n+1$ for all n.

The replacement of $(0,\alpha]$ by other sets yields an easy way to construct $\mathbf{Z}$-words of low complexity. We will not go into this now.

2.5.3. Christoffel words. Let $\alpha \in \mathbf{R}^+$ and let l be the line with equation $y = \alpha x$. We define a path $\gamma = (P_n)_0^\infty$ in $\mathbf{Z}^2$ by $P_0 = (0,0)$

$$P_{n+1} = \begin{cases} P_n + (0,1) \text{ if } P_n + (0,1) \text{ does not lie above } l \\ P_n + (1,0) \text{ otherwise.} \end{cases}$$

Hence γ remains as close as possible under or on the line l. The path γ can be used to define a word z with domain $\mathbf{N}$, namely by setting $z_i = a \iff P_{i+1} = P_i + (1,0)$. Hence horizontal steps in γ are encoded by a and vertical ones by b. The word z will be called the infinite Christoffel word of slope α and we write $z = \mathrm{CH}(\alpha)$.

We now consider the special case that α is rational. Hence let us suppose that $\alpha = \frac{p}{q}$ where p,q are coprime positive integers. The first point after P_0 where γ intersects the line l will be $P_{p+q} = (q,p)$. The path from P_0 to P_{p+q} induces a prefix of z of length $p+q$. We will call this word z' the primitive Christoffel word of slope $\frac{p}{q}$. Note that it is indeed primitive since $c(z') = q, |z'| = p+q$ and $(q, p+q) = 1$. We write $\mathrm{CP}(\frac{p}{q})$ for this primitive Christoffel word. It is customary to define $\mathrm{CP}(0) = a$ and $\mathrm{CP}(\infty) = b$. We will also write CP for the class of primitive Christoffel words.

Proposition 2.12. a) *If* $\alpha \in \mathbf{R}^+$, *then* $\mathrm{CH}(\alpha) = \{\lfloor i(\alpha+1)\rfloor\}_0^\infty \subset \mathbf{N}$.
b) *Let* s,r *be coprime positive integers and let* $\frac{l}{s}, \frac{k}{r}$ *be the left and right density of* $\mathrm{PER}(s,r,1)$ *respectively. Then* $a\ \mathrm{per}(s,r,1)\ b = \mathrm{CP}(\frac{r+s}{k+l} - 1)$.
c) $\mathrm{CP} = \{a,b\} \cup a\mathcal{H}b$.

Proof. *a)* Let $z = \mathrm{CH}(\alpha)$. Then $z_n = a$ if and only if $P_n = (i, \lfloor i\alpha \rfloor)$ for some $i \in \mathbf{N}$. The next a then corresponds to $(i{+}1, \lfloor (i{+}1)\alpha \rfloor)$ and the number of b's in between equals $\psi_i := \lfloor (i+1)\alpha \rfloor - \lfloor i\alpha \rfloor$. Therefore we have $z = \Pi_0^\infty(ab^{\psi_i})$. The claimed equality now follows from $\lfloor (i+1)(\alpha+1)\rfloor - \lfloor i(\alpha+1)\rfloor = \psi_i + 1$ for all $i \geq 0$.

b) Let $p := r+s-(k+l), q := k+l$. Then $p,q \in \mathbf{N}^+$ and $lp-(s-l)q = 1$, hence p,q are coprime. Part a) then shows that $\mathrm{CH}(\frac{p}{q}) = \{\lfloor i(\frac{r+s}{k+l})\rfloor\}_0^\infty \subset \mathbf{N}$. Restricting domains yields $\mathrm{CP}(\frac{p}{q}) = \{\lfloor i(\frac{r+s}{k+l})\rfloor\}_0^{k+l-1} \subset [0, r+s-1]$. Since $\lfloor (k+l-1)(\frac{r+s}{k+l})\rfloor < r+s-1$, it follows that $\mathrm{CP}(\frac{p}{q}) = axb$ where $x = \{\lfloor i(\frac{r+s}{k+l})\rfloor\}_1^{k+l-1} \subset [1, r+s-2]$. By the remark after Lemma 2.8.2, finally, we have $x = \mathrm{per}(s,r,1)$.

c) Let p,q be coprime positive integers and let (λ,μ) be the unique pair of integers satisfying $\lambda(p+q) - \mu q = 1, 0 < \mu < p+q$. Define $l := \lambda, k := q-\lambda, s := \mu, r := p+q-s$. Direct verification shows $0 \le k \le r, 0 \le l \le s, lr - ks = 1$ and $\frac{r+s}{k+l} - 1 = \frac{p}{q}$. This shows that every primitive Christoffel word not equal to a,b can be obtained as in b).□

Part b) already appears in [Be/dL] as Proposition 4.4, but the proof there is only valid if $\frac{p}{q} \le 1$. Part c) also appears in [Be/dL], as Theorem 4.1. For more information on Christoffel words, see for instance [Be/dL],[Bo/L]. In [Be] algorithms are given for the construction of Christoffel words.

2.5.4. Sturmian substitutions. A substitution is a mapping $T : \{a,b\}^* \to \{a,b\}^*$ which satisfies $T(xy) = T(x)T(y)$ for all $x,y \in \{a,b\}^*$. Of course T is uniquely determined by Ta, Tb and if $Ta = X, Tb = Y$, then we will write $T = (X,Y)$. The mapping T has a natural extension to the collection of all words, finite and infinite, and we will identify T with this extension. We now consider the substitutions $L = (ba,b), R = (ab,b), C = (b,a)$. For notational convenience later on we write $R' := (a,ba) = CRC$.

Lemma 2.13.1. *Let w be a $\mathbf{Z}$-word. Then Lw is balanced (stiff) if and only if w is balanced (stiff). The same holds if L is replaced by C or R.*

Proof. By Lemma 2.6.1 we know that a $\mathbf{Z}$-word x is balanced if and only if there exists no word y with $aya, byb \subset w$. By Proposition 2.7 we also know that a word w is stiff if and only if it has no strictly bispecial factor y, i.e. a word y such that $aya, ayb, bya, byb \subset w$. We will now use these characterisations.

Suppose w is not balanced. Let y be such that $aya, byb \subset w$. Then $baL(y)ba, bL(y)bb \subset Lw$ and Lw is not balanced. Now assume that Lw is not balanced, hence $aya, byb \subset Lw$ for some y. We have $baya = L(a\xi a)$ and substitution of $y = L(\xi)b$ gives $aL(\xi)ba, bL(\xi)b^2 \subset Lw$. This implies $a\xi a, b\xi b \subset w$, whence w is not balanced.

Now assume that w is not stiff, hence $aya, ayb, bya, byb \subset w$ for some y. Then $baL(y)ba, baL(y)bb, bL(y)ba, bL(y)bb \subset Lw$. This implies that $L(y)b$ is strictly bispecial in Lw and Lw is not stiff. Now assume that Lw is not stiff, then $aya, ayb, bya, byb \subset Lw$. We have $baya = L(a\xi a)$ and this implies $aL(\xi)ba, aL(\xi)b^2, bL(\xi)ba, bL(\xi)b^2 \subset Lw$. Then ξ is SBS in w and w is not stiff.

This proves the lemma for L. Since $Lw = Rw$ for any $\mathbf{Z}$-word w the lemma also holds for R. It is trivial for C. □

Proposition 2.13. *Let w be a $\mathbf{Z}$-word. If w is sturmian (infinite Hedlund), then Lw, Rw, Cw is sturmian (infinite Hedlund).*

Proof. First assume that w is infinite Hedlund. Then Lw, Rw, Cw are stiff and not balanced by Lemma 2.13.1. Hence they are infinite Hedlund. Now assume that w is sturmian. Then Lw, Rw, Cw are balanced, recurrent and have irrational density as a small calculation shows. Hence they are sturmian. □

Remark. If $w = \{\lfloor \zeta i + \phi \rfloor\}$ then $Lw = Rw = \{\lfloor (\zeta + 1)i + \phi \rfloor\}$ and $Cw = \{\lceil \frac{\zeta i - \phi}{\zeta - 1} \rceil\}$. Similar formulas hold for $w = \{\lceil \zeta i + \phi \rceil\}$.

Lemma 2.14.1. *Let w be a balanced $\mathbf{Z}$-word. Then $w = Lx$ for some balanced $\mathbf{Z}$-word x or $w = R'x$ for some balanced $\mathbf{Z}$-word x. If $w = Lx = R'y$ for $\mathbf{Z}$-words x, y, then $w = (ab)^\infty$.*

Proof. We have $aa \not\subset w \vee bb \not\subset w$ because w is balanced. In the first case we have $w = Lx$ for some $\mathbf{Z}$-word x and x is balanced by Lemma 2.13.1. The second case is similar. If $w = Lx = R'y$ for $\mathbf{Z}$-words x, y we have $aa, bb \not\subset w$. Hence $w = (ab)^\infty$. □

The collection of all substitutions form a monoid (halfgroup) and we let $\mathcal{M}$ be the submonoid generated by $\{L, R, C\}$. Note that $\{L, R', C\}$ is another generating set. Then every $T \in \mathcal{M}$ maps sturmian $\mathbf{Z}$-words into sturmian $\mathbf{Z}$-words and the converse is also true. This is shown in [Mi/S] but it also follows from Theorem 3.1 in this thesis. Elements from $\mathcal{M}$ are called sturmian substitutions, transformations or morphisms. Some other basic properties of $\mathcal{M}$ are mentioned in the next theorem. We recall that a sequence $x_1, x_2, \cdots$ is said to converge to an infinite word x if there exists a sequence $m_n \to \infty$ with $|x_n| \geq m_n$ and x_n, x agreeing on the first m_n

symbols for all n.

Theorem 2.14. **a)** *If w is a balanced $\mathbf{Z}$-word of rational density, then $w = Tx$ where $T \in \mathcal{M}, x \in \{a^\infty, a^\infty b a^\infty\}$.*
b) *If w is an infinite Hedlund word we have $w = T(a^\infty b^\infty)$ for some $T \in \mathcal{M}$.*
c) *Let w be an infinite Hedlund word with left period cycle π_1 and right period cycle π_2. Let T be substitution. If $T\pi_i$ is primitive for $i = 1, 2$ and Tw is also an infinite Hedlund word, then $T \in \mathcal{M}$.*
d) *(Adaptation of Lemma 2.13.1.) Let w be an $\mathbf{N}$-word with $w_0 = a$ and $T \in \{L, R, C\}$. Then w is balanced if and only if Tw is balanced.*
e) *If w is a balanced word with domain $\mathbf{N}$, there exists a sequence $T_n \in \mathcal{M}$ with $T_n(a) \to w$.*
f) *Let $\mathcal{C}$ be the collection of finite primitive words π such that π^∞ is balanced. Then $\mathcal{C} = \cup_{T \in \mathcal{M}} \{T(a)\}$.*

Proof. **a)** Write $\alpha(w) = \frac{k}{n}$ where $k \geq 0, n \geq 1$ are coprime. If $\alpha \in \{0, 1\}$ the statement is obvious, hence we assume $\alpha \in (0, 1)$. By the previous lemma we have $w = Lx$ or $w = R'x$ for some balanced $\mathbf{Z}$-word x. In the first case we have $\alpha(w) \leq \frac{1}{2}, \alpha(x) = \frac{k}{n-k}$ as an elementary calculation shows and in the second case we have $\alpha(w) \geq \frac{1}{2}, \alpha(x) = \frac{2k-n}{k}$. We can now apply the same reasoning to x as to w. If we write $\alpha(x) = \frac{p}{q}$ where $p \geq 0, q \geq 1$ are coprime then $p + q < k + n$ in both cases. It follows that this process ends in some $\mathbf{Z}$-word y and then $\alpha(y) \in \{0, 1\}$. If $\alpha(y) = 0$ we are done and if $\alpha(y) = 1$ we write $y = Cz$ where $\alpha(z) = 0$.
b) First let a quadruple (k, l, r, s) of integers be given with $0 \leq k \leq r, 0 \leq l \leq s$ and $lr - ks \in \{-1, 1\}$. We will show by induction on $\max(s, r)$ that there exists a sturmian substitution $T = (X, Y)$ with $(c(X), c(Y), |X|, |Y|) = (l, k, s, r)$. The induction basis is given by $r = s = 1$, in which case $\{k, l\} = \{0, 1\}$ and then $T = (a, b)$ or $T = C$ will do. In all other cases $s \neq r$. If $s < r$ then $(l, k - l, s, r - s)$ is another such quadruple and the induction hypothesis shows that there exists a sturmian $T' = (X', Y')$ with $(c(X'), c(Y'), |X'|, |Y'|) = (l, k - l, s, r - s)$. If $(X, Y) = T := T' \circ (ab, b)$ we will have $(c(X), c(Y), |X|, |Y|) = (l, k, s, r)$ as required. The case $r < s$ is similar.

Now suppose that $w = \mathrm{PER}(s, r, \Delta)$ is infinite Hedlund. By the previous paragraph there exists a sturmian $T = (X, Y)$ with $(c(X), c(Y), |X|, |Y|) = (l, k, s, r)$. But then $T(a^\infty b^\infty)$ is infinite Hedlund by Proposition 2.13 and has left and right density $\frac{l}{s}, \frac{k}{r}$, respectively. In particular it has the same

parameters (s, r, Δ) as w and therefore it equals w.
c) We will use induction on $|Ta|+|Tb|$. If $|Ta| = |Tb| = 1$ we have $T = (a, b)$ or $T = C$ and then clearly $T \in \mathcal{M}$. Now suppose that $|Ta|+|Tb| > 2$. Write $T = (X, Y)$ and assume for the moment that between X and Y there exist no prefix or suffix relations. If A, B are the greatest common prefix and suffix of X, Y, respectively, we have $|A|, |B| < \min(|X|, |Y|)$.

Since w, Tw are both infinite Hedlund we can write $w = \mathrm{PER}(s, r, \Delta)$ and $Tw = \mathrm{PER}(s', r', \Delta')$ and we define k, l, k', l' as usual. Considering the maximal overlap of the periodic parts in Tw we find $\mathrm{per}(s', r', \Delta') = BT(\mathrm{per}(s, r, \Delta))A$, hence $r' + s' - 2 = |ABX^{k+l-1}Y^{r+s-1-k-l}|$. Now $T\pi_1$ is a minimal left period cycle since it is primitive by assumption. Hence $s' = |T\pi| = |X^lY^{s-l}|$. Similarly $r' = |X^kY^{r-k}|$ and substitution yields $|AB| = |XY| - 2$. This implies $|A| = |X| - 1, |B| = |Y| - 1$, hence $X = \sigma X' = X''\tau$ and $Y = \overline{\sigma}X' = X''\overline{\tau}$ for symbols σ, τ and words X', X''. Then $X' = X'' = \emptyset$ and $|Ta| = |Tb| = 1$, contradicting our hypothesis.

Therefore there exists some prefix or suffix relation between X and Y. Suppose first that $X = YZ$ for some Z. Then $T = (Z, Y) \circ (ba, b)$. By assumption we have $T\pi_i = (Z, Y)((ba, b)\pi_i) \in \mathrm{PRIM}$, which implies $(ba, b)\pi_i \in \mathrm{PRIM}$. Hence $\tilde{\pi}_i := (ba, b)\pi_i$ is a left and right minimal period, respectively, for the infinite Hedlund word $(ba, b)w$. For the pair $w' := (ba, b)w, T' := (Z, Y)$ we can now apply the induction hypothesis since $|ZY| < |XY|$ and we conclude that $T' \in \mathcal{M}$. Then also $T \in \mathcal{M}$. The other cases can be dealt with similarly.
d) Let w be an $\mathbf{N}$-word and $T \in \{L, R, C\}$. If w is balanced, then $w \subset w^*$ where w^* is a balanced $\mathbf{Z}$-word as can be seen by applying theorems 2.3 and 2.4. Applying Lemma 2.13.1 we find that Tw^* is balanced and then the same is true for Tw. Now suppose that it is given that Tw is balanced. We can find a left-infinite word $\tilde{w}$ such that $\tilde{w}Tw$ is balanced. We claim that $\tilde{w} = T(w^*)$ for some left-infinite word w^*. For $T = C$ this is immediate. If $T = L$ and $aa \not\subset \tilde{w}$ this is also clear and if $T = L, aa \subset \tilde{w}$ we have $bb, aa \not\subset Tw$, whence $w = a^\infty$. In this case it is already clear that w is balanced. If $T = R$ one uses $aa \not\subset \tilde{w}$ and the assumption that $w_0 = a$.

All in all we have that $T(w^*w)$ is balanced and applying Lemma 2.13.1 we find that w^*w is balanced and the same is then true for w. The example $w = bba^\infty$ shows that the condition $w_0 = a$ in the R-case cannot be omitted in general.
e) We define $w^{(0)} := w$. If $aa \subset w^{(n)}$ we define $w^{(n+1)} := Cw^{(n)}$. If $aa \not\subset w^{(n)}$ and $w_0^{(n)} = a, b$, then we define $w^{(n+1)}$ by $w^{(n)} = Rw^{(n+1)}, Lw^{(n+1)}$, respec-

tively. In all cases $w^{(n+1)}$ is balanced by $\mathbf{d}$. We define $\Psi_n = C, R, L$ in the respective cases and $\Lambda_n := \Psi_1 \cdots \Psi_n \in \mathcal{M}$. Then $w = \Lambda_n w^{(n)}$ for all $n \in \mathbf{N}$. Note that $|\Lambda_n(a)| \to \infty$. If $w_0^{(n)} = a$ for infinitely many n we can extract a subsequence n_k such that $w_0^{(n_k)} = a$ for all k. Grouping some consecutive Ψ's together we find $w = (\Phi_1 \cdots \Phi_k) w^{(n_k)}$ where $\Phi_i \in \mathcal{M}$ and $\Phi_1 \cdots \Phi_k(a) \to w$. Suppose now that $w_0^{(n)} = b$ for $n \geq N$. If $w^{(N)} \neq b^\infty$, then the number χ of initial b's satisfies $1 \leq \chi < \infty$ and a simple induction shows that $w^{(N+i)}$ has exactly $\chi - i$ initial b's if $0 \leq i < \chi$. In particular $w^{(N+\chi-1)}$ starts with ba and then $w^{(N+\chi)}$ starts with a, a contradiction. Therefore $w^{(N)} = b^\infty$ and writing $\Lambda_N =: (\pi, \rho)$ we have $w = \rho^\infty$. In this case the sequence $T_n := \Lambda_N L^n$ will do.

f) If $T = (X, Y) \in \mathcal{M}$ with $(c(X), c(Y), |X|, |Y|) = (l, k, s, r)$ an easy induction argument shows that $lr - ks \in \{-1, 1\}$. In particular $(k, r) = 1$ whence X is primitive. Also $X^\infty Y^\infty$ is infinite Hedlund by Proposition 2.13 and it follows that $X \in \mathcal{C}$. Now suppose that $X \in \mathcal{C}$ and write $c(X) =: l, |X| := s$. From the periodic part of Theorem 2.5 it follows that $|X|$, the minimal period of X^∞, equals the least possible denominator of $\alpha(X^\infty) = \frac{l}{s}$. Hence $(l, s) = 1$ and we can choose numbers $r \in \mathbf{N}^+, \Delta \in \{-1, 1\}$ such that $w = \mathrm{PER}(s, r, \Delta)$ has left density $\frac{l}{s}$. If the overlap of the periodic parts is situated at $[1, r+s-2]$, then every $0 \leq i \leq r+s-2$ gives rise to a bisection $w_{(-\infty,i]}, w_{[i+1,\infty)}$ of w into two parts with periods s, r, respectively. Because $r+s-1 \geq s$ we can choose such an i with $w_{[i-s+1,i]} = X$ and this means that a primitive word Y of length r exists with $X^\infty Y^\infty = w$. Then $(X, Y) \in \mathcal{M}$ by part **c**. $\square$

2.5.5. Sturmian words and continued fractions. In part **e** of the previous theorem we saw that that the right-infinite closure of the collection $\{T(a)\}_{T \in \mathcal{M}}$ equals the set of right-infinite balanced words. The next theorem shows, basically, which closure one obtains when $\mathcal{M}$ is restricted to $\mathcal{M}' =< L, C >$, i.e. the submonoid generated by C and L. The theorem already appears in [AR, p. 206] but the proof below is based on ideas by Stolarsky. Lemma 2.14.3 has been taken from [S].

Theorem 2.15. *Let $\zeta > 1$ be irrational with continued fraction expansion $\zeta = [1 + d_0, d_1, \cdots]$. Let w be the standard sturmian word $w := \{\lfloor \zeta i \rfloor\}_{i=1}^\infty \subset \mathbf{N}^+$ and let the sequence (s_n) of finite words be defined by $s_{-2} = a, s_{-1} = b$ and $s_{n+2} = s_{n+1}^{d_{n+2}} s_n$ for $n \geq -2$. Then $s_n \to w$ if $n \to \infty$.*

Proof. Let $\frac{p_n}{q_n}$ be the n-th convergent of ζ and define S_n as the restriction of w to the interval $[1, p_n]$. The proof is based on three small lemmas. The first one states that the convergents of ζ are alternatingly smaller and greater than ζ, which is proved in [R, Thms. 10.9,10.11] and the second one describes a well-known "best approximation property" of the convergents. A proof is given in [R, Thm. 10.15]. We assume as known the recurrence relations for p_n, q_n which can be found in [R, Thm 10.7].

Lemma 2.15.1. *Let $\frac{p_n}{q_n}$ be the convergents of an irrational number ζ. Then $\frac{p_n}{q_n} < \zeta \iff n \equiv 0 \bmod 2$.*

Lemma 2.15.2. *Let p_n, q_n, ζ be as above. If k, l are integers with $|k - l\zeta| < |p_n - q_n\zeta|$, then $l \geq q_{n+1}$.*

Lemma 2.15.3. *Let p_n, q_n, ζ be as above. Then $\lfloor \zeta(i + q_n) \rfloor = \lfloor \zeta i \rfloor + p_n$ for $1 \leq i < q_{n+1}$.*

Proof of Lemma 2.15.3. If $n \equiv 0 \bmod 2$ we use Lemma 2.15.2 with $(k, l) = (\lceil \zeta i \rceil, i)$ to obtain $1 > \lceil \zeta i \rceil - i\zeta \geq q_n\zeta - p_n \geq 0$. Subtraction yields $\lfloor \lceil \zeta i \rceil + p_n - (i + q_n)\zeta \rfloor = 0$ or $\lfloor (i + q_n)\zeta \rfloor = \lfloor \zeta i \rfloor + p_n$ which was to be shown. If $n \equiv 1 \bmod 2$ we use Lemma 2.15.2 with $(k, l) = (\lfloor \zeta i \rfloor, i)$ to obtain $1 > \zeta i - \lfloor \zeta i \rfloor \geq p_n - q_n\zeta \geq 0$. Again subtraction gives the required equality.□

Proof of Theorem 2.15. Applying Lemma 2.15.2 with $k = \lfloor \zeta \rfloor, l = 1$ we find for each natural n that $|\zeta q_n - p_n| \leq \mathrm{fr}(\zeta) < 1$. In particular $\lfloor \zeta q_n \rfloor \leq p_n$. Also Lemma 2.15.3 with $i = 1$ gives $\lfloor \zeta(q_n + 1) \rfloor > p_n$, hence $c(S_n) = q_n$ for all $n \in \mathbf{N}$. We will now show that $S_{n+2} = S_{n+1}^{d_{n+2}} S_n$ for all $n \in \mathbf{N}$. The lengths are equal and therefore we only have to check

$$\lfloor \zeta(i + \lambda q_{n+1}) \rfloor = \lfloor \zeta i \rfloor + \lambda p_{n+1} \text{ for } 1 \leq i \leq q_{n+1}, 0 \leq \lambda < d_{n+2},$$
$$\lfloor \zeta(i + d_{n+2} q_{n+1}) \rfloor = \lfloor \zeta i \rfloor + d_{n+2} p_{n+1} \text{ for } 1 \leq i \leq q_n \text{ and}$$
$$\lfloor \zeta(q_n + 1 + d_{n+2} q_{n+1}) \rfloor > p_{n+2}.$$

The first line follows from Lemma 2.15.3 and $q_{n+1} + (d_{n+2} - 2)q_{n+1} < q_{n+2}$, the second line follows from Lemma 2.15.3 and $q_n + (d_{n+2} - 1)q_{n+1} < q_{n+2}$ and the third line was proved before since $q_n + d_{n+2}q_{n+1} = q_{n+2}$. It is clear that $S_0 = b^{\lfloor \zeta \rfloor} a = s_0$. The first line is also valid for $n = -1$ and combining this with $c(S_1) = q_1 = d_1 = c(S_0^{d_1})$ we find $S_1 = S_0^{d_1} b = s_0^{d_1} s_{-1}$. Induction now shows that $S_n = s_n$ for all $n \in \mathbf{N}$. In particular $s_n \to w$. □

Corollary. *Let $\mathcal{M}'$ be the monoid generated by $\{L, C\}$ and let w be an infinite word with domain $\mathbf{N}^+$. There exists a sequence $T_n \in \mathcal{M}'$ with $T_n(a) \to w$ if and only if w is a standard Beatty word.*

Proof. Let w be a standard Beatty word, hence $w = b^\infty$ or $w = \{\lfloor \zeta i \rfloor\}_1^\infty \subset \mathbf{N}^+$ for some $\zeta \geq 1$ or $w = \{\lfloor \frac{in-1}{k} \rfloor\}_1^\infty$, where $0 < k < n$ and $(k, n) = 1$. We let $\mathcal{W}$ be the collection of $\mathbf{N}^+$-words $w_\zeta := \{\lfloor \zeta i \rfloor\}_1^\infty$ with $\zeta \notin \mathbf{Q}$. It is not hard to check that every standard Beatty word is contained in the closure $\overline{\mathcal{W}}$, where the topology is the usual (metric) one on N^+-words. Therefore we can assume that $w = w_\zeta$ for some irrational ζ. We define s_n as in Theorem 2.14 and $T_n = (s_n, s_{n+1})$. Then $T_{n+1} = T_n L^{d_{n+2}} C$, hence $T_n \in \mathcal{M}'$ for all $n \geq -2$. Also $T_n(a) = s_n \to w$. Now assume that $T_n(a) \to w$ where $T_n \in \mathcal{M}'$. If T is any substitution with $aT(a), bT(a) \in \text{Bal}$, then $T' = CT$ and $T' = LT$ are substitutions with the same property. It follows that $Ta \in \text{MLE(Bal)}$ for all $T \in \mathcal{M}'$. Then $T_n(a) \in \text{MLE(Bal)}$ for all n and $w \in \text{MLE(Bal)}$. Hence w is a standard Beatty word by the final remark of Section 2.2. □

Remark. We call $\mathcal{M}'$ the monoid of standard sturmian morphisms.

2.5.6. Generalized Robinson triples. We first quote Problem E 3156 of the American Mathematical Monthly, see also [Ro].

E 3156. *Suppose that r, s, t are integers with $r \geq 0, s \geq 0, t = r + s \geq 2$. Is there a word W of length t in the alphabet $\{a, b\}$ such that $W = AB = Cab$, where A, B, C are palindromes, and the lengths of A and B are r and s? Show that such a word W exists if and only if $r + 2$ is prime to $s - 2$, and that in this case it is unique.*

Clearly this problem was solved in the Monthly, see [Pd]. It is known that the set of possible C equals $\mathcal{H}$. This can be seen, for instance, by combining propositions 1 and 2 in [dL]. In this section we want to show how the solution triples (A, B, C) can, in a way, be parametrized by a submonoid of $\mathcal{M}$. The above result that $\{C\} = \mathcal{H}$ will then follow as a corollary. We will investigate the slightly more general equation $AB = C \overleftarrow{x}$ where $x \in \{a, b\}^*$ is fixed.

Let $x \in \{a, b\}^*$ and define $\Omega := \{(A, B, C) \in \text{PAL}^3 | AB = C \overleftarrow{x}\}$. We let $\hat{\Omega}, \tilde{\Omega}$ be the sets of triples $(A, B, C) \in \Omega$ with $|B| \leq |x|$ and $|B| \geq |x|$, respectively. Note that $\hat{\Omega}$ and $\tilde{\Omega}$ are disjoint if and only if x is not a palindrome. In Lemmas 2.16.1 and 2.16.2 we construct a lot of solutions and then we show that they are all.

Lemma 2.16.1. *Let* $\overleftarrow{x} = \phi\chi\psi$ *be a trisection of* $\overleftarrow{x}$ *into palindromes. Then* $(\chi\phi\chi, \psi, \chi) \in \hat{\Omega}$ *and* $(\chi, \psi\chi\phi\chi\psi, \chi\psi\chi) \in \tilde{\Omega}$.

Proof. Trivial. □

Let $(A, B, C) \in \hat{\Omega}$. From $AB = C \overleftarrow{x}$ and $|B| \leq |x|$ we deduce $\overleftarrow{x} = \lambda B$ for some word λ. This is the λ in the next lemma. Now let $(A, B, C) \in \tilde{\Omega}$. From $AB = C \overleftarrow{x}$ it follows that $BA = xC$. Since $|B| \geq |x|$ we have $B = x\beta$ for some word β.

Lemma 2.16.2. *We have mappings* $\mathcal{L} : \Omega \to \tilde{\Omega}, \mathcal{R} : \tilde{\Omega} \to \tilde{\Omega}, \mathcal{S} : \hat{\Omega} \to \hat{\Omega}$ *given by* $\mathcal{L}(A, B, C) = (A, xAB, AxC), \mathcal{R}(A, x\beta, C) = (\beta A, x\beta, \beta C)$ *and* $\mathcal{S}(A, B, C) = (A\lambda, B, C\lambda)$.

Proof. First we deal with $\mathcal{L}$. From $xAB = xC \overleftarrow{x}, AxC = ABA$ it follows that xAB, AxC are palindromes and we have $AxAB = AxC \overleftarrow{x}$. Therefore $(A, xAB, xC) \in \tilde{\Omega}$. In the $\mathcal{R}$-case we have $\beta A = C, x\beta = B, x(\beta C) \overleftarrow{x} = BAB$ hence $\beta A, x\beta, \beta C \in \mathrm{PAL}$ and $\beta A x \beta = \beta AB = \beta C \overleftarrow{x}$. In the $\mathcal{S}$-case we have $A\lambda = \overleftarrow{\lambda} C\lambda, C\lambda = A$ hence $A\lambda, C\lambda \in \mathrm{PAL}$ and $A\lambda B = A \overleftarrow{x} = C\lambda \overleftarrow{x}$.□

We have only finitely many possibilities (ϕ, χ, ψ) for the trisection appearing in Lemma 2.16.1 and, in principle, one can find all of them when x is given. We let $\Sigma \subset \tilde{\Omega}$ and $\Sigma' \subset \hat{\Omega}$ be the set of solutions obtained in this way. Then Σ, Σ' are also finite. We define $\Sigma'' := \hat{\Omega} \cap \tilde{\Omega}$, hence $\Sigma'' = \{(A, x, A) | A \in \mathrm{PAL}\}$ if $x \in \mathrm{PAL}$ and $\Sigma'' = \emptyset$ otherwise. Also we define $\Sigma''' := \{(\emptyset, A, A) | A \in \mathrm{PAL}\}$ if $x = \emptyset$ and $\Sigma''' = \emptyset$ otherwise.

Theorem 2.16. *We have* $\tilde{\Omega} = \mathcal{L}(\Omega) \cup \mathcal{R}(\tilde{\Omega}) \cup \Sigma$ *and* $\hat{\Omega} = \mathcal{S}(\hat{\Omega}) \cup \Sigma' \cup \Sigma''$. *Every solution* $(A, B, C) \in \Omega$ *is the image of a solution in* $\Sigma \cup \Sigma' \cup \Sigma'' \cup \Sigma'''$ *under an element of* $< \mathcal{L}, \mathcal{R}, \mathcal{S} >$.

Proof. Assume $(A, x\beta, C) \in \tilde{\Omega}$. We have $Ax\beta = C \overleftarrow{x} = \beta A \overleftarrow{x}$ hence a prefix-relation exists between A and β.

If $|\beta| \leq |A|$ then $A = \beta\alpha$ for some word α and $(A, x\beta, C) = (\beta\alpha, x\beta, \beta A) = \mathcal{R}(\alpha, x\beta, A)$, provided we show that $(\alpha, x\beta, A) \in \tilde{\Omega}$. This follows from $x\beta(\alpha)x\beta = xAx\beta = xC\overleftarrow{x} \in \mathrm{PAL}$ and $\beta(\alpha x\beta) = Ax\beta = C\overleftarrow{x} = \beta(A\overleftarrow{x})$. Hence $(\alpha, x\beta, A) \in \tilde{\Omega}$ and $(A, x\beta, C) \in \mathcal{R}(\tilde{\Omega})$.

If $|A| < |\beta|$ then $\beta = A\gamma$ for some γ and $(A, x\beta, C) = (A, xA\gamma, A\gamma A)$. Since $A\gamma A = C$ we have $\gamma \in \mathrm{PAL}$. The equality $Ax\beta = \beta A\overleftarrow{x}$ above implies $x\beta = \gamma A\overleftarrow{x}$ and therefore a prefix-relation between x and γA exists.

If $|x| \leq |A\gamma|$ we have $\gamma A = x\delta$ for some δ and $(A, x\beta, C) = (A, xA\gamma, Ax\delta) = \mathcal{L}(A, \gamma, \delta)$. The last step follows from $\gamma \in \mathrm{PAL}, Ax(A\gamma) = Ax(\delta\overleftarrow{x})$ and $x\delta\overleftarrow{x} = xA\gamma = x\beta \in \mathrm{PAL}$. Suppose now that $|A\gamma| < |x|$, hence $x = \gamma A\epsilon$. Then $(A, x\beta, C) = (A, xA\gamma, A\gamma A) = (A, \gamma A\epsilon A\gamma, A\gamma A)$. We immediately see that also $\epsilon \in \mathrm{PAL}$, hence $\epsilon A\gamma = \overleftarrow{x}$ and $(A, x\beta, C) \in \Sigma$.

We now deal with the case that $(A, B, C) \in \hat{\Omega}$ and as usual we write $\overleftarrow{x} = \lambda B$. We have $AB = C\overleftarrow{x}$, hence $A = C\lambda = \overleftarrow{\lambda} C$. We assume $\lambda \neq \emptyset$, otherwise $(A, B, C) \in \Sigma''$. Induction gives $C\lambda^n = (\overleftarrow{\lambda})^n C$ for all $n \in \mathbf{N}$ and this implies that C is a suffix of the left-infinite word λ^∞. Hence $C = Q\lambda^n$ where Q is a strict suffix of λ and $n \in \mathbf{N}$. Writing $\lambda = PQ$ we find $\overleftarrow{\lambda} = QP$, i.e. λ and $\overleftarrow{\lambda}$ are conjugate. By $PQ = \lambda = \overleftarrow{P}\overleftarrow{Q}$ we have $P, Q \in \mathrm{PAL}$. Then $\overleftarrow{x} = \lambda B = PQB$ and $(A, B, C) = (C\lambda, B, C) = (Q\lambda^{n+1}, B, Q\lambda^n)$. If $n \geq 1$ we have $(A, B, C) = \mathcal{S}(Q\lambda^n, B, Q\lambda^{n-1})$ and if $n = 0$ we have $(A, B, C) \in \Sigma'$.

This proves the two equalities enunciated in the theorem. Now suppose that $(A, B, C) \in \Omega$ is not contained in $\Sigma \cup \Sigma' \cup \Sigma'' \cup \Sigma'''$. If $(A, B, C) \in \hat{\Omega}$, then $(A, B, C) = \mathcal{S}(P, Q, R)$ where $(P, Q, R) \in \hat{\Omega}$. Then $(A, B, C) = (P\lambda, Q, R\lambda)$ hence $|PQR| \leq |ABC|$ with equality if and only if $\lambda = \emptyset$. If indeed $\lambda = \emptyset$ we have $B = x \in \mathrm{PAL}$ and $(A, B, C) = (A, x, A) \in \Sigma''$, a contradiction. Therefore (A, B, C) is the image under $\mathcal{S}$ of a "smaller" triple. The case $(A, B, C) \in \tilde{\Omega}$ is analogous. □

Example 1. Consider $AB = Caababb$. The possible trisections of $aababb$ into palindromes are $(\phi, \chi, \psi) = (a, aba, bb), (aa, bab, b)$. Therefore the basic solutions are given by $\Sigma = \{(aba, bbabaaababb, ababbaba),$
$(bab, bbabaababb, babbbab)\}$ and $\Sigma' = \{(abaaaba, bb, aba), ((babaabab, b, bab)\}$.

Example 2. Consider $AB = Caababbaa$. No triples (ϕ, χ, ψ) exist, hence this equation has no solution. It is not hard to verify that $|x| = 8$ is minimal, i.e. that every word of length at most 7 is contained in PAL^3. By the fact that the number of palindromes of length p equals $2^{[(p+1)/2]}$ we have $|\{a,b\}^n \cap \mathrm{PAL}^3| \leq \sum_{k+l+m=n} 2^{\frac{k+1}{2}+\frac{l+1}{2}+\frac{m+1}{2}} = \binom{n+2}{2} 2^{\frac{n+3}{2}}$. Hence for all large enough n there exist words x of length n such that our equation has no solution.

Example 3. Consider $AB = Cab$. The trisections of ab are $(\emptyset, a, b), (a, \emptyset, b)$, $(a, b, \emptyset)$, hence $\Sigma = \{(a, baab, aba), (\emptyset, bab, b), (b, bab, bb)\}$ and $\Sigma' = \{(aa, b, a), (a, b, \emptyset), (bab, \emptyset, b)\}$. Note that Σ, Σ' are not minimal since Theorem 2.16 remains true when we delete $(b, bab, bb) = \mathcal{R}(\emptyset, bab, b)$ and $(aa, b, a) = \mathcal{S}(a, b, \emptyset)$ from them, respectively. See also Stelling 6.

From now on we only consider $AB = Cab$ as in Example 3. As in Section 2.5.4 we define $L := (ba, b)$ and also $L' := CLC = (a, ab)$. We let $\mathcal{M}'' :=< L, L' >$, note that $\mathcal{M}'' \cup C\mathcal{M}'' = \mathcal{M}'$ where $\mathcal{M}'$ is the monoid of standard sturmian morphisms from Section 2.5.5. Also we set $\mathcal{M}^* = \{(X,Y) \in \mathcal{M}'' | |X| \geq 2\}$. Note that every $(X,Y) \in \mathcal{M}^*$ is of the form $(\xi ba, Y)$.

Lemma 2.17.1. **a)** *If $(A, ba\beta, C) \in \tilde{\Omega}$, then $(Aba, \beta) \in \mathcal{M}^*$.*
b) *If $(\xi ba, Y) \in \mathcal{M}^*$, then a Z exists with $(\xi, baY, Z) \in \tilde{\Omega}$.*
c) *Suppose that $(X,Y) \in \mathcal{M}'', |X| =: s, |Y| =: r$ and that $Z = \mathrm{per}(s, r, 1)$. Then $XY = Zab$ and $YX = Zba$.*

Proof. **a)** Theorem 2.16 and Example 3 together allow us to use induction. The induction basis consists of the elements of $\tilde{\Omega}$ which are minimal with respect to application of $\mathcal{L}, \mathcal{R}$. All these are contained in $\cup_0^\infty \mathcal{LS}^n(\Sigma') \cup \Sigma$. If $\pi = (aa, b, a) \in \Sigma'$ we have $\mathcal{LS}^n(\pi) = (a^{n+2}, ba^{n+3}b, a^{n+2}ba^{n+2})$ and clearly $(a^{n+2}ba, a^{n+2}b) \in \mathcal{M}^*$. The other two cases are similar. For $\pi \in \Sigma$ the result follows by inspection. For the induction step we assume $(A, ba\beta, C) \in \tilde{\Omega}$ and $(Aba, \beta) \in \mathcal{M}^*$. For $\mathcal{L}(A, ba\beta, C) = (A, baAba\beta, AbaC)$ the statement follows since $(Aba, Aba\beta) \in \mathcal{M}^*$ and for $\mathcal{R}(A, ba\beta, C) = (\beta A, ba\beta, \beta C)$ it follows since $(\beta Aba, \beta) \in \mathcal{M}^*$.
b) Here we use induction on $\mathcal{M}^*$, now with respect to L and L'. The minimal elements are $(L')^n L = (a^n ba, a^n b)$ and we have $(a^n, ba^{n+1}b, a^n ba^n) \in \tilde{\Omega}$. Now suppose $(\xi ba, Y) \in \mathcal{M}^*$ and $(\xi, baY, Z) \in \tilde{\Omega}$ for some Z. Then $\mathcal{L}(\xi, baY, Z) = (\xi, ba\xi baY, \xi baZ) \in \tilde{\Omega}$ implies the statement for $(\xi ba, Y)L'$ and using $\mathcal{R}$ one finds the statement for $(\xi ba, Y)L$.
c) We know that $w := X^\infty Y^\infty$ is an infinite Hedlund word. With induction it is easy to show that $\alpha(X) > \alpha(Y)$ when $(X,Y) \in \mathcal{M}''$, hence

$w = \mathrm{PER}(s, r, 1)$. Since $[X]^1 = a, [Y]^1 = b$ it follows that Y^∞ is the maximal right-periodic subword of w. Hence $Z = [X^\infty]_{r+s-2} = [Y^\infty]_{r+s-2} = X[Y^\infty]_{s-2} = Y[X^\infty]_{r-2} = X[Y]_{r-2} = Y[X]_{s-2}$. An easy induction also shows $[XY]^2 = ab, [YX]^2 = ba$ and the result follows. □

The following theorem follows at once from **a** and **b** in the previous lemma.

Theorem 2.17. *Consider the equation $AB = Cab$ in palindromes A, B, C. Let $\tilde{\Omega}, \mathcal{M}^*$ be as defined above. An element of $\tilde{\Omega}$ will be written in the form $(A, ba\beta, C)$ and an element of $\mathcal{M}^*$ in the form $(\xi ba, Y)$, as before. Then there exist mutual inverses $F : \tilde{\Omega} \to \mathcal{M}^*, G : \mathcal{M}^* \to \tilde{\Omega}$ given by $F(A, ba\beta, C) = (Aba, \beta), G(\xi ba, Y) = (\xi, baY, Z)$. Here Z is defined by* **b** *in the previous lemma.*

Corollary. *If $(A, B, C) \in \Omega$ then $C \in \mathcal{H}$. Conversely, if $C \in \mathcal{H}$, then there exist A, B such that $(A, B, C) \in \Omega$.*

Proof. Let us assume that $(A, B, C) \in \Omega$. If $|B| \leq 1$ we have $(A, B, C) \in \hat{\Omega} = \cup_0^\infty \mathcal{S}^n(\Sigma') = \{(a^{n+1}, b, a^n), ((ba)^{n+1}b, \emptyset, (ba)^n b)\}$, hence $C \in \mathcal{H}$. If $|B| \geq 2$ we have $(A, B, C) = (A, ba\beta, C) \in \tilde{\Omega}$ and by **a** we have $(X, Y) := (Aba, \beta) \in \mathcal{M}^*$. Defining s, r, Z as in **c** we have $Zab = XY = Aba\beta = AB = Cab$ and $C = Z \in \mathcal{H}$.

Finally let us assume that $C \in \mathcal{H}$. We can write $C = \mathrm{per}(s, r, 1)$ for uniquely determined s, r by the remark at the end of Section 2.3. Define $(X, Y) \in \mathcal{M}''$ by $|X| = s, |Y| = r$, then $XY = Cab$ by **c**. If $|X| \leq 1$ we have $(X, Y) = (a, a^n b), C = a^n$ and for these C we have the triple $(a^{n+1}, b, a^n) \in \Omega$. If $|X| \geq 2$ we have $(\xi ba, Y) := (X, Y) \in \mathcal{M}^*$ and by **b** a Z exists with $(\xi, baY, Z) \in \tilde{\Omega}$. Then $Cab = XY = \xi baY = Zab$, hence $C = Z$ and $(\xi, baY, C) \in \Omega$. □

Chapter 3.
Z-words of minimal block growth.

3.1. Classification. Let $k \in \mathbf{N}^+$ and suppose that w is a k-stiff $\mathbf{Z}$-word. Then either w is periodic or $P(w,n) \geq n+1$ for all n by Theorem 2.1. If we assume that w is not periodic, then Theorem 2.1 also implies that $P(n+1) - P(n) \geq 1$ for all n and it is clear that $P(n+1) - P(n) = 1$ for all but finitely many n. It follows that there exist integers k', N with $k' \leq k$ such that $P(w,n) = n + k'$ for $n \geq N$.

Definition 3.1. *Let w be a $\mathbf{Z}$-word. Then w has minimal block growth if there exist integers k, N such that $P(w,n) = n + k$ for all $n \geq N$. The integer $k \geq 1$ is called the stiffness of w. We write $k(w)$.*

We give some more definitions. A $\mathbf{Z}$-word w is called recurrent if every finite factor x appears at least twice in w. It then appears infinitely often, but not a priori infinitely often in both directions. Two finite words x, y are called conjugate if they are cyclic permutations of each other and we write $x \sim y$. Another characterization is that there exist words A, B with $x = AB, y = BA$ and yet another one is that x, y have the same length and generate the same $\mathbf{Z}$-word $x^\infty = y^\infty$. This last characterization shows that conjugacy is an equivalence relation. A $\mathbf{Z}$-word w is called left periodic if there exist integers $p \geq 1$ and N such that $w_i = w_{i-p}$ for $i \leq N$. The least possible p is then called the left period for w and any p-factor x of w with domain contained in $(-\infty, N]$ is called a left period cycle. Note that x is determined up to conjugacy only. Of course analogous definitions can be given with right instead of left.

First we describe the structure of non-recurrent $\mathbf{Z}$-words of minimal block growth. Proofs can be found in [C, Section 3]. We note, however, that Theorem 3.10 in [C] is not entirely correct. To avoid confusion we will give our own proofs and afterwards we indicate what (minor) changes have to be made in Theorem 3.10 in order for it to be true.

Theorem A. *Let w be k-stiff ($k \geq 1$) and not recurrent. Then w is left periodic and right periodic.*

Now let w be an arbitrary $\mathbf{Z}$-word which is periodic in both directions but not periodic as a whole, for instance $w = \cdots(ab)(ab)(aba)(aba)\cdots$ We denote the left- and right period by s, r respectively and the corresponding maximal periodic domains by $(-\infty, \lambda], [\mu, \infty)$. We define the overlap of w to be $g = \lambda - \mu + 1$ (this overlap can be 0 or negative). Also we define $\chi = r = s$ if the period cycles are conjugate and $\chi = s + r$ otherwise.

Theorem B. *Let w be a $\mathbf{Z}$-word which is left periodic and right periodic but not periodic. Then w is not recurrent, it has minimal block growth and $k(w) = \chi - 1 - g$.*

Important for the recurrent case are substitutions which were already introduced in Section 2.5.4. If $T = (AB, AC)$ and $T' = (BA, CA)$, then $T\sigma = T'\sigma$ for all $\mathbf{Z}$-words σ. It follows that we can identify T and T' if we are only interested in their action on $\mathbf{Z}$-words. Now suppose that $T = (X, Y)$ where X, Y are non-empty of length $m, n \geq 1$, respectively. We define the right infinite words $x = X^\infty, y = Y^\infty$. As before, T may be identified with $T_k = (x_k \cdots x_{k+m-1}, y_k \cdots y_{k+n-1})$ if $x_i = y_i$ for $1 \leq i < k$. It follows that either $x = y$, in which case X, Y are powers of the same word π, or else that T is equivalent to T' where $[T'a]_1 \neq [T'b]_1$. In the first case we call T trivial and $T\sigma = \pi^\infty$ for all σ. We see, therefore, that every non-trivial T can be identified with a T where Ta, Tb have different initial symbols.

From now on we assume that $T = (X, Y)$ is a substitution with $[X]_1 = a, [Y]_1 = b$. It might happen that T is of the form (X, ZX) or of the form (ZY, Y). Of course these cases are incompatible since $X \neq Y$. In the first case we define the reduction T^{red} of T as (X, Z) and in the second case as (Z, Y). If neither of these cases apply we call T irreducible and we just define $T^{\text{red}} = T$. Now let T be any substitution. After a finite number of reductions one obtains an irreducible substitution and we denote the result by T^{RED}. For example, if $T = (abba, bba)$ then $T^{\text{RED}} = (a, bb)$. Note that an irreducible substitution T is of the form $(A\sigma C, B\overline{\sigma}C)$ with uniquely determined A, B, σ, and, conversely, that every substitution of this form with $[A\sigma]_1 = a, [B\overline{\sigma}]_1 = b$ is irreducible. We define the stiffness $k(T)$ of an irreducible substitution $T = (A\sigma C, B\overline{\sigma}C)$ by $k(T) = |ABC| + 1$. If T is reducible then we define $k(T) = k(T^{\text{RED}})$. The term "stiffness" will become clear from the next theorem and Lemma 3.4.

Theorem 3.1. *Let w be a recurrent k-stiff $\mathbf{Z}$-word. Then there exist a recurrent stiff $\mathbf{Z}$-word σ and a substitution T such that $T\sigma = w$. Conversely, if σ is sturmian and T a non-trivial substitution, then $T\sigma$ is recurrent, has* *minimal block growth and* $k(T\sigma) = k(T)$.

The first assertion can be found in [A, Chapitre 3] for words with domain $\mathbf{N}^+$ over any finite alphabet Σ. The formula for the stiffness in the second assertion is new. Related results on recurrent k-stiff words can be found in Didier [D], Paul [P] and Coven [C].

Proof of Theorem A. Assume that the subword x with domain $[1, n]$ does not occur elsewhere in w. Then every subword containing this one does not occur elsewhere in w either. Let $w' = w_2 w_3 \cdots$. For $N \geq n$ there exist $N - n + 1$ intervals of length N containing $[1, n]$. Hence $P(w', N) \leq (N + k) - (N - n + 1) = k + n - 1$. Since $P(w', N)$ is bounded we have that w' and also w are right periodic. By a symmetry argument w is left periodic. □

Proof of Theorem B. First we deal with $g < 0$. Hence we can write $w = A^\infty B C^\infty$ where $|A| = s, |B| = -g, |C| = r, A$ and C are primitive (not powers of a smaller word), $B \neq \emptyset, A$ and B have different initial symbols and B, C have different terminal symbols. If B starts with σ and ends in τ, then $A\sigma$ does not appear in A^∞ and τC does not appear in C^∞. It follows that $ABC = A\sigma \cdots \tau C$ appears only once in w, say in position $[1, n]$. In particular w is not recurrent.

Now let D be any subinterval of $\mathbf{Z}$ of length N and x the subword of w with domain D. We say that x has property * if $x \subset A^\infty$ after deleting at most $|BC| - 1$ symbols from the right and we say that x has property ** if $x \subset C^\infty$ after deleting at most $|AB| - 1$ symbols from the left. If $D \subset (-\infty, n - 1]$ then x has property * and if $D \subset [2, \infty)$ then x has property **. The remaining intervals D are exactly those containing $[1, n]$.

Now suppose that x satisfies * and ** simultaneously where

$$N \geq |A| \cdot |C| + |ABBC| - 2.$$

Performing both deletions we find a word x' with $|x'| \geq |A| \cdot |C|$ such that x' is contained in A^∞ and C^∞. Let x'' be a subword of x' of length $|A| \cdot |C|$. Then $x'' = \tilde{A}^{|C|} = \tilde{C}^{|A|}$ where $\tilde{A} \sim A, \tilde{C} \sim C$. So the $\mathbf{Z}$-words $\tilde{A}^\infty, \tilde{C}^\infty$ are equal, hence $A^\infty = C^\infty$ and by primitivity we have $A \sim C$. Now x is obtained by extending x'' with period $|A| = |C|$ in both directions. Therefore, if $N \geq |A| \cdot |C| + |ABBC| - 2$ then every word x satisfying * and ** is contained in both periodic parts of w.

We note that for N large exactly χ words of length N are contained in some periodic part of w and that exactly $N-n+1$ subwords of w contain ABC. We will now count the N-factors x of w which do not contain ABC and are not contained in any of the two periodic parts. By the previous paragraph these subwords satisfy exactly one of the conditions * and **. First suppose that x satisfies only *. Then any domain D of x satisfies $D \subset (-\infty, n-1], D \not\subset (-\infty, |A|]$, and there exist $|BC|-1$ such intervals of length N. If D is such an interval, then the corresponding word x is not contained in the left periodic part (x contains $A\sigma$), not contained in the right periodic part (then x would satisfy *,** simultaneously, hence be contained in the left periodic part), and does not contain ABC. Moreover, all such D yield different x, since the x's can be distinguished by the first appearance of $A\sigma$. Hence the number of words not contained in a periodic part, not containing ABC and satisfying only * equals $|BC|-1$. Similarly the number of such x satisfying only ** equals $|AB|-1$. Hence

$$P(w, N) = \chi + (N - |ABC| + 1) + |BC| - 1 + |AB| - 1 = N + \chi - 1 - g$$

Now we consider the case $g \geq 0$. Then we can write $w = A^\infty B C^\infty$ where $|A| = s, |B| = g, |C| = r$ where A, C are primitive and where the maximal periodic parts are given by $A^\infty B$ and BC^∞. In analogy with the previous argument we find that ABC appears only once in w and that $P(w, N) = \chi + (N - |ABC| + 1) + |C| - 1 + |A| - 1 = N + \chi - 1 - g$ for N large. □

Remark. A formula for $k(w)$ in Theorem B already appears in [C, Theorem 3.10]. Coven defines, in our terminology, that $\chi = r = s$ if w is asymptotically symmetric and $\chi = r + s$ otherwise. Here a **Z**-word w is called asymptotically symmetric if there exist i, j such that $w_{i-k} = w_{j+k}$ for all $k \geq 0$. The reader should be aware that the condition of asymptotic symmetry appearing in [C] is *not* equivalent to our condition of equal period cycles on either side as the example $w = (aababb)^\infty (baabab)^\infty$ shows. If Coven's condition is replaced by ours, then the proof in [C] is correct.

Proof of Theorem 3.1. Let $T = (X, Y)$ be a substitution with $[X]_1 = a, [Y]_1 = b$. We associate with T a directed graph $G(T)$, each edge of which is labelled with a or b. The graph consists of two directed cycles α, β of lengths $|X|, |Y|$ whose only intersection is a vertex O, the origin. Also, if one follows α from O to itself the labels read X and if one follows β from O to itself the labels read Y. We call $G(T)$ the representing graph for T.

An acceptable path in $G(T)$ is a directed path whose labels form the initial segment of a right-infinite word on $\{X, Y\}$. In other words, an acceptable path is a path whose labels can be obtained starting from O. If an acceptable path has label x, then there is a unique ξ of minimal length such that x is a left-factor of $T\xi$. We call ξ the coding for x. It indicates the order in which X, Y appear following the path. The choice is unique because X, Y start with different symbols.

Lemma 3.1.1 *Let T be an irreducible substitution and G its representing graph. From every vertex $P \neq O$ there is at most one acceptable path of given length. Moreover, there exist positive integers M, N such that every acceptable path with starting vertex $\neq O$ has a coding with period M after deleting the first N symbols.*

Proof. Let $P \neq O$ and first assume that there is an infinite acceptable path from P. The first symbol of the coding is fixed and following X, Y accordingly in $G(T)$ we find a path $P \to Q$. The direction to take at O is completely determined by the next symbol. Suppose $Q = O$ and without loss of generality that the final edge of the path $P \to Q$ lies in α. Apparently the word you followed was Y, since otherwise $P = O$. If $|Y| \leq |X|$ then $X = ZY$ for some Z and if $|X| < |Y|$ then $Y = ZX$. Both are impossible because T was irreducible. Hence $Q \neq O$ and repeating the procedure we find a path $P \to Q \to R \to \cdots$ where each arrow has label X or Y. Since G has only $|XY|-2$ vertices different from O, the first $|XY|-1$ elements of the sequence $P, Q, R, \cdots$ cannot be distinct. Hence some vertex appears twice in the sequence and since every point in the sequence determines its successor uniquely it follows that the sequence is eventually periodic. The same is then true for the acceptable path and its coding. Let N be the maximal preperiodic part of the codings (taken over all P) and M the least common multiple of all the periods. These M, N will do. If there exist only finite acceptable paths from P then the uniqueness is shown as above and enlarging N sufficiently the second part of the Lemma becomes trivial for all such P.□

We recall that a finite word x is called primitive if x is not a power of a strictly smaller word. Every finite word can be written uniquely as power of a primitive word. See [B/P, Proposition 3.1] for a proof.

Lemma 3.1.2. *Let T be an irreducible substitution. Then there exist only finitely many finite primitive words x such that Tx is not primitive.*

Proof. Suppose x is primitive and $Tx = \eta^n$ with η primitive and $n \geq 2$. Trace out η in $G(T)$, starting in O and find a path $O \to P$. If $P = O$ then $\eta = T(\xi)$, hence $Tx = T(\xi^n)$ and $x = \xi^n$, a contradiction. Here the injectivity of T is a direct consequence of the fact that X starts with a and Y with b. Thus $P \neq O$ and there is an acceptable path from P with label η^∞. But we know from the proof of Lemma 3.1.1 that η is determined by P and hence the set of possible η is finite. Now suppose that x, y, η are primitive such that Tx, Ty are powers of η. Then we have $Tx = \eta^m, Ty = \eta^n$ for some $m, n \in \mathbf{N}^+$, hence $x^n = y^m$. From the Defect Theorem, (see [B/P, Theorem 2.8]), it follows that x, y are powers of the same word and by primitivity we have $x = y$. We conclude that there are only finitely many η and each η yields at most one x. □

Lemma 3.1.3. *Let T be an irreducible substitution. There are only finitely many primitive x for which there exists a word y with $x \not\sim y$ but $Tx \sim Ty$.*

Proof. Suppose x, y are as in the lemma and write $Tx = \eta^m$ with η primitive and $m \geq 1$. Choose an admissible path γ from O with label $(Ty)^\infty$. Since $(Ty)^\infty$ has primitive period cycle η there is a $P \in \gamma$ such that the induced path from P onwards has label η^∞. If $P = O$ then $T(y^\infty) = T(zx^\infty)$ for some z hence $y^\infty = zx^\infty$. It follows that $x^\infty = y^\infty$ (where now these words are understood to be $\mathbf{Z}$-words), hence $x \sim v^k, y \sim v^l$ for some primitive v and $k, l \geq 1$. Then $T(v)^k \sim Tx \sim Ty \sim T(v)^l$ and by comparing lengths we find $k = l$ and $x \sim y$, a contradiction. Hence $P \neq O$, and since the infinite path from P onwards is admissible we find that η is determined by P. Hence only finitely many η are possible and since every η yields at most one x we are done. □

Let $w = \mathrm{PER}(s, r, \Delta)$ and L, R, C as in Section 2.5.4. We know that Lw, Rw, Cw are infinite Hedlund words as could also be seen by calculating χ, g and applying Theorem B. A density calculation shows $Lw = Rw = \mathrm{PER}(s + l, r + k, \Delta), Cw = \mathrm{PER}(s, r, -\Delta)$. From these formulas it follows that L, R, C induce injections $\mathcal{H} \to \mathcal{H}$ and then the same is also true for any $T \in \mathcal{M}$. We now concern ourselves with the effect of arbitrary substitutions on infinite Hedlund words.

Lemma 3.1.4. *Let T be an irreducible substitution. Then there exists a finite set $V \subset \mathbf{Q}$ such that $P(Tw, n) = n + k(T)$ for n large whenever w is* *an infinite Hedlund word whose left- and right-density avoid V.*

Proof. Choose $w = \mathrm{PER}(s, r, \Delta)$ such that the left and right period cycle are not in the exceptional sets of Lemmas 3.1.2 and 3.1.3. We write $|Ta| = \phi, |Tb| = \psi$ and A, B, C as in the definition of $k(T)$. Then Tw has minimal left-period $l\phi + (s-l)\psi$, minimal right-period $k\phi + (r-k)\psi$ and overlap $(k+l-1)\phi + (r+s-k-l-1)\psi + |C|$. The exclusion made guarantees that the period cycles mentioned are not conjugate and applying Theorem B we have $k(Tw) = \chi - 1 - g = (k+l)\phi + (r+s-k-l)\psi - 1 - (k+l-1)\phi - (r+s-k-l-1)\psi - |C| = \phi + \psi - 1 - |C| = |ABC| + 1 = k(T)$.□

Remark. Let $T = (abba, bb)$ and $w = \mathrm{PER}(2,5,1) = (ab)^\infty(babab)^\infty$. The reader may verify that $k(T) = 5$ and that $k(Tw) = 2$. This shows that the restriction on w cannot be dropped.

Let w be a $\mathbf{Z}$-word of minimal block growth, then $P(w, n) = n + k$ for $n \geq N$ and this means that for $n \geq N$ we have $\mathrm{MLE}_n(w) =: \{C_n\}, \mathrm{MRE}_n(w) =: \{B_n\}$. It is obvious that B_n is a right-factor of B_{n+1} if $n \geq N$, hence there exists a unique left-infinite word B such that B_n equals the last n symbols of B for every $n \geq N$. Of course every right-factor from B has MRE and this allows us to define B_n for every $n \geq 0$. Similar definitions can be given for C and C_n. We note that for stiff words w we have $\mathrm{MRE}(w) = \{B_n\}, \mathrm{MLE}(w) = \{C_n\}$ for all n and that for infinite Hedlund words B, C are equal to the maximal periodic tails. We say that w has a jump at $n \geq 0$ if $|\mathrm{MRE}_n(w)| > 1$ or, equivalently, if $P(w, n+1) - P(w, n) > 1$. Of course $n < N$ for such n. The following lemma has been set apart since it will be used several times in the sequel.

Lemma 3.1.5. *Let w be a $\mathbf{Z}$-word with $P(w, n) = n + 1$ for all n, T an irreducible substitution, γ a bi-infinite path in $G(T)$ with label Tw and let $x \in \mathrm{MRE}(Tw)$ be finite. If all paths along γ with label x have the same endpoint, then x is a right-factor of the left-infinite word $T(B(w))$.*

Proof. If all paths along γ with label x have the same endpoint, then this endpoint must be O. It follows that we can write $x = yT(z)$ where y is a strict right-factor of X or Y and where $z \subset w$. Here y, z are unique because T is irreducible. We have $z \in \mathrm{MRE}(w)$, hence $z = B_i(w)$ for some i and we have $B_{i+1}(w) = \lambda z$ for some symbol λ. If γ contains different paths with label x then y is a common right-factor of X and Y. In particular y is a right-factor of $T\lambda$ and x is a right-factor of $T(B(w))$, as stated. If there is a unique path along γ with label x, then y must be a right-factor of $T\lambda$ and we are done as well. □

Lemma 3.1.6. *Let w be a $\mathbf{Z}$-word with $P(w,n) = n+1$ for all n, T an irreducible substitution, $\mu = \max(|Ta|,|Tb|)$ and M, N as in Lemma 3.1.1 If $x \in \mathrm{MRE}(Tw)$ has length τ and x is no right-factor of $T(B(w))$, then w contains an M-periodic word of length $\lceil \frac{\tau}{\mu} - 1 - N \rceil$.*

Proof. It follows from Lemma 3.1.5 that x has two representing paths $P_0 \cdots P_\tau$, $Q_0 \cdots Q_\tau$ along γ with $P_\tau \neq Q_\tau$ and, consequently, $P_0 \neq Q_0$. We may assume $\tau \geq \mu$ (otherwise the theorem is trivial), hence both paths pass through O. If neither of these paths starts at O then we delete the initial edge of both of them and we repeat until one of them does. The new length ν of both paths satisfies $\tau - \mu \leq \nu \leq \tau$ and both paths have label $\tilde{x} := [x]^\nu$. We denote the new paths, without loss of generality, by $OP'_1 \cdots P'_\nu$ and $Q'_0 \cdots Q'_\nu$. Note that $P'_\nu \neq Q'_\nu$ and $O \neq Q'_0$. Let ξ be the coding of these acceptable paths. From $OP'_1 \cdots P'_\nu \subset \gamma$ we deduce $\xi \subset w$ and from Lemma 3.1.1 applied to the acceptable path $Q'_0 \cdots Q'_\nu$ we deduce that ξ has period M after deleting the first N symbols. The result now follows from $|\xi|\mu \geq |T\xi| \geq \nu \geq \tau - \mu$. □

Proof of Theorem 3.1. (first part) The first part of the theorem is trivial if w is periodic, hence we assume that this is not the case. Let $k = k(w)$. For every $n \geq 1$ we define a directed graph G_n with vertex-set $\mathcal{B}(w,n)$ and edge-set $\mathcal{B}(w,n+1)$ in such a way that every $x \in \mathcal{B}(w,n+1)$ induces an arrow from its first n symbols to its last n symbols. This graph is usually known as the $n-$th Rauzy graph of w. Every subword x of w of length $\geq n$ induces in a natural way a path in G_n, namely the path which has the successive n-factors of x as its succesive vertices. More precisely, a word $x = x_1 \cdots x_{n+p}$ induces the path which has $x_i \cdots x_{i+n-1}$ as its i-th vertex. This path has $|x| - n$ edges and is called the path induced by x. Conversely we call x the label of this path.

The undirected graph underlying G_n is of course connected. Since w has minimal block growth, it follows that for n large there is one point B_n of outdegree 2, one point C_n of indegree 2 and that every point has positive indegree. It follows that we have only three possibilities for the type of G_n for n large.

a) A loop from B_n to itself, a path of positive length from B_n to C_n and a loop from C_n to itself. (Apart from B_n, C_n the different loops and paths have to be disjoint).

b) If $B_n = C_n$ then two distinct loops from B_n to itself.

c) If $B_n \neq C_n$ then two paths from B_n to C_n and a single path from C_n to B_n.

If one is in case **a**) for some n then one is in case **a**) for all larger n and it is easy to see that the pathlength B_nC_n in G_n increases by 1 if n increases by 1. After at most one such step we find a point $P \neq B_n, C_n$ on this path and then there is no path in G_n from P to itself, contradicting the fact that w is recurrent. Hence only **b**) and **c**) occur. If one is in case **c**) then it is not hard either to show that the pathlength C_nB_n in G_n decreases by 1 if n increases by 1. See also [A/R, section 1]. It follows that case **b**) appears for infinitely many n and without loss of generality we take some n such that **b**) applies and such that there are no jumps at positions $\geq n$. The loops α, β in G_n have labels $B_nX = \xi B_n$ and $B_nY = \eta B_n$, respectively, where $[X]_1 \neq [Y]_1$ and $[\xi]^1 \neq [\eta]^1$. We assume $[X]_1 = a$ and $[Y]_1 = b$. It is immediate that $w = T(\sigma)$ for some **Z**-word σ. We define $\chi(a) = \alpha, \chi(b) = \beta$ and extend χ by concatenation to a mapping sending words to paths in G_n. Then the infinite path γ induced by w (read off its successive n-factors) equals $\chi(\sigma)$. In particular it follows that σ is determined up to shift. We recall that for each $p \geq 0$ there exists a natural bijection $\mathcal{B}(w, n+p) \leftrightarrow \{$ paths of length p along $\gamma\}$, again by reading off the successive n-factors. The cardinality of both sets equals $p + n + k$. It follows that for every $p \geq 0$ there is a unique path γ_p of length p with two right-extensions in γ. In particular, every two such paths are comparable, i.e., γ_p is a left-extension of γ_q if $p \geq q$.

Now suppose that $x, \tilde{x} \in \mathrm{MRE}(\sigma')$. Then $\chi(x), \chi(\tilde{x})$ have m.r.e. in γ. Therefore these paths are comparable and the same follows for $x, \tilde{x}$. Since σ' is not periodic it follows that $|\mathrm{MRE}_i(\sigma')| = 1$ for each i and hence that $P(\sigma', i) = i + 1$ for all i. If σ' is not recurrent, then σ' is left- and right-periodic with a finite overlap. The same is then true for w, but then Theorem B shows that w is not recurrent. Hence σ' is recurrent and by the classification of stiff **Z**-words it is sturmian.

(second part) Let σ be a sturmian **Z**-word and T a non-trivial substitution. We can assume without loss of generality that T is irreducible. If $x \subset T(\sigma)$ then we can write $yxz = T\xi$ where $\xi \subset \sigma$. Since ξ appears infinitely often in σ, the same holds for x in $T\sigma$. Hence $T(\sigma)$ is recurrent. We write

$\mu = \max(|Ta|, |Tb|)$ as before. Choose $n \geq 1$ and let $\xi_n \subset \sigma$ be a finite subword such that ξ_n contains all $(n+2\mu)-$factors from σ and let $w = w(n)$ be an infinite Hedlund word containing ξ_n. (Since ξ_n is balanced it is contained in a rational Beatty word by Theorem 2.8, hence also in an infinite Hedlund word). Then $\mathcal{B}(\sigma, n+2\mu) = \mathcal{B}(w, n+2\mu)$, hence $\mathcal{B}(T\sigma, n) = \mathcal{B}(Tw, n)$ and $P(T\sigma, n) = P(Tw, n)$. For every $p \geq 1$ there exists a constant C_p such that σ contains no C_p-factors of period p. Otherwise, let ξ_n be a p-periodic subword of length n for every n. Then $d(\frac{c(\xi_n)}{n}, \frac{\mathbf{Z}}{p}) \to 0$ and $\frac{c(\xi_n)}{n} \to \alpha(s)$ as $n \to \infty$, a contradiction. It follows that the left and right density of $w = w(n)$ lie outside the set V from Lemma 3.1.4 for $n \geq n_1$. Taking $n \geq n_1$ and applying Lemma 3.1.4 we have $k(Tw) = k(T)$.
Suppose now that $\tau \geq n$ is such that $|\mathrm{MRE}_\tau(Tw)| > 1$. Then w contains an M-periodic word of length $\lceil \frac{n}{\mu} - 1 - N \rceil$ by Lemma 3.1.6 and since $\lceil \frac{n}{\mu} - 1 - N \rceil \leq n$ the same word appears in σ. Hence $\lceil \frac{n}{\mu} - 1 - N \rceil \leq C_M$ whence $n \leq \mu(N + C_M + 1)$. Taking n larger than this value it follows that Tw has no jumps at places $\tau \geq n$, hence $P(T\sigma, n) = P(Tw, n) = n + k(Tw) = n + k(T)$. □

Remark. Let w be a recurrent $\mathbf{Z}$-word of minimal block growth. It follows from the previous proof that $G_i(w)$ is strongly connected for large i. From this we conclude that $G_i(w)$ is strongly connected for all i.

3.2. Differences and similarities between $k = 1$ and $k > 1$. Let x be a finite balanced word. Then a balanced $\mathbf{Z}$-word w exists with $x \subset w$. The same is true when balanced is replaced by k-balanced (Theorem 2.3) or by stiff (Theorem 2.4). It is however not true for k-stiff when $k > 1$. The first proof we gave used de Bruijn cycles and can be found in [HT]. The following proposition is a little stronger.

Proposition 3.2. *For every $k \geq 1$ there exists a finite 2-stiff word which is not contained in any k-stiff $\mathbf{Z}$-word.*

Proof. Suppose that the proposition is not true for some $k > 1$. Let σ be a sturmian $\mathbf{N}$-word with $\sigma_0 = a, aa \not\subset \sigma$ and let $\tilde{\sigma} = a\sigma$. By hypothesis we have for every $n \geq 1$ a k-stiff $\mathbf{Z}$-word $w^{(n)}$ with $w^{(n)}|_{[1,n]} = [\tilde{\sigma}]_n$. Choosing a subsequence $w^{(n_i)}$ that converges on $(-\infty, 0]$ we find a $\mathbf{Z}$-word w that is k-stiff and such that $w|_{[1,\infty)} = \tilde{\sigma}$. Since w has irrational right density it follows from Theorem A that w is recurrent and then $w = Tx$ for some sturmian

$\mathbf{Z}$-word x by Theorem 3.1. In particular it follows that every finite factor of w appears arbitrarily far to the right and more in particular this holds for $aa \subset w$. We infer $aa \subset \sigma$, a contradiction. This proves the proposition.□

For $k \geq 1$ we define $\mathcal{S}_k$ to be the collection of $\mathbf{Z}$-words w such that $P(w,n) \leq n+k$ for all n and such that $P(w,n) = n+k$ for at least one n. We define

$$\mathcal{S}_k^{\mathrm{per}} = \{w \in \mathcal{S}_k | w \text{ is periodic } \}$$

$$\mathcal{S}_k^{\mathrm{rnp}} = \{w \in \mathcal{S}_k | w \text{ is recurrent but not periodic } \}$$

$$\mathcal{S}_k^{\mathrm{nr}} = \{w \in \mathcal{S}_k | w \text{ is non-recurrent } \}$$

It is obvious that $\mathcal{S}_k$ is the disjoint union of these three sets. In the notation from Theorems 2.5 and 2.6 we have $\mathbf{a} = \mathcal{S}_1^{\mathrm{per}}, \mathbf{c} = \mathcal{S}_1^{\mathrm{rnp}}$ and $\mathbf{b} \cup \mathbf{d} = \mathcal{S}_1^{\mathrm{nr}}$. The following propositions are direct generalizations of the $k=1$ case. The factor set $F(\mathcal{S})$ was defined in Section 2.4.

Proposition 3.3. $F(\mathcal{S}_k^{\mathrm{per}}) = F(\mathcal{S}_k^{\mathrm{rnp}}) \subset F(\mathcal{S}_k^{\mathrm{nr}})$.

Proposition 3.4. *If $w \in \mathcal{S}_k^{\mathrm{nr}}$ has the same period cycle in both directions, then $F(w) \subset F(\mathcal{S}_k^{\mathrm{rnp}})$.*

Proposition 3.5. *Let w be a recurrent and k-stiff $\mathbf{Z}$-word. Then w is k-balanced. The latter k cannot be replaced by $k-1$.*

Remarks. Let $k \geq 1$ and consider the $\mathbf{Z}$-word $w = a^\infty(b^k a)^\infty$. This word is in $\mathcal{S}_k^{\mathrm{nr}}$ (apply Theorem B) and the factor $a^{k+2}b^k ab$ is contained in no element of $\mathcal{S}_k^{\mathrm{rnp}}$ as follows from combining Propositions 3.3 and 3.5. This shows that the inclusion in Proposition 3.3 is strict. At the same time it shows that the recurrency condition in Theorem 3.5 is necessary: in fact w is not k'-balanced for any k'.

Proof of Proposition 3.3. First let $w \in \mathcal{S}_k^{\mathrm{per}}$ with primitive period cycle π and choose n maximal such that $P(w,n) = n+k$. Then $P(w,n-1) < P(w,n) = P(w,n+1)$. The graph G_n is a cycle and because $P(w,n-1) < P(w,n)$ it follows that $\mathrm{MRE}_{n-1}(w)$ contains an element A. Choose a subword τA of w. Then only one of the words Aa, Ab succeeds it in G_n, say $A\tau'$. We let G_n' be the graph which is obtained by adding the edge $\tau A\overline{\tau'}$ to G_n. We write $B_n := \tau A, C_n := A\overline{\tau'}$ for its endpoints. The two paths from B_n to itself in G'_n have labels which we designate by X, Y. Note that $\pi \in \{X, Y\}$ and without loss of generality we take $X = \pi$. Let σ be a $\mathbf{Z}$-word of stiffness 1 and $T(\sigma) = w'$ where $T = (X, Y)$. Then w' has minimal block growth by Theorem B or Theorem 3.1 and we note that $G_n(w') = G'_n$.

We will now show that w' has no jumps at places $\geq n$, hence has stiffness k. Suppose that $x \in \mathrm{MRE}(w')$ satisfies $|x| \geq n$ and let $\Gamma : \mathbf{Z} \to G'_n$ and $\gamma : \mathbf{Z} \to G(T^{\mathrm{RED}})$ be bi-infinite paths induced by w'. We normalize these bi-infinite paths by demanding that $\Gamma(0) = B_n, \gamma(0) = O$, and that the edge $\Gamma(t)\Gamma(t+1)$ has the same label as the edge $\gamma(t)\gamma(t+1)$ for all $t \in \mathbf{Z}$. For all $P \in G'_n$ there is a corresponding point $Q \in G(T^{\mathrm{RED}})$ such that $\Gamma(t) = P$ implies $\gamma(t) = Q$. It is enough to show this for $P = B_n$ and $P = C_n$. For $P = B_n$ the statement is clearly true with $Q = O$, because X, Y can be written as words in $T^{\mathrm{RED}}(a), T^{\mathrm{RED}}(b)$. In G'_n we have two simple paths from B_n to C_n, exactly one of which has length 1. Denote this path by δ_1, the other path by δ_2, and let $\phi := \overline{\tau'}$ be the label of δ_1. By induction we have a Q for every $P \in \delta_2 \setminus \{C_n\}$. Let P_0 be the vertex of δ_2 preceding C_n. If the subpath of δ_2 from B_n to P_0 has label X_1 and if the unique simple path from C_n to B_n (possibly empty) has label X_2, then $X = X_1\phi X_2, Y = \phi X_2$ and $T^{\mathrm{red}} = (X_1, \phi X_2)$. Therefore X_1 can also be seen as a word in $T^{\mathrm{RED}}(a), T^{\mathrm{RED}}(b)$ and the word corresponding to P_0 is O. From this the statement follows for $P = C_n$, hence for all $P \in G'_n$.

Now suppose that two finite subpaths of γ have label x of length $\tau \geq n$. The corresponding paths along Γ have the same endpoint P, which corresponds to the last n symbols of x. Then the subpaths along γ all end in Q. Let $\Phi \in \mathcal{M}$ be such that $T = T^{\mathrm{RED}} \circ \Phi$ and set $\tilde{\sigma} := \Phi(\sigma)$. We can now apply Lemma 3.1.5 with T^{RED} and $\tilde{\sigma}$ to find that $x = B_t(w')$. Hence w' has no jumps at any $t \geq n$ and w' has stiffness k, as claimed.

If x is a finite factor of $w = \pi^\infty = T(a^\infty)$, then x is also a factor of $w' = T(\sigma)$ whenever σ is a sturmian word with a density sufficiently close to 1. Also, x is a factor of w' if σ is an infinite Hedlund word with one of its densities close enough to 1. Then $T\sigma \in \mathcal{S}_k^{\mathrm{rnp}}$ or $T\sigma \in \mathcal{S}_k^{\mathrm{nr}}$, respectively, and this shows that $F(\mathcal{S}_k^{\mathrm{per}})$ is contained in $F(\mathcal{S}_k^{\mathrm{rnp}})$ and in $F(\mathcal{S}_k^{\mathrm{nr}})$.

Now for the other part. Let $w \in \mathcal{S}_k^{\mathrm{rnp}}$ and $x \subset w$ a finite factor. We take n above all jumps such that $B_n = C_n$ and we may assume $x \subset B_n$. If the loops in G_n have labels X, Y, we set $w' := (XY)^\infty$ as a $\mathbf{Z}$-word and this w' contains x. We define the path $\gamma : \mathbf{Z} \to G_n$ by the requirements $\gamma_0 = B_n, \gamma|_{(-\infty,0]}$ has label $(XY)^\infty$ and $\gamma|_{[0,\infty)}$ has label $(XY)^\infty$. For every $k \in \mathbf{N}$ there exists a natural bijection $\mathcal{B}(w', k+n) \to \{$ paths of length k along $\gamma\}$, namely by taking a word in $\mathcal{B}(w', k+n)$ and then reading off its successive n-factors. Also it is clear that each path along γ of positive length has a unique right extension along γ. We conclude $\mathrm{MRE}_n(w') = \{B_n\}$ and $\mathrm{MRE}_{n+1}(w') = \emptyset$. Since $P(w', n) = P(w, n) = n+k$ this implies $w' \in \mathcal{S}_k^{\mathrm{per}}$.□

Proof of Proposition 3.4. Let $w \in \mathcal{S}_k^{\mathrm{nr}}$ with equal period cycles in both directions and $x \subset w$ a finite factor. Again we take n above all jumps of w. The common period cycle in w induces a cycle in G_n which contains B_n, C_n since B_n is contained in the left periodic part and C_n is contained in the right one. It follows that we are in case **b**) or **c**), as described in the proof of Theorem 3.1. Without loss of generality we assume that we are in case **b**) and that $x \subset B_n$. If the loops in G_n have label X, Y we set $w' := (XY)^\infty$ as before and this $w' \in \mathcal{S}_k^{\mathrm{per}}$ contains x. □

Proof of Proposition 3.5. Suppose that w is not k-balanced. We can find in w two subwords A, B with $|A| = |B| = n$ and $c(A) \geq c(B) + k + 1$. We take A, B such that n is minimal. Then $c(A) = c(B) + k + 1$. The graph $G_n(w)$ is strongly connected, as remarked after the proof of Theorem 3.1. Let α be a directed path from A to B of minimal length in G_n and β a directed path from B to A of minimal length. The path α has length $\geq n$ since otherwise a non-empty right-factor of A would equal a left-factor of B, contradicting the minimality of n. Likewise β has length $\geq n$. We define the effect of an arrow $x \to y$ in G_n as $c(y) - c(x)$, which is contained in $\{-1, 0, 1\}$. The effect of a set of edges is simply the sum of the individual effects. We denote the number of edges in $\alpha \setminus \beta$ by s and their total effect by σ. We denote the number of edges in $\beta \setminus \alpha$ by t and their effect by τ. We denote the number of common edges by f and their total effect by ϕ. Then

$$s + t + f \leq n + k + 1.$$

Also $\sigma + \phi = -(k+1), \tau + \phi = k + 1$, hence

$$s + t \geq |\sigma| + |\tau| = |\phi + k + 1| + |k + 1 - \phi| = 2\max(k + 1, |\phi|) \geq 2(k + 1).$$

Combining these inequalities with $s + f \geq n$ and $t + f \geq n$ we find that equality holds everywhere. This implies that

- the path $\alpha\beta$ is induced by the word ABA;
- the number of $(n+1)-$words appearing in $\alpha\beta$ equals $n + k + 1$ and hence that these words form $\mathcal{B}(w, n+1)$;

- all edges in $\alpha \setminus \beta$ are of the form $a * b$ with $*$ a word of length $n-1$;

- all edges in $\beta \setminus \alpha$ are of the form $b * a$.

Now suppose that an edge from α also appears in β. Let $P \to Q$ be the first such edge (following α). Then $P \neq A$ because β finishes when arriving in A. The arrow to P in α is of the form $a * b$. The arrow to P in β is of the form $b * a$. This is a contradiction, because the last symbol of P cannot be a and b at the same time. Hence $f = 0 = \phi, s = t = n$. But then all edges in α are of the form $a * b$, hence $A = a^n, B = b^n$. From $c(A) = c(B) + k + 1$ we read off $n = k+1$, but then $P(s,n) = |\{\text{vertices in } ABA\}| = 2n > n+k$, contradicting that w is k-stiff.

Now let σ be an arbitrary sturmian word containing b^k and T the substitution with $T(a) = a^k, T(b) = b$. Then $T\sigma$ is k-stiff by Theorem 3.1, it is recurrent and obviously not $(k-1)-$balanced since $a^k, b^k \subset T\sigma$. Therefore the second k is indeed sharp. □

Corollary. *If σ is a balanced $\mathbf{Z}$-word, then $T\sigma$ is $k(T)$-balanced.*
Indeed, every finite factor $x \subset \sigma$ is contained in a sturmian $\mathbf{Z}$-word σ' by Proposition 3.3. Now $T\sigma'$ is recurrent and by Theorem 3.1 it is $k(T)$-stiff. Applying Proposition 3.5 we find that $T\sigma'$ is $k(T)$-balanced and the same then holds for Tx. Since x was any factor, the result follows. □

3.3. More counting theorems. In this section we use Theorems A,B and 3.1 to give, for fixed k and variable n, an upper bound for the number of words x of length n which are contained in some k-stiff $\mathbf{Z}$-word w, hence for $|F_n(\mathcal{S}_k)|$. First we deal with recurrent w and by Proposition 3.3 we may restrict ourselves to w which are not periodic.

Theorem 3.6. *For every $k \in \mathbf{N}$ there exists a constant $\mathcal{C}_k$ such that $|F_n(\mathcal{S}_k^{\mathrm{rnp}})| \leq \mathcal{C}_k n^3$.*

The next theorem estimates the number of words of length n which appear in a non-recurrent w of stiffness k, but not in a k-stiff recurrent $\mathbf{Z}$-word.

Theorem 3.7. $|F_n(\mathcal{S}_k^{\mathrm{nr}}) \setminus \cup_{i=1}^{k} F_n(\mathcal{S}_i^{\mathrm{rnp}})| \leq 2^k (n+k)^3$.

Combining these theorems we see that $|F_n(\mathcal{S}_k)| = O(n^3)$ when k is fixed. Of course this does not help very much to estimate the total number of k-stiff words of length n.

The properties stiff and k-stiff do not seem to lie far apart. The situation is very different for balanced and k-balanced since $\mathrm{bal}_k(n)$, the number of k-balanced words of length n, is exponentially large in n when $k \geq 2$. We prove the following theorem.

Theorem 3.8. *For every $k \geq 2$ there exist positive constants c, d, C_k, D_k with $3^{1/3} \leq C_k \leq D_k < 2$ such that $c \cdot C_k^n \leq \mathrm{bal}_k(n) \leq d \cdot D_k^n$ for all n and such that $\lim_{k\to\infty} C_k = 2$.*

Lemma 3.6.1. *For $k \geq 1$ there exist exactly $\nu := (k^2 + k + 2)2^{k-3}$ irreducible substitutions with $k(T) = k$.*

Proof. Write $Ta = A\sigma C, Tb = B\overline{\sigma}C$ as before. Then $|ABC| = k - 1$. It immediately follows that there exist only finitely many T and for the explicit calculation we distinguish between four cases. If $A, B = \emptyset$ then $Ta = aC, Tb = bC$ where $|C| = k - 1$ and we have 2^{k-1} choices. If $A = \emptyset \neq B$ then $Ta = aC, Tb = b\tilde{B}bC$ where $|\tilde{B}C| = k - 2$ and we have $2^{k-2}(k - 1)$ choices. The case $B = \emptyset \neq A$ is similar and gives the same number. If $A, B \neq \emptyset$ then $Ta = a\tilde{A}\sigma C, Tb = b\tilde{B}\overline{\sigma}C$ where $|\tilde{A}\tilde{B}C| = k - 3$ and we have $2^{k-2}\binom{k-1}{2}$ choices. The total number then becomes $2^{k-1} + 2^{k-1}(k - 1) + 2^{k-2}\binom{k-1}{2} = (k^2 + k + 2)2^{k-3}$. □

Proof of Theorem 3.6. Let $w \in \mathcal{S}_k^{\mathrm{rnp}}$ and $x \in \mathcal{B}_n(w)$. We know by Theorem 3.1 that $w = T(\sigma)$ where σ is sturmian, T is irreducible and $k(T) = k$. We denote the set of these T by $\{T_1, \cdots, T_\nu\}$, where ν is defined in the previous lemma. There exist y, z, ξ such that $xy = zT(\xi)$ where z is a strict right-factor of X or Y and $\xi \subset \sigma$ (possibly empty). Then $x = [zT(\xi)]_n$. We will write Z for the set of possible z and define $\phi : Z \times \{T_i\} \times \mathrm{Bal}(n) \to \{a, b\}^n$ by $\phi(z, T, \xi) = [zT(\xi)]_n$. The image of Φ contains all words in $F_n(\mathcal{S}_k^{\mathrm{rnp}})$. Therefore $|F_n(\mathcal{S}_k^{\mathrm{rnp}})| \leq \nu|Z|\mathrm{bal}(n)$ and the result now follows from Corollary 2.8.1. □

Example. If $k = 2$ then $T \in \{(aa, b), (a, bb), (ab, bb), (aa, ba)\}, Z = \{\emptyset, a, b\}$ and $|F_n(\mathcal{S}_2^{\mathrm{rnp}})| \leq 12\mathrm{bal}(n)$.

Lemma 3.7.1. *Let $r, s \in \mathbf{N}^+, d = (s, r), \phi = r + s - 2$. The total number of $\mathbf{Z}$-words w with left-period s, right-period r (not per se minimal) and exact overlap g equals $\max(2^d, 2^{\phi-g})$ if $g \leq \phi + 1 - d$ and 0 otherwise.*

Proof. We first deal with $g < 0$. Then $w = A^\infty BC^\infty$ with $|A| = s, |B| = -g, |C| = r$, A, B have different initial symbols and B, C have different terminal symbols. Choose the symbols in B arbitrarily. This yields only 2 letter restrictions on A, C, hence we find $2^{|B|+(r-1)+(s-1)} = 2^{\phi-g}$ for the number of possible w. Note that $\phi - g \geq d$.

Now for $g \geq 0$, we first consider the case $d = 1$. The proof is basically a generalisation of Tijdeman's proof in [T, Section 4] when $d = 1$ and $g = \phi$. We will use Tijdeman's result that if $x_0 = 0$ and x_n is inductively defined by

$$x_{n+1} = \begin{cases} x_n + s \text{ if } \mathrm{x_n < r} \\ x_n - r \text{ if } \mathrm{x_n \geq r} \end{cases},$$

then $\{x_0, \cdots, x_{\phi+1}\} = \{0, \cdots, \phi + 1\}$ and also $x_{\phi+1} = r$.
Now first suppose that $g > \phi$ and without loss of generality that $[1, \phi + 1]$ lies in the intersection of the periodic parts. If we define $(x_n)_0^\infty$ as above and write $\sigma_n = w_{x_n}$, then $\sigma_n = \sigma_{n+1}$ for $n \leq \phi$, hence w is constant on $[1, \phi+1]$. Since $\phi+1 = r+s-1 \geq r, s$ we see that the **Z**-word w is constant, a contradiction.
Now let $g \leq \phi$ and assume, without loss of generality, that the intersection of the periodic parts of w is situated at $[1, g]$. At first we have to fill in $(\sigma_n)_0^{\phi+1}$ subject to the condition that $\sigma_n = \sigma_{n+1}$ when $x_{n+1} \leq x_n$ or $x_{n+1} \leq g$. Then $\sigma_n \neq \sigma_{n+1}$ is only possible when $x_{n+1} > x_n, g$, hence if

$$x_n \in [\max(g - s + 1, 0), r - 1].$$

Note that this interval is not empty because $g \leq \phi$. This yields $1 + |[\max(g - s + 1, 0), r - 1]| = r + 1 - \max(0, g - s + 1)$ choices. Next we have to fill in the $\max(0, s - (g + 1))$ symbols to the left of the 0-position to complete the left period cycle. The total number of choices is then $r + 1 - \max(0, g - s + 1) + \max(0, s - g - 1) = r + 1 + s - g - 1 = \phi + 2 - g$, which is apparently the number of **Z**-words with left-period s, right-period r (we will abbreviate this as s-r **Z**-words) with $[1, g]$ in the overlap. The number of s-r **Z**-words with $[1, g]$ in the overlap but not $[1, g + 1]$ then equals

$$\begin{cases} 2^{\phi+2-g} - 2^{\phi+1-g} = 2^{\phi+1-g}, & \text{if } g < \phi; \\ 2^{\phi+2-g} - 2 = 2, & \text{if } g = \phi. \end{cases}$$

The second formula follows from the fact just proved that w is constant if

$g > \phi$. Hence the number of s-r $\mathbf{Z}$-words with exact overlap $[1, g]$ equals

$$\begin{cases} 2^{\phi+1-g} - 2^{\phi-g} = 2^{\phi-g}, & \text{if } g < \phi; \\ 2, & \text{if } g = \phi. \end{cases}$$

This proves our theorem when $(r, s) = 1$.

Now for the case $(r, s) = d > 1$. We write $r = d\rho, s = d\sigma$ and $t = \rho+\sigma-2$. If an s-r $\mathbf{Z}$-word has finite overlap then the overlap contains at most t elements from a certain residue-class λ mod d, for otherwise the $\mathbf{Z}$-word would be constant on every residue class modulo d, and this would imply periodicity. Hence $g \leq t + (t+1)(d-1) = td + d - 1 = r + s - d - 1 = \phi + 1 - d$. Now suppose that an s-r $\mathbf{Z}$-word has finite overlap containing $[1, g]$ where $g \leq \phi + 1 - d$ and write $c(\lambda) = |(\lambda \text{ mod } d) \cap [1, g]|$ for $\lambda \in \mathbf{Z}$. The number of choices is then $\sum_{\lambda=1}^{d} (t + 2 - c(\lambda)) = d(t+2) - g = \phi + 2 - g$ and the rest is similar to the discussion when $d = 1$. □

Proof of Theorem 3.7. Let $x \in F_n(\mathcal{S}_k^{\text{nr}}) \setminus \cup_{i=1}^{k} F_n(\mathcal{S}_i^{\text{rnp}})$ and w a non-recurrent $\mathbf{Z}$-word of stiffness k containing x. Write s, r as usual for the minimal periods of w. Without loss of generality we write $(-\infty, g], [1, \infty)$ for its maximal periodic domains. By Proposition 3.4 the period cycles are not conjugate. If $[\lambda, \lambda+n-1]$ is a domain for x in w, then $g+2-n \leq \lambda \leq 0$, for x is contained in no periodic part of w. Hence for given w it follows that the number of possible x is at most $n - 1 - g = n + k - \chi \leq n + k$. From the above we also conclude $g \leq n - 2$, hence $r, s \leq r + s = \chi = k + 1 + g \leq n + k$. For fixed s, r the number g is determined by $g = r + s - 1 - k$ and we have at most $\max(2^d, 2^{\phi-g})$ possibilities for w by Lemma 3.7.1. Now $g \leq \phi + 1 - d$, again by Lemma 3.7.1, and substituting $g = \phi + 1 - k$ this gives $d \leq k$. Also $\phi - g = k - 1$ and therefore each choice s, r yields at most 2^k possible w. Combining all previous inequalities we have $|F_n(\mathcal{S}_k^{\text{nr}}) \setminus \cup_{i=1}^{k} F_n(\mathcal{S}_i^{\text{rnp}})| \leq 2^k(n+k)^3$. □

Lemma 3.8.1. *Let N, k, ϕ be positive integers with $2\phi \leq k, N - \phi \leq k$ and $\mathcal{A} = \{w| \ |w| = N, c(w) = \phi\}$. Then every element of $\mathcal{A}^\infty$ is k-balanced.*

Proof. For any $w \in \mathcal{A}^\infty$ there is a partition of $\mathbf{Z}$ into intervals $I_i = p + [Ni, Ni + N - 1]$ such that the word with domain I_i is contained in $\mathcal{A}$ for all i. Now let us assume that $w \in \mathcal{A}^\infty$ is not k-balanced. Choose subwords $A, B \subset w$ of equal length n such that $|c(A) - c(B)| > k$ and choose domains $D(A), D(B)$ for A and B. We can write $D(A) = PQR, D(B) = STU$ where

Q, T are unions of I_i's and where P, R, S, U are strict subintervals of some I_i. We will identify P, Q, R, S, T, U with the subwords they induce and we write $|Q| = \lambda N, |T| = \mu N$ where $\lambda, \mu \geq 0$. Comparing lengths and contents we find

$$\begin{cases} |PR| + \lambda N = |SU| + \mu N \\ |(c(PR) + \lambda\phi) - (c(SU) + \mu\phi)| > k \end{cases}$$

We have $|\lambda - \mu| N = ||PR| - |SU|| \leq \max(|PR|, |SU|) < 2N$ hence $|\lambda - \mu| \leq 1$. If $\lambda = \mu$ then the second inequality would imply $k < |c(PR) - c(SU)| \leq \max(c(PR), c(SU)) \leq 2\phi \leq k$, a contradiction. It follows that it is safe to assume that $\lambda = \mu + 1$, interchanging A and B if necessary. Counting the number of b's in SU we find $|SU| - c(SU) \leq 2(N - \phi)$ hence

$$|PR| \geq c(PR) > k - \phi + c(SU) \geq |SU| + k + \phi - 2N = |PR| + k + \phi - N \geq |PR|$$

This contradiction completes the proof. □

Proof of Theorem 3.8. Let $k \geq 2$ and choose $\phi, N \in \mathbf{N}$ such that $2\phi, N - \phi \leq k$. For instance $\phi = \lfloor \frac{k}{2} \rfloor, N = \phi + k$ will do. Now $\mathrm{bal}_k(i)$ is increasing in i (this follows from Theorem 2.3) and applying the previous lemma we find

$$\mathrm{bal}_k(n) \geq \mathrm{bal}_k(N\lfloor \frac{n}{N} \rfloor) \geq \binom{N}{\phi}^{\lfloor n/N \rfloor} \geq \binom{N}{\phi}^{-1} \binom{N}{\phi}^{n/N} =: c \cdot C_k^n$$

In particular we can take $C_2 = 3^{\frac{1}{3}}$ and because $\mathrm{Bal}_2(n) \subset \mathrm{Bal}_k(n)$ for $k \geq 2$ we can take all $C_k \geq 3^{\frac{1}{3}}$. The choice $\phi = \lfloor \frac{k}{2} \rfloor, N = k$ gives $C_k = \binom{k}{\lfloor k/2 \rfloor}^{1/k}$. The inequalities $2^{2\kappa} \leq (2\kappa + 1)\binom{2\kappa}{\kappa}$ and $2^{2\kappa+1} \leq (2\kappa + 2)\binom{2\kappa+1}{\kappa}$ then show that $C_k \geq \frac{2}{\sqrt[k]{k+1}}$, hence $\lim_{k \to \infty} C_k = 2$. Now for the upper bound, let $N = 2k + 2$. Then $\mathrm{bal}_k(N) =: \lambda < 2^N$, hence

$$\mathrm{bal}_k(n) \leq \mathrm{bal}_k(N\lceil \frac{n}{N} \rceil) \leq \lambda^{\lceil n/N \rceil} \leq \lambda \cdot (\lambda^{1/N})^n =: d \cdot D_k^n$$

with $D_k = \lambda^{1/N} < 2$. □

3.4. The interval coding property. Let $\mathbf{T} = \mathbf{R}/\mathbf{Z}$ be the unit circle and consider θ, ϕ, V where $\theta, \phi \in \mathbf{R}$ and where $V \subset \mathbf{T}$ is a finite union of intervals. It is possible to define a $\mathbf{Z}$-word w by $w_i = a \iff \theta + i\phi \in V$ and if a $\mathbf{Z}$-word w can be obtained in this way, we say that w has the interval coding property (ICP). The interval coding property is closely related, but not identical, to the notion of interval exchange as introduced in [K]. In [D] interval codings with values with values in a g-letter alphabet are considered.

All periodic $\mathbf{Z}$-words have ICP and it follows from Section 2.5.2 that sturmian $\mathbf{Z}$-words have ICP. We will consider the question which recurrent $\mathbf{Z}$-words w of minimal block growth have ICP. We know that $w = T\sigma$ with T an irreducible substitution and σ a sturmian $\mathbf{Z}$-word and we will give a characterization in terms of T. We recall from Section 3.1 that a substitution T is called trivial if and only if Ta, Tb are both powers of some word π.

Definition 3.2. *Let T be a substitution. Then T has ICP if and only if $T\sigma$ has ICP for some sturmian $\mathbf{Z}$-word σ.*

Theorem 3.9. *If $T = (X, Y)$ is a non-trivial substitution, then the following statements are equivalent:* a) $(|X|, |Y|) = 1$*;* b) *T has ICP;* c) *$T\sigma \in$ ICP for all sturmian $\mathbf{Z}$-words σ.*

First we will prove this theorem and then we will give an example to clarify its proof. For the next lemma we define an equivalence relation on the set of substitutions by calling two substitutions $T = (X, Y)$ and $\tilde{T} = (\tilde{X}, \tilde{Y})$ equivalent if $|X| = |\tilde{X}|, |Y| = |\tilde{Y}|$. We write $T \sim \tilde{T}$.

Lemma 3.9.1. *Suppose that $T = (X, Y)$ is irreducible and has ICP. If $\tilde{T} \sim T$, then $\tilde{T}$ has ICP.*

Proof. Let σ be a sturmian $\mathbf{Z}$-word such that $T\sigma =: w$ has ICP. Choose θ, ϕ, V as in the definition. We define the representing graph of T as in the proof of Theorem 3.1. There exists a path $\gamma : \mathbf{Z} \to G(T)$ such that the edge $\gamma_{i-1}\gamma_i$ has label w_i for all $i \in \mathbf{Z}$. Take $n \in \mathbf{N}^+, x \in \mathcal{B}(w, n)$ and assume that two paths along γ exist with label x and different endpoints. We define μ, M, N as in Lemmata 3.1.1,3.1.6 and assume that $n \geq \mu$. Then we can still apply the proof of Lemma 3.1.6 and we conclude that σ contains an M-periodic factor of length $\lceil \frac{n}{\mu} - 1 - N \rceil$. For large n this is a contradiction. Hence, for n large, all paths with a given label $x \in \mathcal{B}(w, n)$ have the same endpoint in $G(T)$. Because $n \geq \mu$ and T is irreducible, this implies the following:

for n large, all paths with a given label $x \in \mathcal{B}(w, n)$ have the same final edge in $E(G(T))$

Therefore Lemma 3.9.1 will follow from the following weaker lemma which we have isolated for future reference. Having noted this we conclude the proof of Lemma 3.9.1. □

Lemma 3.9.2. *Let $T = (X, Y)$ be a substitution and σ a sturmian $\mathbf{Z}$-word such that $w := T\sigma \in \mathrm{ICP}$. Suppose that, for n large and $x \in \mathcal{B}(w, n)$, all paths in $G(T)$ with label x have the same final edge. Also assume that $\tilde{T} \sim T$. Then $\tilde{T}\sigma \in \mathrm{ICP}$.*

Proof. Since $w \in \mathrm{ICP}$ we can choose θ, ϕ, V as in the definition. The final edge condition induces a surjective mapping $\chi_n : \mathcal{B}(w, n) \to E(G(T))$. (It is surjective because $a, b \in \sigma$). Since $|X| = |\tilde{X}|, |Y| = |\tilde{Y}|$, we can define Γ as the graph with the same vertex set as $G(T)$ but where the labels of α, β are now $\tilde{X}, \tilde{Y}$. Because $\gamma : \mathbf{Z} \to \Gamma$, we can define $\tilde{w}_i$ as the Γ-label of $\gamma_{i-1}\gamma_i$ for all i and it is clear that $\tilde{w} = \tilde{T}(\sigma)$.

Let Σ be the collection of edges of Γ that have label a and $\Lambda := \chi_n^{-1}(\Sigma) \subset \mathcal{B}(w, n)$. Then

$$\tilde{w}_i = a \iff \gamma_{i-1}\gamma_i \in \Sigma \iff w_{i-(n-1)} \cdots w_i \in \Lambda.$$

We write $V_a := V, V_b := V^c$. For $\lambda := \lambda_1 \cdots \lambda_n \in \{a, b\}^n$ we then have

$$w_{i-(n-1)} \cdots w_i = \lambda \iff \theta + i\phi \in \cap_{t=1}^n ((n-t)\phi + V_{\lambda_t}).$$

It follows that $\tilde{w}_i = a \iff \theta + i\phi \in \cup_{\lambda \in \Lambda} \cap_{t=1}^n (((n-t)\phi + V_{\lambda_t})$ and this representation shows that $\tilde{w}$ has ICP, proving our lemma. □

Lemma 3.9.3. *Let $T = (X, Y)$ be irreducible with $d := (|X|, |Y|) > 1$. Then $T \notin ICP$.*

Proof. Write $|X| =: d\xi, |Y| =: d\eta$ and suppose that T has ICP. By Lemma 3.9.1 we also have $\tilde{T} = (a^{|X|}, (a^{d-1}b)^\eta) \in ICP$. Choose a sturmian $\mathbf{Z}$-word σ such that $\tilde{T}\sigma =: w \in ICP$. Then w is not periodic, hence $\phi \notin \mathbf{Q}$. Also $w_{i_0+d\mathbf{Z}} = a$ for some i_0 and because $\theta + (i_0 + d\mathbf{Z})\phi$ lies dense in $\mathbf{T}$ this means that V^c is finite. But then $w_i = a$ for all but finitely many n and $\alpha(w) = 0$, a contradiction. □

Lemma 3.9.4. *Let $T = (X, Y) \in ICP$ be non-trivial. Then $d = 1$.*

Proof. We can assume $[X]_1 = a, [Y]_1 = b$. We know that $T = T' \circ \Phi$ with T' irreducible and $\Phi \in \mathcal{M}$. Let σ be a sturmian **Z**-word such that $T\sigma \in ICP$. With $\tau := \Phi(\sigma)$ we have $T'(\tau) \in ICP$. By the previous lemma we have $(|X'|, |Y'|) = 1$ and an easy induction shows that $(|X|, |Y|) = (|X'|, |Y'|)$. Hence $d = 1$. □

Proof of Theorem 3.9. Of course c)⇒ b) and from Lemma 3.9.4 it follows that b)⇒ a). Now suppose that S is a substitution with $d = 1$. By parts **b** and **f** of Theorem 2.14 there exists a $T \in \mathcal{M}$ with $T \sim S$ and by the procedure formulated just after Theorem B we can ensure that Ta, Tb have distinct initial symbols. Also we can assume $[Ta]_1 = a, [Tb]_1 = b$. Clearly $T\sigma \in ICP$ for all sturmian **Z**-words σ since $T\sigma$ is sturmian. We will now show that T and σ satisfy the hypotheses of Lemma 3.9.2. Then c) follows for S and we are done. □

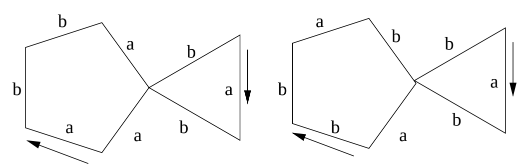
Lemma 3.9.5. *Let $T = (X, Y) \in \mathcal{M}$ with $[X]_1 \neq [Y]_1$, let $G(T)$ be its representing graph and $s := |X|, r := |Y|, g := r + s - 2$. Then any two paths in $G(T)$ with the same label of length $g + 1$ have the same final edge.*

Proof. Let $w = X^\infty Y^\infty$, where X^∞ is left-infinite and Y^∞ is right-infinite. Then w is an infinite Hedlund word by Proposition 2.12 and its finite Hedlund word P is a common suffix of the right-infinite words X^∞, Y^∞ of length g. We consider $G := G_g(w)$, the g-th word graph of w. Every path of length s with initial vertex A is of the form $A\pi$ where $|\pi| = s$. Then $A\pi$ is the label of the path and we call π the reduced label of the path. It indicates which symbols one adds to the right when walking through the path. Now let w_l, w_r be the maximal left- and right-periodic subwords of w, respectively, and define $G_l := G_g(w_l), G_r := G_g(w_r)$. Note that they are cycles in G of length s, r, respectively. Every $x \in \mathcal{B}(w, g + 1)$ is contained in some periodic part of w, hence $s + r = g + 2 = P(w, g + 1) \leq P(w_l, g + 1) + P(w_r, g + 1) \leq r + s$. It follows that $\mathcal{B}(w_l, g + 1) \cap \mathcal{B}(w_r, g + 1) = \emptyset$ or, in other words, that $G_l, G_r \subset G$ are edge-disjoint. Since $P \in w_l, w_r$ we deduce that G consists of two cycles from P to itself of length s, r and considering w_l, w_r we find that these cycles have label PX, PY. Therefore their reduced labels are X, Y. Now define $\Phi : G(T) \to G$ as the unique isomorphism of graphs. If we give each edge in G its reduced label, then Φ respects the labelling of the edges by the previous part. It follows that the lemma for $G(T)$ is equivalent to the lemma for G, where it is trivial. Indeed, by definition of reduced label, each path with label x of length $\geq g+1$ has final edge $[x]^{g+1}$.□

We devote the remainder of this section to the promised example.

Example. Let $\alpha := \frac{1}{\sqrt{2}}$, $\sigma := \{n \in \mathbf{Z} | \mathrm{fr}(n\alpha) \in [0,\alpha)\} = \{\lceil \frac{n}{\alpha} \rceil\}_{n \in \mathbf{Z}}, S := (aabba, bab)$ and let $w := S\sigma$. By Theorem 3.9 we have $w \in ICP$ and we want to give explicit θ, ϕ, V for w. First we construct a sturmian T with $T \sim S$, hence with $(|Ta|, |Tb|) = (5, 3)$.

Consider the sequence $(5,3) \to (2,3) \to (2,1) \to (1,1)$. For $(1,1)$ we obviously have the solution $T = (a,b)$ and working back we find $(a,b) \to (ab,b) \to (ab,bab) \to (abbab,bab)$. Hence we put $T = (abbab,bab)$. The proof of Lemma 3.9.4 shows that P is the common suffix of length 6 of the left-infinite words X^∞, Y^∞, hence $P = babbab$. We put $x := T\sigma$. Then x is sturmian and an explicit calculation shows that $x = \{\lceil \frac{(2\alpha+3)n}{\alpha+1} \rceil\}$. For this one might use the formulas right after Proposition 2.12. Let $\Gamma := G(T)$.



G(S) Γ

For every $p \in \mathcal{B}(x,7)$ there exists an edge so that all paths in Γ with label x end in that edge. The mapping $\chi_7 : \mathcal{B}(x,7) \to E(\Gamma)$ from the proof of Lemma 3.9.1 sends p to this edge and we denote by $\Phi(p)$ the reduced label in $G(S)$ of this final edge. Then $\Phi : \mathcal{B}(x,7) \to \{a,b\}$ and, very specifically, Φ is given by

$$\begin{array}{lcl} babbaba \to a & , & babbabb \to b \\ abbabab \to a & & abbabba \to a \\ bbababb \to b & & bbabbab \to b \\ bababba \to b & & \\ ababbab \to a & & \end{array}$$

The mapping Φ has a natural extension to x, namely by reading off the successive 7-factors of x and then writing down, successively, their images under Φ. It is then clear that $\Phi x = w$.

We have seen that x has ICP and, with $\beta := \frac{\alpha+1}{2\alpha+3}$, that x can be described with $\theta = 0, \phi = \beta, V = [0,\beta)$. As before we define $V_a = V, V_b = V^c$

and for $\lambda \in \{a,b\}^n$ we define $V_\lambda = \cap_{t=1}^n((n-t)\phi + V_{\lambda_t})$. We have also seen in the proof Lemma 3.9.1 that

$$x_{i-(n-1)} \cdots x_i = \lambda \iff \theta + i\phi \in V_\lambda.$$

The numbers $0, 3\beta, 6\beta, \beta, 4\beta, 7\beta, 2\beta, 5\beta, 1$ lie in this order on the unit circle $\mathbf{T}$ and as an example we calculate that for $\lambda = babbaba \subset x$ we have $V_\lambda = V_a \cap (V_b+\beta) \cap (V_a+2\beta) \cap (V_b+3\beta) \cap (V_b+4\beta) \cap (V_a+5\beta) \cap (V_b+6\beta) = [0,\beta) \cap [2\beta, \beta+1) \cap [2\beta, 3\beta) \cap [4\beta, 3\beta+1) \cap [5\beta, 4\beta+1) \cap [5\beta, 6\beta) \cap [7\beta, 6\beta+1) = [0, 3\beta)$. The multiples $(n\beta)_0^7$ divide $\mathbf{T}$ into 8 intervals, therefore it is a priori clear that every $x \in \mathcal{B}(w,7)$ gives us exactly one of these intervals. We find

$$\begin{array}{lcl} babbaba \to [0,3\beta) & , & babbabb \to [5\beta,1) \\ abbabab \to [\beta,4\beta) & & abbabba \to [6\beta,\beta) \\ bbababb \to [2\beta,5\beta) & & bbabbab \to [7\beta,2\beta) \\ bababba \to [3\beta,6\beta) & & \\ ababbab \to [4\beta,7\beta) & & \end{array}$$

We are now ready to give the explicit description for w. Write $\Lambda := \Phi^{-1}(a)$. From $w = \Phi x$ it follows that, modulo shift, we have $w_i = a \iff \Phi(x_{i-(n-1)} \cdots x_i) = a \iff x_{i-(n-1)} \cdots x_i \in \Lambda \iff \theta + i\phi \in \cup_{\lambda \in \Lambda} V_\lambda = [0,3\beta) \cup [\beta,4\beta) \cup [4\beta,7\beta) \cup [6\beta,\beta) = [0,3\beta) \cup [6\beta,7\beta)$. It follows from the equipartition of $\theta + i\phi$ modulo 1 and this representation that $\alpha(w) = |[0,3\beta) \cup [6\beta,7\beta)| = (3\beta-1) + (7\beta-2) - (6\beta-2) = 4\beta - 1 = \frac{2\alpha+1}{2\alpha+3}$. This value also follows directly from the fact that $S(a^k b^l)$ contains $3k+l$ a's and $2(k+l)$ b's, but we leave these details to the reader.

Chapter 4.
The $P(n)/n$-function for bi-infinite words.

In the previous chapter we studied **Z**-words w of minimal block growth and for such w we obviously have $\lim_{n\to\infty} P(n)/n = 1$. The following theorem gives, for every integer $r \geq 1$, an easy construction of a recurrent w with $P(n)/n \to r$. Its proof will be given at the end of the chapter.

Theorem 4.0. *Let σ be a sturmian $\mathbf{Z}$-word and $r \geq 1$ an integer. Define w by $w_i = \sigma_i \iff i \not\equiv 0 \bmod r$. Then $P(w,n) = r(n+1)$ for n large enough.*

Remark. J.-P. Allouche and J. Berstel have posed the problem for which integers α, β there exist infinite words with complexity function $P(w,n) = \alpha n + \beta$. If one requires this to hold for all $n \geq 0$ then obviously $\beta = 1$. In [Ca, Cor. 5.2] this problem is solved, in the sense that Cassaigne shows for which $(\alpha, \beta) \in \mathbf{Z}^2$ there exists a binary sequence (one-sided infinite word on 2 symbols) w with ultimate complexity $\alpha n + \beta$. For $\alpha = 0, 1$ the only condition is $\beta \geq 0$ as we already know and for $\alpha \geq 2$ there is no restriction on β. Theorem 4.0 supplies a construction for $\alpha = \beta = r$ and to our knowledge this construction is new. Finally we mention the paper [Lo/Na] in which geometric methods are used to construct words of ultimate complexity $\alpha n + \beta$.

Remark. The results quoted above usually deal with **N**-words. We emphasize that the theorems and proofs that follow hold equally well for **N**-words, provided one assumes that they are recurrent.

In this chapter we will study the sequence $(a_n)_1^\infty$ with $a_n := P(n)/n$ a little more closely. Put $\Delta_n := P(n+1) - P(n)$, then clearly $\Delta_n = |\mathrm{MRE}_n(w)| = |\mathrm{MLE}_n(w)|$. The following theorem implies that $P(n)/n$ cannot converge to a value $\alpha \in (1,2)$.

Theorem 4.1. *Let w be a $\mathbf{Z}$-word, $P(n)$ its complexity function and $\Delta_n := P(n+1) - P(n)$. Assume that $\Delta_n = 1$ for infinitely many n and that $\Delta_n > 1$ for infinitely many n. Let V be the infinite set $\{n \in \mathbf{N}^+ | \Delta_n = 1, \Delta_{n+1} > 1\}$,*

$\alpha := \liminf_{n\to\infty, n\in V} P(n)/n$ and $d := d(\alpha, \mathbf{Z})$. Then

$$\limsup_{m,n\to\infty} \left|\frac{P(m)}{m} - \frac{P(n)}{n}\right| \geq \frac{d\alpha}{6+7\alpha+2d}.$$

Corollary. *If $\Delta_n = 1$ for infinitely many n and $P(n)/n$ is convergent with limit α, then α is an integer.*

In the next theorem we give bounds which are a little sharper when $1 < \alpha < 2$.

Theorem 4.2. *Maintain the assumptions of the previous theorem and furthermore suppose that $1 < \alpha < 2$. Then*

$$\limsup_{m,n\to\infty} \left|\frac{P(m)}{m} - \frac{P(n)}{n}\right| \geq \frac{(2-\alpha)(\alpha-1)}{\alpha}.$$

The next theorem shows in fact that the bound for $\alpha = 3/2$ as given by Theorem 4.2 is sharp. See also Stelling 5 at the inserted page.

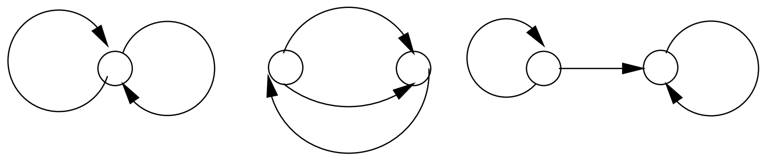
Theorem 4.3. *There exists a recurrent $\mathbf{Z}$-word w with $\liminf(\frac{P(n)}{n}) = 3/2$ and $\limsup(\frac{P(n)}{n}) = 5/3$.*

The final result shows that limit 2 can occur if $\Delta_n = 1$ infinitely often.

Theorem 4.4. *There exists a recurrent $\mathbf{Z}$-word with $\Delta_n = 1$ for infinitely many n such that $P(n)/n \to 2$ as $n \to \infty$.*

Proof of Theorem 4.1. We may assume without loss of generality that $\alpha \notin \mathbf{Z}$, hence that $\alpha \in (t, t+1)$ where t is a positive integer. We assume that $a_n := P(n)/n \in (t, t+1)$ for n large since otherwise we are done because of $d \geq \frac{d\alpha}{6+4\alpha+2d}$. We have $a_{n+1} - a_n = \frac{na_n+\Delta_n}{n+1} - a_n = \frac{\Delta_n - a_n}{n+1}$. By the assumption above it follows that for n large we have $a_{n+1} > a_n \iff \Delta_n > t$. We will use the sequence of word graphs $(G_n)_{n=0}^{\infty}$, as introduced in the proof of Theorem 3.1, for our w. We recall that every factor $x \subset w$ of length $\geq n$ corresponds uniquely to a path of length $|x| - n$ by reading off the consective n-factors. In other words, $x = x_1 \cdots x_{n+k}$ corresponds to the path $x_1 \cdots x_n \to \cdots \to x_k \cdots x_{n+k}$ and x was called the label of this path. We note that $\mathrm{MRE}_n(w)$, the n-factors with m.r.e., correspond to the vertices of G_n with outdegree 2 and, likewise, that $\mathrm{MLE}_n(w)$ corresponds to the vertices of G_n with indegree 2. Suppose that $n \in V$. The underlying graph of

G_n is connected, each vertex has positive indegree and G_n has one vertex with indegree 2 and one vertex of outdegree 2. We then have three types of graphs which are possible for G_n.



In the second and third case we see that $\mathrm{MLE}_n(w) \cap \mathrm{MRE}_n(w) = \emptyset$, hence for every $x \in \mathrm{MRE}_n(w)$ we have a unique symbol σ such that $\sigma x \subset w$ and we must have $\sigma x \in \mathrm{MRE}_{n+1}(w)$. It follows that $\Delta_{n+1} = \Delta_n$, in contradiction with the definition of V. (Actually, if the third case occurs one can show that all following graphs are of this type too. Then $P(i) = i + k$ for some k and all $i \geq n$. This implies $\alpha = 1$, contradicting the assumption at the start of this proof. Hence the third case does not occur.) Therefore we find ourselves in the first case if $n \in V$. Note that $\Delta_{n+1} = 2$. Let B be the unique element of $\mathrm{MRE}_n(w)$ and let X, Y be the words such that BX, BY are the labels of the two cycles in G_n. We note that X, Y must have different initial symbols and that $BX = \xi B, BY = \eta B$ for some words ξ, η. We write $x := |X|, y := |Y|$ and assume without loss of generality that $x \geq y$. Then $x + y = P(n+1) = (n+1)a_{n+1}$ and

$$\frac{a_{n+1}}{2} \leq \frac{x}{n+1} \leq a_{n+1} \qquad (1).$$

Lemma 4.1.1. *Let $n \in V$ and B, X, Y, ξ, η as above.*
a) *Elements of $\mathrm{MRE}_k(w)$ with $n \leq k < n + x$ which have no or two left-extensions in $\mathrm{MRE}_{k+1}(w)$ are of the form BY^* and are suffixes of the left-infinite word Y^∞.*
b) $\Delta_{k+1} \in \{\Delta_k - 1, \Delta_k, \Delta_k + 1\}$ *for $n \leq k < n + x$*
c) *if $\Delta_{k+1} = \Delta_k - 1$ then $\Delta_{k+1} = \cdots = \Delta_{n+x}$.*
d) *There exists a partition of $[n, n+x]$ in at most three intervals on each of which $P(k)/k$ is monotonic.*

Proof. Suppose that $n \leq k \leq n + x$ and that $p \in \mathrm{MRE}_k(w)$. There is a unique path γ of length $k - n$ in G_n with label p. This γ can be uniquely extended to the left until we arrive in the vertex B. This new path $\tilde{\gamma}$ is a path from B to itself with label $\tilde{p}$, say. Note that p has u.l.e. until $\tilde{p}$. If Σ is a collection of finite words, then we recall that Σ^* is the collection of all words which can be formed by concatenating a finite number of elements of Σ. Now from the special form of $\tilde{\gamma}$ we conclude that $\tilde{p} \in B\{X,Y\}^* = \{\xi,\eta\}^* B$. (The label of one cycle is BX or BY and then we just add X, Y to the right.) If $B^{-1}\tilde{p} \notin Y^*$, then p apparently has u.l.e. to a word of length $\geq n + x$. Otherwise we have $\tilde{p} \in BY^*$ which is the first part of a). From $BY = \eta B$ we deduce $BY^i = \eta^i B$ for all $i \geq 0$ and taking i large enough we find that B is a suffix of the left-infinite word Y^∞. Therefore $\tilde{p} \in BY^*$ implies that $\tilde{p}$ is a suffix of the left-infinite word Y^∞ and the same is true for p, which is the second part of a). For every k there exists at most one $\pi \in \mathrm{MRE}_k(w)$ as in a). If we denote the number of left-extensions in $\mathrm{MRE}_{k+1}(w)$ by d we have $\Delta_{k+1} - \Delta_k = d - 1$. This implies b). If we have $d = 0$ then no strict left-extension of π is contained in $\mathrm{MRE}(w)$ and in particular this is true for longer suffixes of Y^∞. This implies c). Finally c) implies d) since $\Delta_k > t, \Delta_{k+1} \leq t$ can occur for at most one k. □

Lemma 4.1.2. *Suppose that $t \in \mathbf{Z}$ and $a_k \in (t, t+1)$ for all $k \geq n$. If a_k is monotonic on the interval $[n, n+l]$, then we have $|a_{n+l} - a_n| \geq \frac{ld(a_n,\mathbf{Z})}{l+n}$.*

Proof. If (a_k) is increasing, which is equivalent to $\Delta_k > t$ for all k, then $|a_{n+l} - a_n| \geq |\frac{na_n + l(t+1)}{n+l} - a_n| = \frac{l|t+1-a_n|}{n+l}$. If a_k is decreasing we find $|a_{n+l} - a_n| \geq \frac{l|a_n - t|}{n+l}$. Now take the minimum of these two. □

We now proceed with the proof of Theorem 4.1. Suppose that $\limsup_{m,n\to\infty} |a_m - a_n| < \epsilon := \frac{d\alpha}{6+7\alpha+2d}$. For large n we then have $a_n \in (\alpha - \epsilon, \alpha + \epsilon) \subset (t, t+1)$, whether $n \in V$ or not. Now take $n \in V$ so large that this is satisfied. The interval $[n, n+x]$ contains a subinterval $[n_0, n_0 + l]$ of length $l \geq \frac{x}{3}$ on which a_k is monotonic by Lemma 4.1.1. Note that $l \geq \frac{x}{3} \geq \frac{(n+1)a_{n+1}}{6}$ by formula (1) and that $n_0 + l \leq n + x$. Applying Lemma 4.1.2 we find

$$|a_{n_0+l} - a_{n_0}| \geq \frac{ld(a_{n_0},\mathbf{Z})}{l+n_0} \geq \frac{l(d(\alpha,\mathbf{Z}) - \epsilon)}{x+n} \geq$$

$$\frac{(n+1)a_{n+1}(d-\epsilon)}{6(x+n)} \geq \frac{(\alpha-\epsilon)(d-\epsilon)}{6(1+a_{n+1})} > \frac{d\alpha - \epsilon(\alpha+d)}{6(1+\alpha+\epsilon)} \geq \frac{d\alpha - \epsilon(\alpha+d)}{6(1+\alpha+d/7)} > \epsilon,$$

which is a contradiction since we can let $n \in V$ tend to infinity. This finishes the proof of Theorem 4.1. □

Proof of Theorem 4.2. We may assume that $a_n \in (1, 2)$ for n large because $d \geq \frac{(2-\alpha)(\alpha-1)}{\alpha}$. We then have $(a_n < a_{n-1}, a_{n+1}) \iff (\Delta_{n-1} = 1, \Delta_n > 1) \iff n-1 \in V \iff n \in V+1$. In other words, the local minima for a_n are attained exactly in $V+1$. Note that $\Delta_n = 1$ for $n \in V$, hence the general formula $a_{n+1} - a_n = \frac{\Delta_n - a_n}{n+1}$ implies that $a_{n+1} - a_n \to 0$ when $n \to \infty$ in V. Therefore $\alpha := \liminf_{n\to\infty, n\in V}(a_n) = \liminf_{n\to\infty, n\in V+1}(a_n) = \liminf_{n\to\infty}(a_n)$. With $\beta := \limsup_{n\to\infty}(a_n)$ we have to show that $\beta - \alpha \geq \frac{(2-\alpha)(\alpha-1)}{\alpha}$. Choose a sequence Σ of $n \in V$ with $a_n \to \alpha$ and note that also $a_{n+1} \to \alpha$. For every n we have the parameters x, y as in the beginning of the proof of Theorem 4.1. We distinguish between two cases.

a) $\Delta_k \geq 2$ on $[n+1, n+x)$.
Then a_k is monotonically increasing on $[n+1, n+x]$ and the proof of Lemma 4.1.2 combined with (1) shows that
$|a_{n+x} - a_{n+1}| \geq \frac{(x-1)(2-a_{n+1})}{n+x} \geq \frac{(x-1)(2-a_{n+1})}{n+x+1} \geq \frac{(a_{n+1}-\frac{2}{n+1})(2-a_{n+1})}{a_{n+1}+2}$.
The right-hand side has limit $\frac{\alpha(2-\alpha)}{2+\alpha} > \frac{(2-\alpha)(\alpha-1)}{\alpha}$ as $n \to \infty$ in Σ. We can therefore assume without loss of generality that case a) happens only finitely many times in Σ. We now consider the other case.

b) In this case $\Delta_k = 1$ for some $k \in [n+1, n+x)$ and we define $s \in [0, x)$ to be minimal such that $\Delta_{n+1+s} = 1$. Since $n \in V$ we have $\Delta_{n+1} = 2$, hence $s \in [1, x)$. Firstly s must be a multiple of y by Lemma 4.1.1 a). Hence $s \geq y$. Secondly, $\Delta_k \in \{1, 2\}$ for all $n \leq k \leq n+x$ and we obtain that

$$\Delta_i = \begin{cases} 2 \text{ if } n < i \leq n+s \\ 1 \text{ if } n+s < i \leq n+x. \end{cases}$$

Writing $a_{n+1} = a$, $\frac{x}{n+1} = \xi$, $\frac{y}{n+1} = \eta$, $\frac{s}{n+1} = \sigma$ we have $a_{n+1+s} = \frac{(n+1)a+2s}{n+1+s} = \frac{a+2\sigma}{\sigma+1}$ and $a_{n+1+x} = \frac{a+\xi+\sigma}{\xi+1}$. Choose $0 < \epsilon < \alpha - 1$ arbitrary. For $n \in \Sigma$ large enough we have $\frac{\alpha+\epsilon+\xi+\sigma}{\xi+1} > \frac{a+\xi+\sigma}{\xi+1} = a_{n+1+x} > \alpha - \epsilon$ and leaving out the middle-terms this simplifies to $2\epsilon + \sigma > (\alpha - 1 - \epsilon)\xi = (\alpha - 1 - \epsilon)(a - \eta)$. By $s \geq y$ we have $\sigma \geq \eta$. Thus $2\epsilon + \sigma \geq (\alpha - 1 - \epsilon)(a - \sigma)$ whence $\sigma \geq \frac{-2\epsilon + a(\alpha-1-\epsilon)}{\alpha-\epsilon}$. Since $\epsilon \in (0, \alpha - 1)$ was arbitrary and $a \to \alpha$ we find that $\liminf_\Sigma(\sigma) \geq \alpha - 1$. Then $|a_{n+1} - a_{n+1+s}| = \frac{(2-a)\sigma}{\sigma+1}$ implies $\limsup_{m,n\to\infty} |a_m - a_n| \geq \frac{(2-\alpha)(\alpha-1)}{\alpha}$, which is the bound stated in the theorem. $\square$

Proof of Theorem 4.3. We will write $\{a,b\}^n$ for the directed graph with all n-words on a,b as vertices and all $(n+1)$-words on a,b as edges. Let $(G_n)_0^\infty$ be a sequence of graphs such that for each n we have:

- G_n is a strongly connected subgraph of $\{a,b\}^n$
- $E(G_n) = V(G_{n+1})$

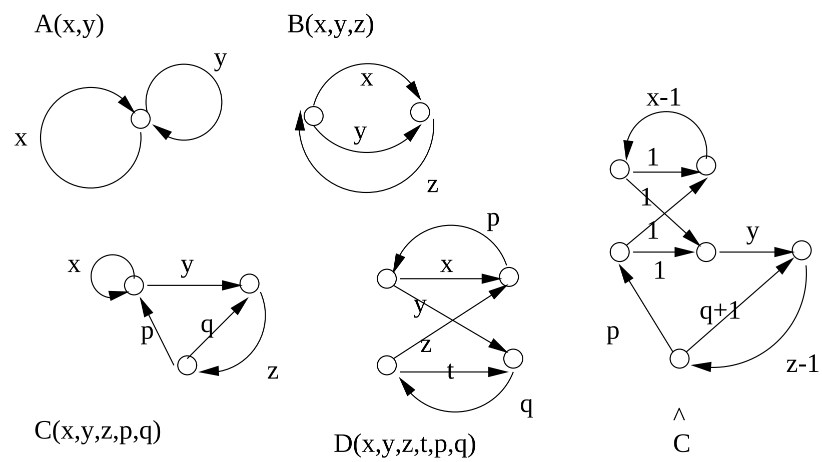
We wonder if this implies that there exists a recurrent $\mathbf{Z}$-word w with $\mathcal{B}(w,n) = V(G_n)$ for all n. In general this is not the case. There exists a sequence $(G_n)_0^\infty$ as above such that the only $\mathbf{Z}$-words inducing it are non-recurrent. An example is given by the word graph sequence (G_n) of a skew balanced $\mathbf{Z}$-word. A proof that such (G_n) indeed has the stated property will be given at the end of this chapter. We prove that the answer is positive for a restricted class of wordgraph sequences. Define $\delta_n := |E(G_n)| - |V(G_n)|$.

Lemma 4.3.1. *Let G_n be a sequence of graphs as above and suppose that $\delta_n = 1$ for infinitely many n and that $\delta_n > 1$ for infinitely many n. Then there exists a recurrent $\mathbf{Z}$-word w with $\mathcal{B}(w,n) = G_n$ for all n.*

Proof. Define the language $L := \cup_0^\infty(G_n)$ and let $\mathcal{P}$ be the set of positive integers n with $\delta_n = 1$. For $n \in \mathcal{P}$ we have a unique word $x \in \mathrm{MRE}(L,n)$ and we call this word B_n. If $m,n \in \mathcal{P}$ and $m \leq n$, then B_m is a suffix of B_n. Hence there is a unique left-infinite word B which has each $B_n, n \in \mathcal{P}$ as a suffix. Now we define B_n for every n as the suffix of B of length n, whence $B_n \in \mathrm{MRE}(L,n)$ for all n. Choose a symbol σ such that $B\sigma$ is not periodic. Since $B\sigma \neq B$ there exists a suffix $B_n\sigma$ of $B\sigma$ with u.r.e. in L, say that only $B_n\sigma\sigma_1$ appears in L. Each finite factor of $B\sigma\sigma_1$ is then contained in L. With the same reasoning there exists a unique symbol σ_2 such that every factor of $B\sigma\sigma_1\sigma_2$ is contained in L and inductively one finds a $\mathbf{Z}$-word $w = B\sigma\sigma_1\sigma_2\cdots$ such that every finite factor is contained in L as well, hence $\mathcal{B}(w,n) \subset L$ for all n and $\mathcal{B}(w,n) \subset G_n$ for all n. Take $k \in \mathcal{P}$ such that $k+1 \notin \mathcal{P}$. Then G_k consists of two loops joined together in one point, B_k, as we have seen already in the proof of Theorem 4.1. Since $\mathcal{B}(w,k) \subset G_k$ we see that w induces a path $\gamma : \mathbf{Z} \to G_k$ by reading off the consecutive k-factors. Since w is not periodic ($B\sigma$ already is not), we find that γ passes through both loops. This implies that $\mathcal{B}(w,k) = G_k$ for all $k \in V$ with $k+1 \notin V$. If $x \in L$ then $x \subset y \in G_k$ for such a $k \geq x$ and because $y \subset w$ we have $x \subset w$ and we conclude that $\mathcal{B}(w,n) = G_n$ for all n.

Now suppose that w is not recurrent. Then there is an $x \in L$ which appears only once in w. Extending x if necessary we can assume that $x \in G_k$ for some k with $k \in \mathcal{P}, k+1 \notin \mathcal{P}$. Then $x \neq B_k$ and if we indicate the labels of the loops of G_k by B_kX, B_kY we have without loss of generality that $w = X^\infty Y X^\infty$. But by Theorem B from Chapter 3 such a word either is periodic, which is false, or has minimal block growth, i.e. $P(w,n) = n+k$ for some constant k and n large enough. This contradicts the assumptions of the Lemma, which shows that w is recurrent. □

Let G be a strongly connected directed graph. We define the blow-up of G as the directed graph $\hat{G}$ with $V(\hat{G}) = E(G)$ and $E(\hat{G}) = \{(x,y) \in E(G)^2 |$ terminal vertex $(x) =$ initial vertex $(y)\}$. It is clear that $\hat{G}$ is also strongly connected and that we can identify paths $\mathbf{Z} \to G$ with paths $\mathbf{Z} \to \hat{G}$. Assume now that $G \subset \{a,b\}^n$. Then $\hat{G} \subset \{a,b\}^{n+1}, \hat{G}$ is strongly connected and $E(G) = V(\hat{G})$. This helps us to construct sequences G_n as in Lemma 4.3.1. We now introduce some notation for (isomorphy types of) directed graphs.



Here the label of an arrow indicates the actual length of the path. All pathlenghts in A, B, C, D are assumed to be positive and for these graphs the blow-up graph can be easily determined. We have $\hat{A} = D(1,1,1,1,x-1,y-1), \hat{B} = B(x+1, y+1, z-1), \hat{D} = D(x+1, y+1, z+1, t+1, p-1, q-1)$ and $\hat{C}$ has been included in the picture above. Sometimes a graph degenerates in the sense that a label might become 0. To deal with this we have the following rules: $D(x,y,z,t,p,q) = D(t,z,y,x,q,p), D(x,y,z,t,p,0) = C(t,z,p,y,x)$ and $B(x,y,0) = A(x,y)$. If we delete from $\hat{C}$ the two horizontal arrows of length 1, then the graph changes into $C^* = B(x+y+p+1, q+1, z-1)$ and we have $E(C) = V(C^*)$. We are now ready to inductively define our sequence G_n. We set $G_0 = A(1,1)$, the only vertex is $\emptyset$ and the only edges are a, b. Also we set $G_1 = A(2,1)$ with vertices a, b and edges ba, ab, bb. From here on we define

$$G_{n+1} = \begin{cases} \hat{G}_n \text{ if } G_n \text{ is of type } A, B, D \\ G_n^* \text{ if } G_n \text{ is of type } C \end{cases}$$

Applying the rules above we find

$$\begin{array}{ll} G_1 = A(2,1) & G_2 = C(1,1,1,1,1) \\ G_3 = A(4,2) & G_5 = C(2,2,2,2,2), \\ G_7 = A(8,4) & G_{11} = C(4,4,4,4,4) \end{array}$$

and it is not hard to prove inductively that $G_{2^n-1} = A(2^n, 2^{n-1})$ and $G_{3 \cdot 2^{n-1}-1} = C(2^{n-1}, \cdots, 2^{n-1})$. Note that this implies $P(2^n) = 3 \cdot 2^{n-1}$ and $P(3 \cdot 2^{n-1} = 5 \cdot 2^{n-1}$. By virtue of Lemma 4.3.1 there exists a recurrent $\mathbf{Z}$-word w with $\mathcal{B}(w,n) = G_n$ for all n. Since $\delta_n \in \{1,2\}$ for all n we see that also $\Delta_n = \delta_n \in \{1,2\}$ for all n. Therefore $a_k := P(k)/k \in (1,2)$ for all $k \geq 2$ and we see $a_{k+1} > a_k \iff \Delta_k = 2$ as in the beginning of the proof of Theorem 4.1. It follows that a_k takes on local minima in $k = 2^n$ and local maxima in $k = 3 \cdot 2^{n-1}$, where $n \in \mathbf{N}^+$. Therefore $\alpha = \lim_{n\to\infty} \frac{P(2^n)}{2^n} = 3/2$ and $\beta = \lim_{n\to\infty} \frac{P(3 \cdot 2^{n-1})}{3 \cdot 2^{n-1}} = 5/3$. □

It is possible here to give a more explicit description of w and to do so we consider the labels in the graphs more closely. First some notational matters. In the previous pictures we used the notation $P \xrightarrow{x} Q$ to denote that the path in the graph has length x and then its label is of the form $PX = \xi Q$ where $|X| = |\xi| = x$. We call X the reduced label of the path. If in $A(x,y)$ we want to indicate what the reduced labels of the loops are, we write $A(X,Y)$ where X, Y are now words. Now suppose that $n \in V$. Then G_n is of type A and G_{n+1} is of type D. Denote the reduced labels of the loops by X, Y and assume that $x > y$. From $G_n = A(X,Y)$ we deduce $G_{n+1} = D([X]_1, [Y]_1, [X]_1, [Y]_1, [X]^{x-1}, [Y]^{y-1})$, $G_{n+y} = C(Y, [X]_y, [X]^{x-y}, Y, [X]_y)$, $G_{n+y+1} = B(Y^2[X]_{y+1}, [X]_{y+1}, [X]^{x-y-1})$ and finally $G_{n+x} = A(Y^2X, X)$. Hence to go from one A-graph to the next we may simply replace (X,Y) by (Y^2X, X). We now prove some little facts about X_n, Y_n where X_n, Y_n are the reduced labels of G_{2^n-1} with $|X_n| = 2^n$.

Lemma 4.3.2. a)$X_n = \tilde{Y_n} Y_n$ *where* $\tilde{}$ *means changing the first symbol only.*
b) $X_{2n} \to b\phi, Y_{2n} \to a\phi$ *where* ϕ *is a right-infinite word.*
c) $X_n = \sigma_n \pi_n, Y_n = \overline{\sigma_n} \rho_n$ *where* σ_n *is a letter and* π_n, ρ_n *are palindromes.*
d) *The words* $w = \overleftarrow{\phi}\, a\phi$ *and* $w = \overleftarrow{\phi}\, b\phi$ *are recurrent and for both we have* $\mathcal{B}(w, n) = G_n$ *for all* n*. Here* $\leftarrow$ *stands for reversing the order of the symbols, hence* $\overleftarrow{\phi}$ *is a left-infinite word.*

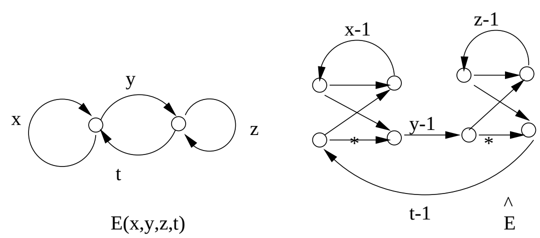
Proof. a) It is true for $n = 1$ since $X_1 = ab, Y_1 = b$. If it is true for n then we have $\tilde{Y}_{n+1} Y_{n+1} = \tilde{X}_n X_n = Y_n^2 X_n = X_{n+1}$.
b) One has $X_{n+1} = Y_n^2 X_n, Y_{n+1} = X_n$ and applying this rule again we find $X_{n+2} = X_n^2 Y_n^2 X_n$ and $Y_{n+2} = Y_n^2 X_n$. Hence $X_{2n} \to \xi, Y_{2n} \to \eta$ for some right-infinite words ξ, η and taking the limit $n \to \infty$ in $X_{2n} = \tilde{Y_{2n}} Y_{2n}$ one finds $\xi = \tilde{\eta}$.
c) It is true for $n = 1$. Now suppose it is true for some $n \in \mathbf{N}^+$. Then $X_{n+1} = \tilde{Y}_{n+1} Y_{n+1} = \tilde{X}_n X_n = \overline{\sigma_n} \pi_n \sigma_n \pi_n$ and $\pi_n \sigma_n \pi_n$ is a palindrome. Also we have $Y_{n+1} = X_n = \sigma_n \pi_n$.
d) We will follow the construction in the proof of Lemma 4.3.1 for our sequence $(G_n)_0^\infty$. The word B has all X_n as a suffix, hence $\overleftarrow{B}$ has all $\overleftarrow{X_n}$ as a prefix. Now $\overleftarrow{X_n} = \pi_n \sigma_n$ by c), hence $\overleftarrow{B} = \phi$ and $B = \overleftarrow{\phi}$. Now assume that G_N is of type A where the loops have label $B_N X, B_N Y$ and we note that every N-factor of w is contained in $B_N X$ or in $B_N Y$. In the next A-graph the triple (B_N, X, Y) is replaced by $(B_N X, Y^2 X, X)$ and in the next one by $(B_N X Y^2 X, X^2 Y^2 X, Y^2 X)$. The words $B_N X, B_N Y$ are both contained in $B_N X Y^2 X = \xi B_N Y^2 X$ and we conclude that every N-factor of w is contained in some B_M where $M > N$. This means that $L = \cup_0^\infty \mathcal{B}(B, n)$, and since $P(L, n) \to \infty$ it follows that B is not periodic. This implies that the σ from the proof of Lemma 4.3.1 can be chosen arbitrarily in the present case. Fix $\sigma \in \{a, b\}$. Let $k \in \mathbf{N}^+$, choose $n > k$ such that $|X_n|, |Y_n| > k$ and let N be the integer with $G_N = A(X_n, Y_n)$. Without loss of generality we assume that X_n starts with σ, for otherwise we may replace n by $n + 1$. Since $\overleftarrow{\phi}\, \sigma \sigma_1 \cdots \sigma_k$ appears in w and the path in G_N induced by $\overleftarrow{\phi}$ ends in B_N, we have that σ_k equals the $(k+1)^{\text{st}}$ symbol of X_n. This obviously is the k^{th} symbol of π_n, which in turn equals the k^{th} symbol of ϕ. Therefore the constructed w equals $\overleftarrow{\phi}\, \sigma\phi$, showing that $\overleftarrow{\phi}\, \sigma\phi$ is recurrent and has the right word graphs. $\square$

Remark. We would like to thank Julien Cassaigne for pointing out to us that ϕ is also the fixed point of $T := (bb, ba)$ and that ϕ is known as the "period-doubling sequence". Since it is known how to compute the complexity function of such fixed points, see [Ca], this yields another proof of Theorem 4.3. We now outline a proof that $T\phi = \phi$.

We can write $X_n = C^n(b)\pi_n$ where $C := (b, a)$ as usual and π is as in Lemma 4.3.2 c). In the proof of part c) it was shown that $X_{n+1} = \tilde{X}_n X_n$, hence $\pi_{n+1} = \pi_n C^n(b)\pi_n$. Now let ϕ^* be the unique right-infinite fixed point of T and $\psi_n := [\phi^*]_{2^n-1}$ for all $n \geq 0$. We claim that ψ_n satisfies the same recurrence as does π_n. Since $\pi_0 = \emptyset = \psi_0$ this implies $\pi_n = \psi_n$ for all $n \geq 0$ and from $\pi_n \to \phi, \psi_n \to \phi^*$ we deduce $\phi = \phi^*$. Now for the claim. It is easily seen that $\psi_{n+1} = T(\psi_n)b$ for all n. We have $\psi_{n+1} = \psi_n C^n(b)\psi_n$ for $n = 0$ and if it is true for some n then we have

$$\psi_{n+2} = T(\psi_{n+1})b = T(\psi_n)bC^{n+1}(b)T(\psi_n)b = \psi_{n+1}C^{n+1}(b)\psi_{n+1}$$

Proof of Theorem 4.4. We extend the notation of Theorem 4.3 with $E(x, y, z, t)$ where E is as below and where we have already depicted $\hat{E}$.



The unlabelled edges are just single edges and we define $\tilde{E}$ by deleting the edges with a star from $\hat{E}$. Hence $\tilde{E} = D(1, y + 1, t + 1, 1, x - 1, z - 1)$. Notice that the edgeset of E equals $V(\tilde{E})$. We add the rule $C(x, y, 0, p, q) = E(x, y, q, p)$. If we delete from $\hat{C}$ only the lower horizontal edge of length 1, the result is $D(1, y+1, p+1, q+1, x-1, z-1)$. We will write C^{**} for this graph. It is now clear that we can create a sequence G_n of graphs with the same beginning as the sequence in the proof of Lemma 4.3.1 and with

$$G_{n+1} = \begin{cases} \hat{G}_n \text{ if } G_n \text{ is of type } A, B, D \\ \tilde{G}_n \text{ if } G_n \text{ is of type } E \\ G_n^* \text{ or } G_n^{**} \text{ if } G_n \text{ is of type } C \end{cases} \qquad (2)$$

Hence all appearing G_n are of the form $A, \cdots, E$. One only has a choice for G_{n+1} if G_n is of type C. In the next lemma, if $f : S_1 \to T_1, g : S_2 \to T_2$ are functions with disjoint domains, then we define $f \cup g$ as the function whose domain is $S_1 \cup S_2$ such that f, g are restrictions.

Lemma 4.4.1. a) *If G_n is of type C and $G_{n+1} = G_n^{**}$ then the parameters of the next C-graph are given by $\phi(x, y, z, p, q)$. Here $\phi : (\mathbf{R}^+)^5 \to (\mathbf{R}^+)^5$ is given by $\phi = \phi_1 \cup \phi_2 \cup \phi_3$ where $\phi_1 = (q+z, p+z, x-z, y+z, z)$, $\phi_2 = (x, x+y, z-x, x+p, x+q)$ and $\phi_3 = (x, 2x+y, q, 2x+p, x)$ and where the domains of ϕ_1, ϕ_2, ϕ_3 are by definition the subsets of $(\mathbf{R}^+)^5$ where $x > z, x < z, x = z$ respectively.*
b) *If $G_{n+1} = G_n^*$ then the first graph after G_n of type A is given by $G_{n+z} = A(x + y + p + z, q + z)$.*

Proof. As an example we do the case $x > z$ of part a). Then $C(x, y, z, p, q) \to D(1, y + 1, p + 1, q + 1, x - 1, z - 1) \to D(z, y + z, p + z, q + z, x - z, 0) = C(q + z, p + z, x - z, y + z, z)$ which corresponds to the given formula. □

Important is that the "jump" $\delta_n := |E(G_n)| - |V(G_n)|$ can be derived directly from the type of G_n, because we have $\delta_n \in \{1, 2\}$ for all n and $\delta_n = 1$ if and only if G_n is of type A or B. It is also important that the number of jumps 1 ($\delta_n = 1$) after a C-graph in case b) is equal to z. To get further we calculate some domains explicitly.

Lemma 4.4.2. *Let $\psi := \phi_1^2$ hence $\psi = (x, x + y, 2z + q - x, x + p, x - z)$ with domain $D(\psi) = \{z < x < 2z + q\} \subset (\mathbf{R}^+)^5$. Also let $n \in \mathbf{N}$.*
a) *$D(\psi^n) = \cap_{k=1}^n \{kz + (k - 1)q < kx < (k + 1)z + kq\}$ and $D(\phi_1\psi^n) = D(\psi^n) \cap \{(n + 1)z + nq < (n + 1)x\}$.*
b) *$D(\phi_1\psi^n\phi_i) = D(\psi^{n+1}\phi_i)$ if $i = 2, 3$.*

Proof. a) The first formula holds for $n = 1$. Now we assume it holds for some $n \geq 1$. Then $D(\psi^{n+1}) = \psi^{-1}(D\psi^n) = D(\psi) \cap \{(x, x + y, 2z + q - x, x + p, x - z) \in D(\psi^n)\} = D(\psi) \cap \cap_1^n \{k(2z + q - x) + (k - 1)(x - z) < kx < (k + 1)(2z + q - x) + k(x - z)\} = D(\psi) \cap \cap_1^n \{(k + 1)z + kq < (k + 1)x < (k + 2)z + (k + 1)q\} = \cap_1^{n+1} \{kz + (k - 1)q < kx < (k + 1)z + kq\}$ as can be seen by shifting the indices whereas $D(\psi)$ yields the condition for $k = 1$. Also we have $D(\phi_1\psi^n) = D(\psi^n) \cap \cap_1^n \{k(x - z) + (k - 1)z < k(q + z) < (k + 1)(x - z) + kz\} = D(\psi^n) \cap \cap_1^n \{kx < kq + (k + 1)z < (k + 1)x\} = D(\psi^n) \cap \{nq + (n + 1)z < (n + 1)x\}$, as the other conditions are already fulfilled in $D(\psi^n)$. This gives the required answer.
b) Suppose first $x < z$ and $\phi_2 = (x, x + y, z - x, x + p, x + q) \in D(\phi_1\psi^n)$. To

show that $\phi_2 \in D(\psi^{n+1})$ we only have to check that $(n+1)x < (n+2)(z - x) + (n+1)(q+x)$ or, equivalently, that $(n+2)x < (n+2)z + (n+1)q$. But this is immediate from $x < z$. Now suppose that $x = z$ and $\phi_3 = (x, 2x + y, q, 2x+p, x) \in D(\phi_1\psi^n)$. We have to show that $(n+1)x < (n+2)q + (n+1)x$, and this is obvious. □

Lemma 4.4.3. *Suppose that* $v = (x, y, z, p, q) \in (\mathbf{R}^+)^5$ *and define* $(x_n, y_n, z_n, p_n, q_n) := \phi^n(v)$. *Then for each* $\epsilon > 0$ *there exists an* $n \in \mathbf{N}$ *with* $\frac{z_n}{x_n+y_n+z_n+p_n+q_n} < \epsilon$.

Proof. We consider the sequence $v_n := \phi^n(v)$ and define $\gamma : \mathbf{N} \to \{1, 2, 3\}$ by $\gamma(k) = i \iff v_k \in D(\phi_i)$. I will write $\overline{1}$ for an arbitrary symbol unequal to 1, hence 2 or 3. Then Lemma 4.4.2 b) implies

$$\text{if } \overline{1}\ 1^\lambda\ \overline{1} \text{ is a factor of } \gamma \text{ then } \lambda \text{ is even.}$$

Since $\phi_1^2 = \psi, \phi_2$ and ϕ_3 all leave the first coordinate x fixed we see that v_n has a subsequence v_{n_i} with x_{n_i} constant. If $z > x$ then one has to apply ϕ_2, changing (x, z) into $(x, z - x)$. Repeating this often enough one finds a point with $z \leq x$. This means that there exists a (possibly different) subsequence v_{n_i} of v_n with z_{n_i} bounded. Also we easily check that $x_{n+1} + y_{n+1} + z_{n+1} + p_{n+1} + q_{n+1} \geq x_n + y_n + z_n + p_n + q_n + 2\min(x_n, z_n)$. This implies that $x_n + y_n + z_n + p_n + q_n$ tends to infinity with n, in which case we are done, or else that $\sum_{n=0}^{\infty} \min(x_n, z_n)$ converges. In that case we have a subsequence $z_{n_i} \to 0$ and we are done as well. □

We are now able to give a construction which proves Theorem 4.4. We take $G_0 = A(1,1), G_1 = A(2,1)$ as before and henceforth we abide by the stated rules for G_n. Then $G_2 = C(1,1,1,1,1)$. We set $n_1 = 2$. Suppose that $G_{n_k} = C(x_k, y_k, z_k, p_k, q_k) =: C$. Then $G_{n_k+1} \in \{C^*, C^{**}\}$ and each choice leads to a new C-graph when we follow the rules outlined in (2). Choosing C^{**} a suitable number of times, hence by applying ϕ, we end up in a graph $G_{N_k} = C(X_k, Y_k, Z_k, P_k, Q_k)$ where

$$n_k < N_k, \frac{X_k + Y_k + Z_k + P_k + Q_k}{N_k + 1} \geq 2 - \frac{1}{k}, \frac{Z_k}{N_k + 1} \leq \frac{1}{k}.$$

Now we choose $G_{N_k+1} = G^*_{N_k}$ and then we have $\delta_n = 1$ up to $G_{N_k+Z_k}$ which is a graph of type A. After this we have $\delta_n = 2$ until we arrive in the next graph of type C which we call $G_{n_{k+1}}$ and proceeding in this fashion we find a sequence G_n of graphs. According to Lemma 4.3.1 there exists a recurrent $\mathbf{Z}$-word w with $\mathcal{B}(w, n) = G_n$ for all n. As in the proof of Theorem 4.2 we have $P(n)/n \in (1,2)$ for all $n \geq 2$ and $P(n)/n$ assumes its local minima exactly in those n with $n \in V+1$, hence if G_{n-1} is a graph of type A. In the notation above we then have $G_{n-1} = G_{N_k+Z_k}$ and

$$\frac{P(n)}{n} = \frac{X_k + Y_k + 2Z_k + P_k + Q_k}{N_k + Z_k + 1} \geq \frac{(2-\frac{1}{k})(N_k+1)}{(N_k+1)(1+\frac{1}{k})} = \frac{2k-1}{k+1}$$

Hence $\alpha = \beta = 2$ for this w. □

Proof of Theorem 4.0. Without loss of generality we assume that σ is given by $\sigma_i = a \iff i \in \{\lfloor \zeta k + \phi \rfloor\}_{k\in\mathbf{Z}}$, where $\zeta > 1$ is irrational and $\phi \in \mathbf{R}$. Let $x \in \mathcal{B}(\sigma, n)$. By an appearance of x in σ we shall mean an integer i such that $x = \sigma_i \cdots \sigma_{i+n-1}$. Since an appearance of x in σ can have at most r values modulo r, we have $P(w,n) \leq rP(\sigma,n) = r(n+1)$ for all n.

We now show that an appearance of a fixed n-factor of σ can be in any residue class modulo r. For n large enough this factor x contains a's and we denote their number by k. We will assume that the restriction of σ to $[1,n]$ induces x and that the a's appear at $\{\lfloor \zeta i + \phi \rfloor\}_1^k$, which may be obtained by changing ϕ modulo ζ if necessary. With abuse of notation we abbreviate this as $x = \{[\zeta i + \phi]\}_1^k \subset [1,n]$. Now let $\epsilon := \min_0^{k+1}(1 - \mathrm{fr}(\zeta i + \phi))$ and let t be any integer. Choose $\lambda \in \mathbf{N}^+$ such that $\mathrm{fr}(\frac{t+r\lambda}{\zeta}) > 1 - \frac{\epsilon}{\zeta}$ and put $p = \lceil \frac{t+r\lambda}{\zeta} \rceil$. Such a λ can be chosen because ζ is irrational. Then $p - \frac{\epsilon}{\zeta} < \frac{t+r\lambda}{\zeta} < p$, hence $t + r\lambda < p\zeta < t + r\lambda + \epsilon$. This implies $\mathrm{fr}(p\zeta) < \epsilon$ and $[p\zeta] \equiv t \mod r$. Write $q := [\zeta p]$. For $0 \leq i \leq k+1$ we have

$$[\zeta(i+p) + \phi] = [\zeta p] + [\zeta i + \phi + \mathrm{fr}(\zeta p)] = q + [\zeta i + \phi],$$

showing that $1 + q$ is also an appearance for x. This proves our claim.

For each $0 \leq k < r$ we define a mapping ϕ_k on finite words of length $n \geq r$ by demanding that $\phi_k(x)$ equals x, except on the k^{th} elements modulo r. Then for each n we have a mapping $\psi_n : \mathcal{B}(\sigma,n) \times \{1, \cdots, r\} \twoheadrightarrow \mathcal{B}(w,n)$ given by $\psi_n(x,k) = \phi_k(x)$. The mapping is surjective because of the previous claim.

Suppose that ψ_n is not injective for infinitely n. For such n we have $\phi_k(x) = \phi_l(y)$ where $(x,k) \neq (y,l)$ and $x, y \in \mathcal{B}(\sigma,n)$. Then $x \neq y, k \neq l$. Writing $x = x_1 \cdots x_n$ we define $\tilde{x}$ as the word obtained by writing down only the x_i with $i \equiv k, l \mod r$ and similarly we define $\tilde{y}$. Then $\tilde{x}$ and $\tilde{y}$ are eachother's complement by hypothesis. We claim that the density of the a's in $\tilde{x}, \tilde{y}$ will converge to $\alpha := \frac{1}{\zeta}$ as $n \to \infty$, uniformly in x and y. Since $\tilde{x}$

and $\tilde{y}$ are complementary words our claim implies $\alpha = 1 - \alpha$, hence $\alpha = \frac{1}{2}$, which is impossible. Hence the mapping is a bijection for n large and we are done. Now for our claim. Section 2.5.2 shows that σ may also be given by $\sigma_i = a \iff \alpha i + \beta \in (1-\alpha, 1] \bmod 1$ where $\alpha := \frac{1}{\zeta}$ and $\beta := \frac{-\phi}{\zeta}$. If we restrict ourselves to an arithmetic subsequence $i = \lambda + kr$ then we may describe the result by $\tilde{\sigma}_k = a \iff \alpha kr + \alpha\lambda + \beta \in (1-\alpha, 1] \bmod 1$. By the uniform distribution of $\alpha kr + \alpha\lambda + \beta \bmod 1$, cf. [KN, Example 2.1], the relative density of the a's within an n-factor of $\tilde{\sigma}$ will tend to α as $n \to \infty$. Since there are at most r different $\tilde{\sigma}$'s the result follows. □

Finally we prove the claim concerning the word graphs of a skew balanced **Z**-word that we made just before proving Lemma 4.3.1. Suppose that w is a balanced skew **Z**-word. It is non-recurrent and in the proof of Proposition 3.4 it is shown that all its word graphs are strongly connected. Now suppose that a **Z**-word w' has the same finite factors. We will show that w and w' are shifts of each other, which implies what we want to show. The word w' is also balanced and since its density $\alpha(w')$ is defined in terms of its finite factors we have $\alpha(w') = \alpha(w) \in \mathbf{Q}$. Since $P(w', n) = P(w, n) = n + 1$ for all n we see that w' cannot be periodic and by the classification in Theorem 2.5 it must be skew of some type. This type can also be recovered from the finite factors and, since w, w' are of the same type, the classification shows that they are shift-related. □

Appendix: balanced words and differential equations.

Let $\phi \in C(\mathbf{R})$, hence ϕ is continuous on $\mathbf{R}$. We consider the linear space $V = \{f \in C^2(\mathbf{R}) | f'' = f\phi\}$. It is well-known, see Lemma 5.1.3, that for every pair $(\alpha, \beta) \in \mathbf{R}^2$ there exists a unique $f \in V$ with $f(0) = \alpha, f'(0) = \beta$. It follows that V is 2-dimensional and that for every $f \in V \setminus \{0\}$ its zero-set $Z(f)$ is a discrete subset of $\mathbf{R}$. Indeed, for an accumulation point α of $Z(f)$ we would have $f(\alpha) = f'(\alpha) = 0$ and then $f \equiv 0$. Now assume that $f \in V \setminus \{0\}$ and define $\psi_n := |Z(f) \cap [n, n+1)|$ for all $n \in \mathbf{Z}$. We can then associate to f the biinfinite word $w(f) = \cdots ab^{\psi_{-1}} ab^{\psi_0} ab^{\psi_1} \cdots$

Lemma 5.1.1. *Let $f, g \in V$. The Wronskian $W(f,g) := f'g - fg'$ is a constant function. Furthermore $W(f,g) = 0$ if and only if f, g are linearly dependent.*

Proof. If $f, g \in V$ we have $W' = f''g - fg'' = \phi(fg - fg) = 0$, hence W is indeed constant. If $f \equiv 0$ or $g \equiv 0$ the equivalence clearly holds. Hence we assume that f, g are not the zero-function. If f, g are linearly dependent we have $f = \lambda g$ for some $\lambda \in \mathbf{R}$ and then $W(f,g) = \lambda W(g,g) = 0$. Conversely, let us assume that $W(f,g) = 0$. The function $\rho = \frac{f}{g}$ is defined on any open interval disjoint with $Z(g)$ and there we have $\rho' = \frac{W(f,g)}{g^2} = 0$. Hence $\rho = \lambda$ and $f - \lambda g = 0$ on that interval, for some constant λ. Also $f - \lambda g \in V$ and by the previous we have $f - \lambda g = 0 \in V$. Therefore f, g are linearly dependent.□

Lemma 5.1.2. [Sturmian separation theorem]. *Let $f, g \in V$ be linearly independent and let α, β be two consecutive zeroes of f. Then $g(x) = 0$ for some $x \in (\alpha, \beta)$.*

Proof. The space $V' := \{h \in V | h(\alpha) = 0\}$ is 1-dimensional and $f \in V'$. From the linear independence of f, g we deduce $g \notin V'$, hence $g(\alpha) \neq 0$. Similarly, $g(\beta) \neq 0$. If $Z(g) \cap (\alpha, \beta) = \emptyset$, then $\rho := \frac{f}{g}$ is defined on $[\alpha, \beta]$ and $\rho(\alpha) = \rho(\beta) = 0$. By Rolle's theorem there exists an $x \in (\alpha, \beta)$ with $0 = \rho'(x) = \frac{W(f,g)}{g(x)^2}$. This implies that f, g are linearly dependent, which is a contradiction. □

We now make the assumption that ϕ has period 1. Then $f(x) \in V \to f(x+p) \in V$ for all $p \in \mathbf{Z}$. Assume that $f \in V \setminus \{0\}$. Applying the sturmian separation theorem to $f(x)$ and $f(x+i-j)$ on $[j, j+s)$ we find $||Z(f) \cap [i, i+$

$s)| - |Z(f) \cap [j, j+s)|| \leq 1$, where $i, j \in \mathbf{Z}$ and $s \in \mathbf{N}^+$. Translated to $w(f)$ this means that for any two subwords $ab^{k_1} \cdots ab^{k_s}a$ and $ab^{l_1} \cdots ab^{l_s}a$ of $w(f)$ we have $|\sum_1^s(k_i) - \sum_1^s(l_i)| \leq 1$. Morse and Hedlund called this the "comparison condition" and words of the form $\cdots ab^{\psi_{-1}}ab^{\psi_0}ab^{\psi_1}\cdots$ (i.e. having infinitely many a's) satisfying it were called sturmian sequences. From the comparison condition one easily deduces that no words P exist with $aPa, bPb \subset w(f)$. Hence, by Lemma 2.6.1, all words $w(f)$ are balanced. With only two exceptions the converse is also true, as follows from the next theorem by Morse and Hedlund.

Theorem 5.1. [MH, Thm. 7.1] *If $f \in V \setminus \{0\}$, then $w(f)$ is balanced. If w is balanced and contains infinitely many a's, then a choice of ϕ, f exists with $f \in V$ and $w = w(f)$.*

We conclude this section with the promised lemma.

Lemma 5.1.3. *Let $\phi \in C(\mathbf{R})$. For every pair $(\alpha, \beta) \in \mathbf{R}^2$ there exists a unique $f \in C^2(\mathbf{R})$ satisfying $f'' = f\phi$ with $f(0) = \alpha, f'(0) = \beta$.*

Proof. The standard way to prove this, it seems, is to associate to $f \in C^2(\mathbf{R})$ the vector valued function $F = (f, f') : \mathbf{R} \to \mathbf{R}^2$. Solutions f to $f'' = f\phi$ then correspond to solutions F of $F'(t) = \Phi(F(t), t)$ where $\Phi : \mathbf{R}^3 \to \mathbf{R}^2$ is given by $\Phi(u, v, w) := (v, u\phi(w))$. One can then use a version of the Cauchy-Lipschitz theorem, see [Le, Thm. 2.1]. For the benefit of the reader who is not acquainted with this theorem we give a proof adapted to this special instance.

It is easy to see that $f \in \mathcal{C}^2(\mathbf{R}), f'' = f\phi, f(0) = \alpha, f'(0) = \beta$ is equivalent to $f \in C(\mathbf{R}), f(x) = \alpha + \beta x + \int_0^x (x-t)f(t)\phi(t)\mathrm{d}t$. We define a sequence $(f_n)_0^\infty$ in $C(\mathbf{R})$ by $f_0 = 0$ and $f_{n+1}(x) = \alpha + \beta x + \int_0^x (x-t)f_n(t)\phi(t)\mathrm{d}t$. Fix $M \in \mathbf{R}^+$. We write $A := ||\phi||_\infty$, $g_n := f_{n+1} - f_n$ and $\mu_n := ||g_n||_\infty$ where the infinity-norm is on the interval $[-M, M]$. We have $g_{n+1}(x) = \int_0^x (x-t)g_n(t)\phi(t)\mathrm{d}t$, hence

$$|g_{n+1}(x)| \leq \frac{\mu_n A x^2}{2}, \ |x| \leq M.$$

Iteration yields $|g_{n+a}(x)| \leq \frac{\mu_n A^a x^{2a}}{(2a)!}$ for $a \in \mathbf{N}^+, |x| \leq M$ and in particular we find $\mu_a \leq \frac{\mu_0 (AM^2)^a}{(2a)!}$. Since $\sum_0^\infty ||g_a||_\infty < \infty$ we conclude that f_n is a Cauchy sequence in the Banach space $(C[-M,M], || \; ||_\infty)$ and therefore f_n converges to an element $f \in C[-M,M]$. Passing to the limit in the recursive formula for f_{n+1} we find that $f(x) = \alpha + \beta x + \int_0^x (x-t)f(t)\phi(t)\mathrm{dt}$ for $|x| \leq M$, hence f is a solution on the interval $(-M,M)$. If g is another solution on $(-M,M)$, then $h := f - g$ satisfies $h(x) = \int_0^x (x-t)h(t)\phi(t)\mathrm{dt}$. This is also clear by linearity. Write $\mu := ||h||_\infty$, a similar reasoning as before yields $\mu \leq \frac{\mu(AM^2)^a}{(2a)!}$ for all $a \in \mathbf{N}^+$. Sending $a \to \infty$ we find $h = 0$, hence the solution f we found is unique. The rest is now easy. For each positive integer n we let f_n be the unique solution on $(-n,n)$. Since $f_m|_{(-n,n)} = f_n$ if $m \geq n$ we find that a unique $f \in C(\mathbf{R})$ exists with restrictions f_n and clearly f is a solution. If $g \in C(\mathbf{R})$ is another solution, then f, g are equal on each interval $(-n,n)$, hence $f = g$. $\square$

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Samenvatting.

Dit proefschrift gaat over combinatoriek op woorden. Een woord is een opeenvolging van symbolen uit een eindig alfabet, in ons geval $\mathcal{A} = \{a, b\}$. Woorden kunnen eindig zijn, oneindig in één richting of oneindig in twee richtingen. In het laatste geval spreken we van **Z**-woorden. Een eindig patroon x dat in een ander woord w voorkomt noemen we een factor of deelwoord van w en we schrijven $x \subset w$. In dit proefschrift gebruiken we getaltheorie, combinatoriek en een beetje grafentheorie om de factorverzameling van woorden, en meer algemeen van talen, te bestuderen. Een woord x heeft een lengte $|x|$ en een inhoud $c(x)$. De lengte van x is het aantal symbolen waaruit x bestaat en de inhoud van x is per definitie het aantal a's dat x bevat. De eerste eigenschap van woorden die we zullen bekijken heet "gebalanceerdheid". Een woord w heet gebalanceerd als $|c(x) - c(y)| \leq 1$ voor elk tweetal factoren $x, y \subset w$ van gelijke lengte. Het blijkt dat deze eigenschap vrij beperkend is. Het is mogelijk om alle gebalanceerde **Z**-woorden te classificeren en dit doen we o.a. in Hoofdstuk 2. Gebalanceerde woorden zijn ingevoerd onder de naam "Sturmse woorden" door Morse en Hedlund.

Een ander belangrijk object is de complexiteitsfunctie van een woord. Als w een woord is dan geven we het aantal verschillende factoren $x \subset w$ met lengte n aan met $P(w, n)$. De afbeelding $P(w, .) : \mathbf{N} \to \mathbf{N}$ heet de complexiteitsfunctie van w en bevat in het algemeen belangrijke informatie over w. Voor **Z**-woorden w, bijvoorbeeld, is $P(w, n)$ uiteindelijk constant precies als w periodiek is. Voor niet-periodieke woorden w geldt kennelijk $P(w, n) \geq n+1$ voor alle n. We noemen een woord x stijf als $P(x, n) \leq n+1$ voor alle n.Het is ook mogelijk om de stijve **Z**-woorden te classificeren. Elk gebalanceerd woord is stijf maar er zijn stijve **Z**-woorden die niet gebalanceerd zijn. We noemen ze oneindige Hedlund woorden en ook deze worden bestudeerd in Hoofdstuk 2. Verder geven we in Hoofdstuk 2 een alternatief bewijs voor de formule voor $\mathrm{bal}(n)$, het aantal gebalanceerde woorden ter lengte n en we leiden een formule af voor $\mathrm{st}(n)$, het aantal stijve woorden ter lengte n. Daarna geven we wat andere beschrijvingen van gebalanceerde **Z**-woorden, voornamelijk met Beatty-woorden of biljartwoorden, en we leggen het verband met sturmse morfismes en kettingbreuken. We sluiten Hoofdstuk 2 af met een kleine generalisatie van de Robinsonvergelijking. Deze luidt $AB = Cab$ waarbij A, B, C palindromen zijn. (Een palindroom is een woord dat achterstevoren gelezen niet verandert).

In Hoofdstuk 3 bekijken we k-stijve woorden. Deze voldoen aan $P(w,n) \leq n+k$ voor alle n. Als w een k-stijf $\mathbf{Z}$-woord is en w is niet periodiek, dan hebben we $P(w,n) = n+c$ voor n groot genoeg. We zeggen dan, in goed Nederlands, dat w minimal block growth heeft. De kwalitatieve beschrijving van $\mathbf{Z}$-woorden met minimal block growth is bekend en wordt in dit proefschrift kwantitatief gemaakt. Verder zullen we de overeenkomsten en verschillen beschouwen tussen het geval $k = 1$ en het geval $k > 1$. We gebruiken de resultaten om een schatting te maken voor het aantal factoren van k-stijve $\mathbf{Z}$-woorden ter lengte n. Hierbij is k vast en n variabel. In Sectie 3.4 bekijken we welke $\mathbf{Z}$-woorden met minimal block growth door een codering van intervallen kunnen worden beschreven.

In Hoofdstuk 4 bestuderen we de functie $P(w,n)/n$ voor $\mathbf{Z}$-woorden w. Het is bekend dat deze functie naar elk positief geheel getal kan convergeren als $n \to \infty$. Hiervan geven we nieuwe voorbeelden. Het is echter niet mogelijk dat $P(n)/n$ een limiet $\alpha \in (1,2)$ heeft. Dit bewijzen we met behulp van woordgrafen. Het is ons onbekend of er überhaupt niet gehele limietwaarden kunnen optreden. Tot besluit geven we een voorbeeld van een $\mathbf{Z}$-woord w met $\lim_{n\to\infty} P(n)/n = 2$, terwijl $P(n+1) - P(n) = 2$ voor oneindig veel n.

Curriculum vitae.

Ik ben geboren op 20 september 1971 te Beverwijk. In juni 1989 behaalde ik mijn VWO-diploma aan het Berlingh College te Beverwijk en daarna begon ik met mijn wiskundestudie aan de Universiteit van Amsterdam. Ik studeerde daar cum laude af in oktober 1995. Mijn scriptie in de richting Analyse, getiteld "Representing measures for the disk algebra" schreef ik onder begeleiding van dr. J.J.O.O. Wiegerinck. Van 1 september 1996 to 1 september 2000 was ik werkzaam aan de Rijksuniversiteit Leiden als assistent in opleiding. Het onderzoek, waarvan de resultaten in dit proefschrift beschreven staan, werd begeleid door prof. dr. R. Tijdeman.